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Uniqueness of positive weak solutions of second order parabolic equations

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Dedicated to the memory
of Professor Witold Pogorzelski

§ 1. Introduction. In 1944 D. V. Widder [8] proved the uniqueness of positive solutions of the Cauchy problem for the equation of heat conduction. J. B. Serrin [7] and Avner Friedman [2] have given extensions of Widder's result to classical solutions of certain second order linear parabolic equations with variable coefficients. In this paper we extend Widder's result to weak solutions of the Cauchy problem for uniformly parabolic equations of the form

$$Lu \equiv \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left\{a_{ij}(x, t)\frac{\partial u}{\partial x_i}\right\} = 0 , \tag{1.1}$$

where we assume only boundedness and measurability of the coefficients a_{ij}. Our proof makes essential use of the existence theory for (1.1) given in [1] and of the Harnack inequality for solutions of (1.1) which was recently proved by Jürgen Moser [5].

Before giving a more precise description of our results we shall introduce some notation and make certain definitions. We use the symbol x to denote a point $(x_1, \ldots, x_n) \in E^n$ and t to denote a point on the real line. Let $S = E^n \times (0, T]$ and $\bar{S} = E^n \times [0, T]$ for some fixed $T > 0$. A function $f(x, t) \in L^1_{\mathrm{loc}}(S)$ is said to be *strongly differentiable* with respect to x in S if there exist n functions $f_{x_i} \in L^1_{\mathrm{loc}}(S)$ such that

$$\int_0^t d\tau \int_{E^n} f\varphi_{x_i} dx = -\int_0^t d\tau \int_{E^n} f_{x_i}\varphi dx \qquad (i = 1, \ldots, n)$$

for all $t \in (0, T]$ and for all $\varphi \in C^1(\bar{S})$ with compact support in E^n.

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Let $u_0(x) \in L^2_{\text{loc}}(E^n)$ be given. The *Cauchy problem* for L is the problem of finding a function $u = u(x, t)$ such that

$$(1.2) \qquad Lu = 0 \quad \text{for} \quad (x, t) \in S, \qquad u(x, 0) = u_0(x) \quad \text{for} \quad x \in E^n .$$

A function $u = u(x, t)$ is said to be a *weak solution* of the Cauchy problem (1.2) if (i) u is measurable in $\bar{S}$ and

$$\max_{[0,T]} \int_\Omega u^2(x, t)\, dx < \infty$$

for every open sphere $\Omega \subset E^n$, (ii) u is strongly differentiable with respect to x in S and

$$\int_0^t d\tau \int_\Omega \Big(u^2 + \sum_{i=1}^n u_{x_i}^2\Big) dx < \infty$$

for every $t \in (0, T]$, and (iii) u satisfies

$$(1.3) \qquad \int_{E^n} u(x, t)\varphi(x, t)\, dx + \int_0^t d\tau \int_{E^n} \Big(-u\varphi_\tau + \sum_{i,j=1}^n a_{ij} u_{x_i} \varphi_{x_j}\Big) dx = \int_{E^n} u_0(x)\varphi(x, 0)\, dx$$

for all $t \in [0, T]$ and all $\varphi \in C^1(\bar{S})$ with compact support in E^n.

Throughout this paper we shall assume that the coefficients a_{ij} of L are measurable in $\bar{S}$, that $a_{ij} = a_{ji}$, and that there exists a constant $\nu > 0$ such that

$$(1.4) \qquad \nu^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^n a_{ij}(x, t)\, \xi_i \xi_j \leqslant \nu |\xi|^2$$

almost everywhere in $\bar{S}$, where $|\xi|^2 = \sum_{i=1}^n \xi_i^2$ and the ξ_i are arbitrary real numbers.

Our first result (Theorem I) is a general uniqueness theorem for weak solutions of (1.2). In particular, we prove that if u is a weak solution of (1.2) with $u_0(x) = 0$ which satisfies

$$(1.5) \qquad \int_0^T dt \int_{E^n} e^{-2a|x|^2} u^2(x, t)\, dx < \infty$$

for some $a \geqslant 0$, then $u \equiv 0$. This result, which we prove in § 2, is a generalization of a result proved by Il'in, Kalašnikov and Oleĭnik in [4]. In § 3, we observe that Moser's Harnack inequality ([5]) implies that a non-negative weak solution of (1.2) with $u_0(x) = 0$ satisfies the growth condition (1.5) [1]. Thus we obtain the following generalization of Widder's Theorem 5 ([8]). If u is a non-negative weak solution of (1.2) with $u_0(x) = 0$ then $u \equiv 0$ (Theorem II). Our main result (Theorem III) is proved in § 5. We show there that every continuous non-negative weak solution of (1.2) is uniquely determined by its initial values. This result is a direct generalization of the theorem which Serrin announced in [7] [2]. The proof of Theorem III is based on Theorem II and a Maximum Principle which we prove in § 4. Finally, in § 6, we derive an estimate for the gradient of a non-negative weak solution under the assumption that the initial data satisfies a certain growth condition (Theorem IV).

We shall have occasion to refer to various function spaces in the remainder of this paper. For convenience we list these spaces here. Let Ω be an arbitrary open sphere in E^n. $H^{1,2}(\Omega)$ is the closure of the $C^\infty(\Omega)$ functions with respect to the norm

$$\|\varphi\|_{H^{1,2}} \equiv \Big\{ \int_\Omega \Big(\varphi^2 + \sum_i \varphi_{x_i}^2\Big) dx \Big\}^{1/2} .$$

$H^{1,2}(\Omega)$ is a Hilbert space with the obvious scalar product. It is known ([3]) that $H^{1,2}(\Omega)$ is equal to the space of functions φ which are strongly differentiable on Ω and for which

$$\int_\Omega \Big(\varphi^2 + \sum_i \varphi_{x_i}^2\Big) dx < \infty .$$

The closure with respect to the $H^{1,2}$-norm of the $C^\infty(\Omega)$ functions which have compact support in Ω will be denoted by $H_0^{1,2}(\Omega)$. $L^2[0, T; H^{1,2}(\Omega)]$ is the space of functions $\varphi(x, t)$ with the following properties; (i) φ is defined and measurable in $\Omega \times (0, T)$, (ii) for almost all $t \in (0, T)$, $\varphi(x, t) \in H^{1,2}(\Omega)$, and (iii) $\|\varphi\|_{H^{1,2}} \in L^2(0, T)$. The space $L^2[0, T; H_0^{1,2}(\Omega)]$ is similarly defined with (ii) replaced by: (ii)′ for almost all $t \in (0, T)$, $\varphi(x, t) \in H_0^{1,2}(\Omega)$. $L^2[0, T; H^{1,2}(\Omega)]$ and $L^2[0, T; H_0^{1,2}(\Omega)]$ are known to be Banach spaces with respect to the norm

$$\Big(\int_0^T \|\varphi\|_{H^{1,2}}^2 dt\Big)^{1/2} .$$

$L^\infty[0, T; L^2(\Omega)]$ is the space of functions $\varphi(x, t)$ such that (i) φ is defined and measurable on $\Omega \times (0, T)$, (ii) for almost all $t \in (0, T)$, $\varphi(x, t) \in L^2(\Omega)$,

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(1) In [5] Moser proves the Harnack inequality for weak solutions which have square integrable strong derivatives with respect to t. Since we do not assume the existence of a t-derivative here we shall need an extension of the results given in [5]. Such an extension was announced by Moser at the Joint Soviet-American Symposium on Partial Differential Equations at Novosibirsk in August 1963. In the Appendix to this paper we show how this extension can be derived from [5].

(2) The author is greatly indebted to Professor Serrin for giving him access to the unpublished proof of the theorem announced in [7].

and (iii) $\|\varphi\|_{L^2(\Omega)} \in L^\infty(0, T)$. This space is also a Banach space with respect to the norm

$$\operatorname*{ess.max.}_{[0,T]} \|\varphi\|_{L^2(\Omega)} .$$

Finally, we will denote by $H^{1,2}[0, T; L^2(\Omega)]$ the closure of the C^∞ functions on $\Omega \times (0, T)$ with respect to the norm

$$\left\{\int_0^T dt \int_\Omega (\varphi^2 + \varphi_t^2) dx\right\}^{1/2} .$$

We note that by a standard limiting argument we can reformulate our definition of weak solution as follows. A function u is a weak solution of the Cauchy problem for (1.2) if, for every sphere $\Omega \subset E^n$, $u \in L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H^{1,2}(\Omega)]$ and u satisfies (1.3) for all $t \in [0, T]$ and all test functions $\varphi \in L^2[0, T; H_0^{1,2}(\Omega)] \cap H^{1,2}[0, T; L^2(\Omega)]$.

§ 2. General uniqueness theorem. We shall prove the following

THEOREM I. *Let u be a weak solution of* (1.2) *in S with $u_0(x) = 0$. If*

$$\int_0^T dt \int_{E^n} e^{-2a|x|^2} u^2(x, t)\, dx < \infty \tag{2.1}$$

for some $a \geqslant 0$ then $u \equiv 0$ in $\bar{S}$.

As we remarked in § 1, Theorem I generalizes a uniqueness theorem proved in [4]. The proof of Theorem I follows essentially the same lines as the proof of the corresponding result in [4]. The main difference lies in the fact that we do not assume the existence of any t-derivatives of u. In [4] it is assumed that u has a square integrable strong derivative with respect to t.

Let $\gamma_R(x) \in C^1(E^n)$ be such that $\gamma_R(x) \equiv 1$ for $0 \leqslant |x| \leqslant R$, $\gamma_R(x) \equiv 0$ for $|x| \geqslant R+2$, γ_R is a non-increasing function of $|x|$ for $|x| \geqslant 0$, and $\sum_{i=1}^n \gamma_{Rx_i}^2 \leqslant 1$. For arbitrary $\beta \geqslant 0$ let $\bar{u}(x, t) = u(x, t) \exp\{-(a + \frac{1}{2}\beta t)|x|^2\}$, $\tilde{u}(x, t) = \gamma_R(x) \bar{u}(x, t)$ and $\hat{u}(x, t) = |x| \tilde{u}(x, t)$. Let $K_h(t)$ be an *even* averaging kernel with compact support in $|t| < h$ (cf. [3]). For any locally integrable function $f(t)$ we shall write

$$f^h(\tau) = \int_0^t K_h(\tau - \eta) f(\eta)\, d\eta$$

for $0 \leqslant \tau \leqslant t$.

Let

$$\varphi_h(x, \tau) = \gamma_R(x) \tilde{u}^h(x, \tau) \exp\{-(a + \tfrac{1}{2}\beta\tau)|x|^2\} .$$

It is clear that, for all sufficiently small h, φ_h is an admissible test function in (1.3). Thus we have

$$\int_{E^n} u(x,t)\varphi_h(x,t)\,dx + \int_0^t d\tau \int_{E^n}\Big(-\varphi_{h\tau}u + \sum_{i,j} a_{ij}u_{x_i}\varphi_{hx_j}\Big)dx = 0 \tag{2.2}$$

for all $t\in[0,T]$. We investigate the behaviour of the individual terms in (2.2) as $h\to 0$.

It is not difficult to verify that

$$\int_0^t d\tau \int_{E^n} u\varphi_{h\tau}\,dx \tag{2.3}$$

$$= \int_0^t\int_0^t K_h'(\tau-\eta)\Big\{\int_{E^n}\tilde u(x,\tau)\tilde u(x,\eta)\,dx\Big\}d\tau\,d\eta - \tfrac12\beta\int_0^t d\tau\int_{E^n}\hat u(x,\tau)\hat u^h(x,\tau)\,dx\,.$$

In view of the symmetry of K_h, the first term on the right in (2.3) is zero. Thus

$$\int_0^t d\tau \int_{E^n} u\varphi_{h\tau}\,dx = -\tfrac12\beta\|\hat u\|^2 + A\,,$$

where

$$|A| \leqslant \tfrac12\beta\|\hat u\|\,\|\hat u^h - \hat u\|\,.$$

Here

$$\|f\|^2 = \int_0^t d\tau \int_{E^n} f^2\,dx\,.$$

By a well known property of averaging kernels (cf. [3]), $A\to 0$ as $h\to 0$. Therefore

$$-\int_0^t d\tau \int_{E^n} u\varphi_{h\tau}\,dx \to \tfrac12\beta\|\hat u\|^2 \tag{2.4}$$

as $h\to 0$.

Now consider

$$\int_0^t d\tau \int_{E^n}\sum_{i,j} a_{ij}u_{x_i}\varphi_{hx_j}\,dx = \int_0^t d\tau\int_{E^n}\sum_{i,j}a_{ij}u_{x_i}(\gamma_R^2 e^{-(2\alpha+\beta\tau)|x|^2}u)_{x_j}dx + B\,,$$

where

$$B = \int_0^t d\tau\int_{E^n}\sum_{i,j}a_{ij}u_{x_i}\{\gamma_R e^{-(\alpha+\frac12\beta\tau)|x|^2}(\tilde u^h - \tilde u)\}_{x_j}dx\,.$$

Since

$$\{\gamma_R e^{-(\alpha+\frac12\beta\tau)|x|^2}(\tilde u^h-\tilde u)\}_{x_j}$$

$$= \gamma_R e^{-(\alpha+\frac12\beta\tau)|x|^2}\{(\tilde u^h - \tilde u)_{x_j} + 2\gamma_{Rx_j}(\bar u^h - \bar u) - 2(\alpha + \tfrac12\beta\tau)x_j(\tilde u^h - \tilde u)\}$$

we can write B as the sum of three integrals which we shall denote by B_p for $p=1,2,3$. Using the Schwarz inequality and (1.4) we obtain

$$|B_1| \leqslant \nu M \|\nabla(\tilde{u}^h - \tilde{u})\|,$$

where $\nabla f(x,t) = (\partial f/\partial x_1, \ldots, \partial f/\partial x_n)$ and

$$M = \|\gamma_R \nabla u \exp\{-(\alpha+\tfrac{1}{2}\beta\tau)|x|^2\}\|.$$

Similarly, taking into account the properties of γ_R we obtain

$$|B_2| \leqslant \nu M \Big(\int_0^t d\tau \int_{|x|\leqslant R+2} |\overline{u}^h - \overline{u}|^2 dx \Big)^{1/2}.$$

Finally, if $t \leqslant \delta$ we have

$$|B_3| \leqslant 2\nu(\alpha+\tfrac{1}{2}\beta\delta) M \|\hat{u}^h - \hat{u}\|.$$

Thus it follows from the known properties of averaging kernels (cf. [3]) that $B \to 0$ as $h \to 0$. Therefore

$$\int_0^t d\tau \int_{E^n} \sum_{i,j} a_{ij} u_{x_i} \varphi_{hx_j} dx \to \int_0^t d\tau \int_{E^n} \sum_{i,j} a_{ij} u_{x_i} (\gamma_R^2 e^{-(2\alpha+\beta\tau)|x|^2} u)_{x_j} dx \tag{2.5}$$

as $h \to 0$.

Since we can write

$$\varphi_h(x,t) = \gamma_R e^{-(\alpha+\frac{1}{2}\beta t)|x|^2} \int_0^h K_h(\zeta) \gamma_R e^{-(\alpha+\frac{1}{2}\beta(t-\zeta))|x|^2} u(x, t-\zeta)\, d\zeta$$

we have

$$\int_{E^n} u(x,t)\varphi_h(x,t)\,dx = \tfrac{1}{2} \int_{E^n} \tilde{u}^2(x,t)\,dx + C,$$

where

$$C = \int_0^h K_h(\zeta) \Big\{ \int_{E^n} \tilde{u}(x,t)[\tilde{u}(x,t-\zeta) - \tilde{u}(x,t)]\,dx \Big\} d\zeta.$$

Let $\psi(x) \in C_0^1(E^n)$. It follows from (1.3) that

$$\int_{E^n} u(x, t-\zeta)\psi(x)\,dx \to \int_{E^n} u(x,t)\psi(x)\,dx \tag{2.6}$$

as $t-\zeta \to t+$ through values in $[0, T]$. Since $\tilde{u}$ is the product of u and a smooth compact support function, (2.6) also holds with u replaced by $\tilde{u}$. On the other hand, since $\tilde{u}(x,t) \in L^2(E^n)$ and vanishes for $|x| \geqslant R+2$, it can be approximated in the mean with arbitrary accuracy by $C_0^1(E^n)$ functions. Thus $C \to 0$ as $h \to 0$ and we have

$$\int_{E^n} u(x,t)\varphi_h(x,t)\,dx \to \tfrac{1}{2} \int_{E^n} \tilde{u}^2(x,t)\,dx \tag{2.7}$$

as $h \to 0$.

Thus it follows from (2.4), (2.5) and (2.7) that if we let $h \to 0$ in (2.2) we obtain

$$(2.8)\quad \tfrac{1}{2}\int_{E^n} \tilde{u}^2(x,t)\,dx + \tfrac{1}{2}\beta\|\hat{u}\|^2 + \int_0^t d\tau \int_{E^n} \sum_{i,j} a_{ij} u_{x_i} (\gamma_R^2 e^{-(2a+\beta\tau)|x|^2} u)_{x_j} dx = 0$$

for $t \in [0, \min(\delta, T)]$.

Let D denote the third term on the left hand side of (2.8). We have

$$D = \int_0^t d\tau \int_{E^n} \sum_{i,j} a_{ij} v_i v_j\, dx + 2\int_0^t d\tau \int_{E^n} u e^{-(a+\frac{1}{2}\beta\tau)|x|^2} \sum_{i,j} a_{ij} v_i \gamma_{Rx_j}\, dx -$$
$$-2\int_0^t d\tau \int_{E^n} (2a+\beta\tau)\gamma_R u e^{-(a+\frac{1}{2}\beta\tau)|x|^2} \sum_{i,j} a_{ij} v_i x_j\, dx = D_1 + D_2 + D_3,$$

where

$$v_i = \gamma_R e^{-(a+\frac{1}{2}\beta\tau)|x|^2} u_{x_i}.$$

It is easy to verify that

$$|D_2| \leqslant \tfrac{1}{2}D_1 + 2\nu \int_0^t d\tau \int_{R\leqslant|x|\leqslant R+2} u^2 e^{-(2a+\beta\tau)|x|^2} dx$$

and that

$$|D_3| \leqslant \tfrac{1}{2}D_1 + 2(2a+\beta\delta)^2\nu\|\hat{u}\|^2$$

for $0 \leqslant t \leqslant \delta$. Thus, from (2.8), we obtain

$$(2.10)\quad \tfrac{1}{2}\int_{E^n} \tilde{u}^2(x,t)\,dx + \{\tfrac{1}{2}\beta - 2\nu(2a+\beta\delta)^2\}\|\hat{u}\|^2$$
$$\leqslant 2\nu \int_0^t d\tau \int_{R\leqslant|x|\leqslant R+2} u^2 e^{-(2a+\beta\tau)|x|^2} dx$$

for all $t \in [0, \min(\delta, T)]$.

If $a = 0$, we set $\beta = 0$ and $\delta = +\infty$. For $a > 0$, given any δ such that $0 < \delta < 1/32a\nu$ there exists a $\beta = \beta(\delta) > 0$ such that $\beta = 4\nu(2a+\beta\delta)^2$. For β chosen in this manner, the second term on the right hand side in (2.10) is zero. In view of (2.1) and the properties of γ_R it follows from (2.10) that

$$\int_{|x|\leqslant R} e^{-(2a+\beta t)|x|^2} u^2(x,t)\,dx \leqslant 4\nu \int_0^{\min(\delta,T)} d\tau \int_{E^n} e^{-(2a+\beta\tau)|x|^2} u^2 dx$$

for all $t \in [0, \min(\delta, T)]$. Thus if we let $R \to \infty$ we obtain

$$\int_{E^n} e^{-(2a+\beta t)|x|^2} u^2(x,t)\,dx \leqslant 4\nu \int_0^{\min(\delta,T)} d\tau \int_{E^n} e^{-(2a+\beta\tau)|x|^2} u^2 dx$$

on $[0, \min(\delta, T)]$. Finally, if we integrate on t from 0 to $\min(\delta, T)$ we find

$$\{1-4\nu\min(\delta, T)\}\int_0^{\min(\delta,T)} d\tau \int_{E^n} e^{-(2\alpha+\beta\tau)|x|^2}u^2dx \leqslant 0\,.$$

Therefore for $\delta < \min(1/32\alpha\nu, 1/4\nu)$ we have $u \equiv 0$ on $[0, \min(\delta, T)]$. In case $\delta \geqslant T$ this completes the proof. If $\delta < T$, the proof can be completed by a finite number of repetitions of the same argument.

Remark. The uniform parabolicity of L does not enter into the proof of Theorem I. It suffices to assume that the a_{ij} are bounded and measurable and that $\sum_{i,j} a_{ij}(x,t)\xi_i\xi_j \geqslant 0$ almost everywhere in $\bar{S}$. Moreover, we may replace the assumption that

$$\int_0^t d\tau \int_\Omega |\nabla u|^2 dx < \infty$$

for all $t \in (0, T]$ and every open sphere $\Omega \subset E^n$ by

$$\int_0^t d\tau \int_\Omega \sum_{i,j=1}^n a_{ij} u_{x_i} u_{x_j} dx < \infty\,.$$

On the other hand, Theorem I can be easily extended to uniformly parabolic equations with lower order terms (cf. [4]).

§ 3. Non-negative solutions with zero initial data. Let u be a weak solution of (1.2) in S. Moser [5] ([1]) has shown that u can be redefined on a set of measure zero so that the resulting function is Hölder continuous on every compact subdomain of S. We assume that this has been done. Hence we may speak of the value of u at any point of S. We shall assume that $u \geqslant 0$ in S and that $u_0(x) = 0$ for all $x \in E^n$. We extend the definition of u to the half-space $t < 0$ by setting $u(x,t) = 0$ for $t < 0$. It is clear that the resulting function, which we again call u, is a weak solution of the Cauchy problem

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\left\{\tilde{a}_{ij}(x,t)\frac{\partial u}{\partial x_i}\right\} = 0 \quad \text{for} \quad (x,t) \in E^n \times (-\varrho, T],$$

$$u(x, -\varrho) = 0 \quad \text{for} \quad x \in E^n,$$

where $\varrho \in (0, \infty)$ is arbitrary and

$$\tilde{a}_{ij}(x,t) = \begin{cases} a_{ij}(x,t) & \text{for} \quad t \geqslant 0\,, \\ \delta_{ij} & \text{for} \quad t < 0\,. \end{cases}$$

Again, we may assume that u is continuous and $u \geqslant 0$ in $E^n \times (-\varrho, T]$.

According to Theorem 2 of [5] (¹), there exists a constant $c > 0$ which depends only on n and ν such that

$$u(x, t) \leqslant u(0, T)\exp\left\{c\left(\frac{|x|^2}{\delta} + \frac{T}{\varrho} + 1\right)\right\}$$

for $(x, t) \in E^n \times [0, T-\delta]$ (³). Here δ is an arbitrary number such that $0 < \delta < T$. If $u(0, T) = 0$ then $u(x, t) \equiv 0$ on $E^n \times [0, T-\delta]$. Otherwise, we have

$$\int_0^{T-\delta} dt \int_{E^n} e^{-2a|x|^2} u^2(x, t)\,dx < \infty$$

for any $a > c/\delta$, and it follows from Theorem I that $u(x, t) \equiv 0$ on $E^n \times [0, T-\delta]$. Therefore, since $\delta > 0$ is arbitrary and u is continuous, we have

THEOREM II. *If u is a non-negative weak solution of* (1.2) *in S with $u_0(x) = 0$, then $u \equiv 0$ on $\bar{S}$.*

§ 4. Maximum principle. In the proof of our main result (§ 5) we will need to be able to make pointwise comparisons between various weak solutions of (1.2). Specifically, we will need the following generalization of the classical (weak) maximum principle for parabolic equations. Let $\Sigma = (|x| < \varrho) \times (0, T]$, $\Sigma^* = (|x| < \varrho^*) \times (0, T]$ and $\Gamma = \{(|x| < \varrho) \times (t = 0)\} \cup \{(|x| = \varrho) \times [0, T]\}$ for some fixed numbers ϱ and ϱ^*, where $0 < \varrho < \varrho^*$. Let $u \in C^0(\bar{\Sigma}^*)$ be a weak solution of (1.2) in Σ^*, that is, we assume u satisfies (1.3) for test functions with compact support in $|x| < \varrho^*$. Let $v \in C^0(\bar{\Sigma}) \cap L^2[0, T; H_0^{1,2}(|x| < \varrho)]$ be the weak solution of the boundary value problem

$$Lv = 0 \quad \text{for} \quad (x, t) \in \Sigma, \qquad v(x, 0) = v_0(x) \quad \text{for} \quad |x| < \varrho,$$
$$v = 0 \qquad \text{for} \qquad (x, t) \in (|x| = \varrho) \times [0, T],$$

where $v_0(x)$ is assumed to be identically zero in a neighbourhood of $|x| = \varrho$. In particular, v satisfies (1.3) for test functions with compact support in $|x| < \varrho$. The existence and uniqueness of v is proved in [1]. We shall prove the

MAXIMUM PRINCIPLE. *Let $w = u - v$, where u and v are defined above. Then*

(4.1) $$\min_\Gamma w \leqslant w(x, t) \leqslant \max_\Gamma w$$

for all $(x, t) \in \bar{\Sigma}$.

(³) The inequality (1.6) in [5] is not valid for $0 < t < T-\delta$. It holds only for $\delta < t < T-\delta$ and we do not use it here.

Choose numbers ϱ_j for $j = 1, \ldots, 4$ such that $\varrho = \varrho_0 < \varrho_1 < \ldots < \varrho_4 \leqslant \varrho^*$. For arbitrary $\varepsilon \in (0, T)$ let $\Sigma_j = (|x| < \varrho_j) \times ((1-j/4)\varepsilon, T]$. Clearly $\Sigma_0 \subset \Sigma_1 \subset \ldots \subset \Sigma_4 \subseteq \Sigma^*$. Let $\eta = \eta(x, t)$ be a smooth function such that $\eta \equiv 1$ in $\bar{\Sigma}_2$ and $\eta \equiv 0$ in the exterior of Σ_3 for $t \leqslant T$. The function $z = \eta u$ belongs to $C^0(\bar{\Sigma}_4) \cap L^2[0, T; H_0^{1,2}(|x| < \varrho_4)]$ and satisfies

$$\int_{|x|<\varrho_4} \psi(x, t) z(x, t)\, dx + \int_0^t d\tau \int_{|x|<\varrho_4} \Big(-\psi_\tau z + \sum_{i,j} a_{ij} z_{x_i} \psi_{x_j}\Big) dx$$
$$= \int_0^t d\tau \int_{|x|<\varrho_4} \Big\{\psi\Big(\eta_\tau u - \sum_{i,j} a_{ij} u_{x_i} \eta_{x_j}\Big) + u \sum_{i,j} a_{ij} \eta_{x_i} \psi_{x_j}\Big\} dx$$

for all $t \in [0, T]$ and $\psi \in C^1(\bar{\Sigma}_4)$ with compact support in $|x| < \varrho_4$. In other words, z is a weak solution in $C^0(\bar{\Sigma}_4) \cap L^2[0, T; H_0^{1,2}(|x| < \varrho_4)]$ of the boundary value problem

$$\text{(4.2)} \quad Lz = f - \operatorname{div} \mathbf{g} \quad \text{for} \quad (x, t) \in \Sigma_4, \qquad z = 0 \quad \text{for} \quad (x, t) \in \Gamma_4,$$

where $\Gamma_4 = \{(|x| < \varrho_4) \times (t = 0)\} \cup \{(|x| = \varrho_4) \times [0, T]\}$, $f = \eta_t u - \sum_{i,j} a_{ij} u_{x_i} \eta_{x_j}$, $\mathbf{g} = (g_1, \ldots, g_n)$ and $g_j = u \sum_i a_{ij} \eta_{x_i}$. Note that f and $\mathbf{g}$ have support in $\Sigma_3 - \Sigma_2$ and that $f, g_j \in L^2(\Sigma_4)$.

It is known [1] that the boundary value problem (4.2) has a unique weak solution in $\mathfrak{L} = L^\infty[0, T; L^2(|x| < \varrho_4)] \cap L^2[0, T; H_0^{1,2}(|x| < \varrho_4)]$. Thus z is the only solution in this class. We extend the coefficients a_{ij} of L for $t \notin [0, T]$ by setting $a_{ij}(x, t) = \delta_{ij}$ for $t \notin [0, T]$. Let $a_{ij}^{(m)}$ denote the integral average of a_{ij} formed with a kernel whose support lies in $(|x|^2 + t^2)^{1/2} < 1/m$, and similarly for $f^{(m)}$, $\mathbf{g}^{(m)} = (g_1^{(m)}, \ldots, g_n^{(m)})$. The boundary value problem

$$L_m z = \frac{\partial z}{\partial t} - \sum_{i,j=1}^n \frac{\partial}{\partial x_j}\Big(a_{ij}^{(m)} \frac{\partial z}{\partial x_i}\Big) = f^{(m)} - \operatorname{div} \mathbf{g}^{(m)} \quad \text{for} \quad (x, t) \in \Sigma_4.$$
$$z = 0 \quad \text{for} \quad (x, t) \in \Gamma_4$$

has a unique classical solution $z = z_m$ for each $m \geqslant 1$. Moreover,

$$\text{(4.3)} \quad \max_{[0,T]} \int_{|x|<\varrho_4} z_m^2(x, t)\, dx + \frac{1}{\nu} \int_0^T dt \int_{|x|<\varrho_4} |\nabla z_m|^2 dx \leqslant c(\nu, T)(\|f\|^2 + \|\mathbf{g}\|^2).$$

It is shown in [1] that the sequence $\{z_m\}$ converges weakly in $\mathfrak{L}$ to z as $m \to \infty$.

For m sufficiently large the supports of $f^{(m)}$ and $\mathbf{g}^{(m)}$ lie in the exterior of $\bar{\Sigma}_1$. Thus $L_m z_m = 0$ in Σ_1 for sufficiently large m. Moreover, according to (4.3),

$$\iint_{\Sigma_1} z_m^2\, dx\, dt \leqslant c(\nu, T)(\|f\|^2 + \|\mathbf{g}\|^2).$$

It follows from results of Moser ([5]; Theorem 3 and the Corollary to Theorem 1) that the sequence $\{z_m\}$ is uniformly bounded and equicontinuous in $\bar{\Sigma}_0$. Since $z_m \to z$ weakly in $\mathfrak{L}$ and $z \equiv u$ on $\bar{\Sigma}_0$, every convergent subsequence of $\{z_m\}$ converges to u in $\bar{\Sigma}_0$. Hence $z_m \to u$ uniformly in $\bar{\Sigma}_0$.

On the other hand, it is proved in [1] that v is the uniform limit in $\bar{\Sigma}$ of the sequence $\{v_m\}$ of classical solutions of the boundary value problem

$$L_m v_m = 0 \quad \text{for} \quad (x,t) \in \Sigma,$$
$$v_m(x,0) = v_0^{(m)}(x) \quad \text{for} \quad |x| < \varrho,$$
$$v_m = 0 \quad \text{for} \quad (x,t) \in (|x| = \varrho) \times [0, T].$$

Let $\Gamma_0 = \{(|x| < \varrho) \times (t = \varepsilon)\} \cup \{(|x| = \varrho) \times [\varepsilon, T]\}$ and $w_m = z_m - v_m$. Since $L_m w_m = 0$ in Σ_0, the classical maximum principle holds for w_m and we have

$$\min_{\Gamma_0} w_m < w_m(x,t) < \max_{\Gamma_0} w_m$$

for all $(x,t) \in \Sigma_0$. Now $w_m \to w$ uniformly on Γ_0. Hence given any $\delta > 0$ there is an $m(\delta) > 0$ such that

$$\min_{\Gamma_0} w - \delta < w_m(x,t) < \max_{\Gamma_0} w + \delta$$

for all $(x,t) \in \Sigma_0$ and $m > m(\delta)$. If we let $m \to \infty$ we obtain

$$\min_{\Gamma_0} w - \delta < w(x,t) < \max_{\Gamma_0} w + \delta$$

in Σ_0. Finally, since δ is arbitrary, it follows that

$$\min_{\Gamma_0} w \leqslant w(x,t) \leqslant \max_{\Gamma_0} w \tag{4.4}$$

in $\bar{\Sigma}_0$.

Consider a fixed point $(x,t) \in \Sigma$. According to (4.4) we have

$$\begin{aligned} A \equiv \min\{\min_{\sigma_\varepsilon} w, \min_{|x|\leqslant\varrho} w(x,\varepsilon)\} &\leqslant w(x,t) \\ &\leqslant \max\{\max_{\sigma_\varepsilon} w, \max_{|x|\leqslant\varrho} w(x,\varepsilon)\} \equiv B \end{aligned} \tag{4.5}$$

for any $\varepsilon \in (0,t)$, where $\sigma_\varepsilon = (|x| = \varrho) \times [\varepsilon, T]$. Let $\theta(\varepsilon) = \max\limits_{|x|\leqslant\varrho} |w(x,\varepsilon) - w(x,0)|$. Since w is uniformly continuous in $\bar{\Sigma}$, $\theta(\varepsilon) \to 0$ as $\varepsilon \to 0$. Now

$$B \leqslant \max\{\max_{\sigma_0} w, \max_{|x|\leqslant\varrho} w(x,0) + \theta(\varepsilon)\} \leqslant \max_{\Gamma} w + \theta(\varepsilon) \text{ and } A \geqslant \min_{\Gamma} w - \theta(\varepsilon).$$

Hence

$$\min_{\Gamma} w - \theta(\varepsilon) \leqslant w(x,t) \leqslant \max_{\Gamma} w + \theta(\varepsilon)$$

and (4.1) follows by letting $\varepsilon \to 0$.

Remark. Since every weak solution is continuous in the interior of its domain of definition (cf. [5]), (4.5) holds without the hypothesis that $u \in C^0(\bar{\Sigma}^*)$ and $v \in C^0(\bar{\Sigma})$.

§ 5. The representation theorem. We consider a non-negative weak solution u of (1.2) in S. We assume that u is continuous in $\bar{S}$. Let

$$\Sigma_m = (|x| < m) \times (0, T]$$

and

$$\Gamma_m = \{(|x| < m) \times (t = 0)\} \cup \{(|x| = m) \times [0, T]\}.$$

For each integer $m \geqslant 3$, let $\gamma_m = \gamma_m(x)$ denote a smooth function defined for all $x \in E^n$ such that $\gamma_m \equiv 1$ for $|x| \leqslant m-2$, $\gamma_m \equiv 0$ for $|x| \geqslant m-1$, $0 \leqslant \gamma_m \leqslant 1$, and $|\nabla \gamma_m|$ bounded independent of m. Consider the sequence of boundary value problems

$$(5.1)_m \qquad \begin{gathered} Lv = 0 \quad \text{for} \quad (x, t) \in \Sigma_m, \qquad v(x, 0) = \gamma_m(x) u(x, 0) \quad \text{for} \quad |x| \leqslant m, \\ v = 0 \quad \text{for} \quad (x, t) \in (|x| = m) \times [0, T]. \end{gathered}$$

It is shown in [1] that, for each $m \geqslant 3$, $(5.1)_m$ has a unique continuous weak solution. More precisely, there exists a function $v_m(x, t) \in C^0(\bar{\Sigma}_m) \cap \cap L^2[0, T; H_0^{1,2}(|x| < m)]$ which satisfies

$$(5.2)_m \qquad \int\limits_{|x|<m} v_m(x, t)\varphi(x, t)\,dx + \int\limits_0^t d\tau \int\limits_{|x|<m} \Big(-v_m \varphi_\tau + \sum_{i,j} a_{ij} v_{mx_i} \varphi_{x_j}\Big)\,dx = \int\limits_{|x|<m} \gamma_m(x) u(x, 0)\varphi(x, 0)\,dx$$

for all $t \in [0, T]$ and all $\varphi \in L^2[0, T; H_0^{1,2}(|x| < m)] \cap H^{1,2}[0, T; L^2(|x| < m)]$. Moreover, v_m is the only function in $L^\infty[0, T; L^2(|x| < m)] \cap L^2[0, T; H_0^{1,2}(|x| < m)]$ which satisfies $(5.2)_m$, and $v_m \geqslant 0$ on $\bar{\Sigma}_m$. We extend the definition of v_m to all of $\bar{S}$ by setting $v_m = 0$ for $(x, t) \in \bar{S} - \bar{\Sigma}_m$.

We shall now prove

THEOREM III. *Let u be a non-negative weak solution of* (1.2) *in S and let u be continuous in $\bar{S}$. If v_m is the sequence of weak solutions of the boundary value problems* $(5.1)_m$, *then $v_m \to u$ as $m \to \infty$ pointwise in $\bar{S}$ and uniformly in every compact subregion of $\bar{S}$.*

Thus, in particular, a continuous non-negative weak solution of (1.2) is uniquely determined by its initial data.

Since $v_m = 0$ and $v_{m+1} \geqslant 0$ in $\bar{S} - \bar{\Sigma}_m$, we have $v_{m+1} \geqslant v_m$ in $\bar{S} - \bar{\Sigma}_m$. Moreover, $v_{m+1} \geqslant v_m$ on Γ_m. Hence, by the maximum principle, $v_{m+1} \geqslant v_m$ in $\bar{\Sigma}_m$. Therefore, $v_{m+1} \geqslant v_m$ in $\bar{S}$ and the sequence $\{v_m\}$ is non-decreasing. On the other hand, for any m, $u \geqslant v_m$ on $\bar{S} - \bar{\Sigma}_m$ and Γ_m. It follows from the maximum principle that $u \geqslant v_m$ in $\bar{\Sigma}_m$. Thus $u \geqslant v_m$ in $\bar{S}$ for all m, and we have

$$0 \leqslant v_3 \leqslant v_4 \leqslant \ldots \leqslant v_m \leqslant \ldots \leqslant u$$

for all $(x, t) \in \bar{S}$. Therefore, there exists a function $w = w(x, t)$ such that $0 \leqslant w \leqslant u$ in $\bar{S}$ and $v_m \to w$ as $m \to \infty$ pointwise in $\bar{S}$.

Let Ω be an arbitrary open sphere in E^n and Ω^* be a concentric sphere such that $\Omega \subset \Omega^*$. Choose m_0 so large that $\Omega^* \subset (|x| < m_0)$. Then for any $\varphi \in L^2[0, T; H_0^{1,2}(\Omega^*)] \cap H^{1,2}[0, T; L^2(\Omega^*)]$ we have

$$\int_{|x|<m} v_m(x, t)\varphi(x, t)\,dx + \int_0^t d\tau \int_{|x|<m} \Big(-v_m\varphi_\tau + \sum_{i,j} a_{ij} v_{mx_i}\varphi_{x_j}\Big)\,dx = \int_{|x|<m} v_m(x, 0)\varphi(x, 0)\,dx \tag{5.3}$$

for all $t \in [0, T]$ and $m \geqslant m_0$. Let $\mu = \mu(x)$ be a smooth function defined for all $x \in E^n$ such that $\mu \equiv 1$ on $\bar{\Omega}$, μ has compact support in Ω^*, and $0 \leqslant \mu \leqslant 1$. Define

$$\varphi_h(x, t) = \mu^2(x) v_m^h(x, t)$$

(cf. § 2). Clearly φ_h is an admissible test function in (5.3). By the argument employed in the first part of the proof of Theorem I we obtain

$$\tfrac{1}{2}\int_{\Omega^*} \mu^2(x) v_m^2(x, t)\,dx + \int_0^t d\tau \int_{\Omega^*} \mu^2 \sum_{i,j} a_{ij} v_{mx_i} v_{mx_j}\,dx = \tfrac{1}{2}\int_{\Omega^*} \mu^2(x) v_m^2(x, 0)\,dx - 2\int_0^t d\tau \int_{\Omega^*} v_m \mu \sum_{i,j} a_{ij} v_{mx_i}\mu_{x_j}\,dx\,.$$

Thus

$$\int_{\Omega^*} \mu^2(x) v_m^2(x, t)\,dx + \int_0^t d\tau \int_{\Omega^*} \mu^2 \sum_{i,j} a_{ij} v_{mx_i} v_{mx_j}\,dx \leqslant \int_{\Omega^*} \mu^2(x) v_m^2(x, 0)\,dx + 4\int_0^t d\tau \int_{\Omega^*} v_m^2 \sum_{i,j} a_{ij}\mu_{x_i}\mu_{x_j}\,dx\,.$$

Finally, using (1.4), the properties of μ and the fact that $v_m \leqslant u$, we have

$$\int_{\Omega} v_m^2(x, t)\,dx + \frac{1}{\nu}\int_0^t d\tau \int_{\Omega} |\nabla v_m|^2\,dx \leqslant \int_{\Omega^*} u^2(x, 0)\,dx + c\int_0^t d\tau \int_{\Omega^*} u^2\,dx \tag{5.4}$$

for all $t \in [0, T]$ and $m \geqslant m_0$, where c is a positive constant which depends only on ν, Ω and Ω^*. We conclude from (5.4) that there is a subsequence of the v_m which converges weakly to a limit function in $L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H^{1,2}(\Omega)]$. However, since $v_m \to w$ pointwise in $\bar{S}$ it follows that the whole sequence $\{v_m\}$ converges weakly to w in $L^\infty[0, T; L^2(\Omega)] \cap L^2[0, T; H^{1,2}(\Omega)]$.

For any $m \geqslant m_0$, v_m satisfies

$$\int_{E^n} v_m(x, t)\varphi(x, t)\,dx + \int_0^t d\tau \int_{E^n} \Big(-v_m\varphi_\tau + \sum_{i,j} a_{ij} v_{mx_i}\varphi_{x_j}\Big)\,dx = \int_{E^n} \gamma_m(x) u(x, 0)\varphi(x, 0)\,dx \tag{5.5}$$

for all $t \in [0, T]$ and all $\varphi \in L^2[0, T; H_0^{1,2}(\Omega)] \cap H^{1,2}[0, T; L^2(\Omega)]$. In view of the weak convergence and pointwise monotone convergence of v_m to w we can let $m \to \infty$ in (5.5) to obtain

$$\int_{E^n} w(x, t)\varphi(x, t)\,dx + \int_0^t d\tau \int_{E^n} \Big(-w\varphi_\tau + \sum_{i,j} a_{ij} w_{x_i}\varphi_{x_j}\Big)\,dx = \int_{E^n} u(x, 0)\varphi(x, 0)\,dx$$

for all $t \in [0, T]$ and all $\varphi \in L^2[0, T; H_0^{1,2}(\Omega)] \cap H^{1,2}[0, T; L^2(\Omega)]$. Since Ω is arbitrary, it follows that w is a weak solution of the Cauchy problem (1.2) in S with initial data $u(x, 0)$. Therefore $u(x, t) - w(x, t)$ is a non-negative weak solution of (1.2) with zero initial data. We conclude from Theorem II that $w \equiv u$ in $\bar{S}$ and $v_m \to u$ pointwise in $\bar{S}$. The uniformity of the convergence in compact subregions of $\bar{S}$ follows from the continuity of u and Dini's theorem on monotone convergence.

§ 6. Estimates for non-negative solutions. We consider a non-negative weak solution u of (1.2) in S. We assume u is continuous in $\bar{S}$ and that there exists an $\alpha \geqslant 0$ such that

$$\int_{E^n} e^{-2\alpha|x|^2} u^2(x, 0)\,dx < \infty. \tag{6.1}$$

In case $\alpha > 0$ we cannot expect u to exist for arbitrary T. For example, the Cauchy problem

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad t > 0, \qquad u(x, 0) = e^{x^2}$$

has the solution

$$u(x, t) = (1 - 4t)^{-1/2} \exp\left\{\frac{x^2}{1 - 4t}\right\}$$

which is clearly only valid for $t < \frac{1}{4}$.

According to Theorem III, the sequence v_m converges pointwise to u. For each m, v_m satisfies

$$\int_{|x|<m} v_m(x, t)\varphi(x, t)\,dx + \int_0^t d\tau \int_{|x|<m} \Big(-v_m\varphi_\tau + \sum_{i,j} a_{ij} v_{mx_i}\varphi_{x_j}\Big)\,dx = \int_{|x|<m} v_m(x, 0)\varphi(x, 0)\,dx \tag{6.2}$$

for all $t \in [0, T]$ and all $\varphi \in L^2[0, T; H_0^{1,2}(|x| < m)] \cap H^{1,2}[0, T; L^2(|x| < m)]$. For arbitrary $\beta \geqslant 0$ let

$$\bar{v}_m(x, t) = v_m(x, t)\exp\{-(a+\tfrac{1}{2}\beta t)|x|^2\}$$

and $\hat{v}_m = |x|\bar{v}_m$. Then

$$\varphi_h(x, t) = \bar{v}_m^h(x, t)\exp\{-(a+\tfrac{1}{2}\beta t)|x|^2\}$$

(cf. § 2) is an admissible test function in (6.2). By the argument employed in the first part of the proof of Theorem I we obtain

$$\tfrac{1}{2}\int\limits_{|x|<m} \bar{v}_m^2(x, t)\,dx + \tfrac{1}{2}\beta\int\limits_0^t d\tau \int\limits_{|x|<m} \hat{v}_m^2\,dx + \int\limits_0^t d\tau \int\limits_{|x|<m} e^{-(2a+\beta\tau)|x|^2}\sum_{i,j} a_{ij}\,v_{mx_i}v_{mx_j}\,dx$$
$$\leqslant \tfrac{1}{2}\int\limits_{|x|<m} \bar{v}_m^2(x, 0)\,dx + 2\left|\int\limits_0^t d\tau \int\limits_{|x|<m} e^{-(2a+\beta\tau)|x|^2} v_m(2a+\beta\tau)\sum_{i,j} a_{ij}\,v_{mx_i}x_j\,dx\right|.$$

Thus, in view of (1.4),

$$(6.3)\qquad \int\limits_0^t \bar{v}_m^2(x, t)\,dx + 2\left\{\frac{\beta}{2} - 4\nu(2a+\beta T)^2\right\}\int\limits_0^t d\tau \int\limits_{|x|<m} \hat{v}_m^2\,dx +$$
$$+\frac{1}{\nu}\int\limits_0^t d\tau \int\limits_{|x|<m} e^{-(2a+\beta\tau)|x|^2}|\nabla v_m|^2\,dx \leqslant \int\limits_{|x|<m} \bar{v}_m^2(x, 0)\,dx$$

for all $t \in [0, T]$.

If $a = 0$ put $\beta = 0$. For $a > 0$, assume $T \leqslant 1/64a\nu$ and put

$$\beta = \{1 - 32aT - (1 - 64a\nu T)^{1/2}\}/16\nu T^2 .$$

In either case the coefficient of the second integral on the left hand side of (6.3) is zero. Thus, since $v_m(x, 0) = \gamma_m(x)u(x, 0) \leqslant u(x, 0)$, it follows from (6.1) and $v_m = 0$ for $|x| > m$ that

$$(6.4)\qquad \int\limits_{E^n} e^{-(2a+\beta t)|x|^2} v_m^2(x, t)\,dx + \frac{1}{\nu}\int\limits_0^t d\tau \int\limits_{E^n} e^{-(2a+\beta\tau)|x|^2}|\nabla v_m|^2\,dx$$
$$\leqslant \int\limits_{E^n} e^{-2a|x|^2}u^2(x, 0)\,dx$$

for all $t \in [0, T]$.

Let $\mathfrak{B}$ denote the Banach space of measurable functions $f(x, t)$ such that

$$|||f||| = \operatorname*{ess.\,max.}_{[0,T]}\Big\{\int\limits_{E^n} e^{-(2a+\beta t)|x|^2} f^2(x, t)\,dx\Big\}^{1/2} +$$
$$+\Big\{\int\limits_0^T dt \int\limits_{E^n} e^{-(2a+\beta t)|x|^2}|\nabla f|^2\,dx\Big\}^{1/2} < \infty .$$

We conclude from (6.5) that there is a subsequence of the v_m which converge weakly to a limit function in $\mathfrak{B}$. On the other hand, we know that v_m converges pointwise to u. Therefore the whole sequence v_m converges weakly in $\mathfrak{B}$ to u and (6.4) holds with u in place of v_m on the left hand side.

Thus we have proved the following

THEOREM IV. *Let u be a non-negative weak solution of* (1.2) *in S. Assume that u is continuous in $\bar{S}$ and that there exists an $a \geqslant 0$ that such*

$$\int_{E^n} e^{-2a|x|^2} u^2(x, 0)\, dx < \infty .$$

If $a = 0$ then

$$\int_{E^n} u^2(x, t)\, dx + \frac{1}{\nu} \int_0^t d\tau \int_{E^n} |\nabla u|^2 dx \leqslant \int_{E^n} u^2(x, 0)\, dx$$

for all $t \in [0, T]$. If $a > 0$ then

$$\int_{E^n} e^{-(2a+\beta t)|x|^2} u^2(x, t)\, dx + \frac{1}{\nu} \int_0^t d\tau \int_{E^n} e^{-(2a+\beta\tau)|x|^2} |\nabla u|^2 dx \leqslant \int_{E^n} e^{-2a|x|^2} u^2(x, 0)\, dx$$

for all $t \in [0, \hat{T}]$, where $\hat{T} = \min(T, 1/64a\nu)$ and

$$\beta = \{1 - 32a\hat{T} - (1 - 64a\nu\hat{T})^{1/2}\}/16\nu\hat{T}^2 .$$

§ 7. Concluding remarks. The results obtained above are based on the existence theory in [1] and the Harnack inequality proved in [5]. Both of these papers treat only equations of the form (1.1). However, by using the device introduced by Oleĭnik and Kruzhkov in [6] the results of [5] can be extended to equations of the form

$$Lu \equiv \frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij} \frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} = 0$$

for bounded measurable coefficients b_i. The results of [1] can be similarly extended. The corresponding extension of the results of this paper would present no difficulties.

Appendix: Extension of the Harnack inequality. In [5] Moser proves the Harnack inequality for weak solutions of (1.1) of the following type. Let D be a domain in the $(n+1)$-dimensional (x, t)-space. A function $u = u(x, t)$ is said to be a weak solution of (1.1) in D if u

has strong derivatives with respect to both the x-variables and t which are square integrable in D, and if u satisfies

$$\iint_D \Big(\varphi u_t + \sum_{i,j=1}^{n} a_{ij} u_{x_i} \varphi_{x_j}\Big) dx\,dt = 0 \tag{1}$$

for all C^∞ functions $\varphi = \varphi(x, t)$ which for each fixed t have compact support as functions of x. Let $R = (|x_i| < \varrho) \times (0, \tau)$, $R^- = (|x_i| < \varrho') \times \times (\tau_1^-, \tau_2^-)$ and $R^+ = (|x_i| < \varrho') \times (\tau^+, \tau)$, where $0 < \varrho' < \varrho$ and $0 < \tau_1^- < \tau_2^- < \tau^+ < \tau$. The main result of [5] is

THEOREM 1. *If u is a non-negative weak solution of* (1.1) *in R in the sense of* (1), *then $\max\limits_{R^-} u \leqslant \gamma \min\limits_{R^+} u$, where $\gamma > 1$ is a constant which depends only on n, ν and the six geometrical constants $\varrho, \varrho', \tau, \tau_1^-, \tau_2^-, \tau^+$.*

The remaining results of [5] follow directly from Theorem 1 without further reference to (1).

At the Joint Soviet-American Symposium on Partial Differential Equations (Novosibirsk, August 1963) Moser announced an extension of the results obtained in [5] to a more general class of weak solutions of (1.1). Let Ω be a domain in E^n and let $D = \Omega \times (0, T)$. A function $u = u(x, t)$ will be called a weak solution of (1.1) in D if

$$\max_{[0,T]} \int_\Omega u^2 dx + \iint_D \sum_{i=1}^{n} u_{x_i}^2 dx\,dt < \infty$$

and

$$\iint_D \Big(-\varphi_t u + \sum_{i,j=1}^{n} a_{ij} u_{x_i} \varphi_{x_j}\Big) dx\,dt = 0 \tag{2}$$

for every C^∞ function $\varphi = \varphi(x, t)$ with compact support in D. To extend the results of [5] to weak solutions in the sense of (2) it suffices to show that Theorem 1 can be extended. For various technical reasons the proof of Theorem 1 in [5] cannot be applied directly to weak solutions in the sense of (2). However, using [5] together with the results of [1] we shall prove

THEOREM 1'. *If u is a non-negative weak solution of* (1.1) *in R in the sense of* (2), *then Theorem* 1 *holds for u.*

Thus, in particular, the Harnack inequality and its consequences can be applied to the solutions which we deal with in the body of this paper.

For $j = 0, 1, \ldots, 4$ let $\Sigma_j = (|x_i| < \varrho_j) \times (t_j, \tau]$, where $\varrho_j = \frac{\varrho + \varrho'}{2} \times \times \Big(1 - \frac{j}{4}\Big) + \frac{j}{4}\varrho$ and $t_j = \frac{\tau_1^-}{2}\Big(1 - \frac{j}{4}\Big)$. Then $R^+ \cup R^- \subset \Sigma_0 \subset \ldots \subset \Sigma_4$. Let

$\eta = \eta(x, t)$ be a smooth function such that $\eta \equiv 1$ on $\bar{\Sigma}_2$, $\eta \equiv 0$ outside Σ_3 and $0 \leqslant \eta \leqslant 1$. Put $v = \eta u$. Clearly $v \in \mathfrak{L} \equiv L^\infty[0, \tau; L^2(|x_i| < \varrho_4)] \cap \cap L^2[0, \tau; H_0^{1,2}(|x_i| < \varrho_4)]$ and $v \equiv 0$ outiside Σ_3. Moreover

$$\iint_{\Sigma_4} \Big(-\psi_t v + \sum_{i,j} a_{ij} v_{x_i} \psi_{x_j}\Big) dx\, dt = \iint_{\Sigma_4} \Big(\psi f + \sum_j g_j \psi_{x_j}\Big) dx\, dt$$

for all smooth ψ with compact support in Σ_4, where $f = u\eta_t - \sum_{i,j} a_{ij} u_{x_i} \eta_{x_j}$ and $g_j = u \sum_i a_{ij} \eta_{x_i}$. It follows by a standard argument that

$$\int_{|x_i| < \varrho_4} v(x, t)\psi(x, t)\, dx + \int_0^t ds \int_{|x_i| < \varrho_4} \Big(-\psi_s v + \sum_{i,j} a_{ij} v_{x_i} \psi_{x_j}\Big) dx$$
$$= \int_0^t ds \int_{|x_i| < \varrho_4} \Big(\psi f + \sum_j g_j \psi_{x_j}\Big) dx$$

for almost all $t \in [0, \tau]$ and all smooth ψ with compact support in $|x_i| < \varrho_4$. Thus v is a weak solution of the boundary value problem

$$(3) \qquad Lv = f - \operatorname{div} \boldsymbol{g} \quad \text{for} \quad (x, t) \in \Sigma_4, \qquad v = 0 \quad \text{for} \quad (x, t) \in \Gamma_4,$$

where $\boldsymbol{g} = (g_1, \ldots, g_n)$ and Γ_4 is the union of the bottom and lateral faces of Σ_4. Since f and the g_j belong to $L^2(\Sigma_4)$, it follows from [1] that v is the unique weak solution of (3) in $\mathfrak{L}$.

Let a_{ij}, f and g_j be extended for $(x, t) \notin R$ in any convenient manner (always preserving (1.4)), and let $a_{ij}^{(m)}$, $f^{(m)}$ and $g_j^{(m)}$ denote their integral averages formed with an averaging kernel whose support is in $x^2 + t^2 < 1/m^2$. It is shown in [1] that v is the weak limit in $\mathfrak{L}$ of the sequence $\{v_m\}$ of classical solutions of the boundary value problem

$$L_m v \equiv v_t - \sum_{i,j} \{a_{ij}^{(m)} v_{x_i}\}_{x_j} = f^{(m)} - \operatorname{div} \boldsymbol{g}^{(m)} \quad \text{for} \quad (x, t) \in \Sigma_4,$$
$$v = 0 \quad \text{for} \quad (x, t) \in \Gamma_4.$$

Moreover

$$(4) \qquad \iint_{\Sigma_4} v_m^2\, dx\, dt \leqslant \text{const}(\|f\|^2 + \|\boldsymbol{g}\|^2),$$

where the constant depends only on ν and τ. On the other hand, since f and $\boldsymbol{g}$ have support in $\Sigma_3 - \bar{\Sigma}_2$ for $t < \tau$, it follows that for m sufficiently large $f^{(m)}$ and $\boldsymbol{g}^{(m)}$ have support in the exterior of $\bar{\Sigma}_1$. Thus v_m satisfies $L_m v_m = 0$ for $(x, t) \in \Sigma_1$. It is clear that

$$(5) \qquad \iint_{\Sigma_1} \Big(\varphi v_{mt} + \sum_{i,j} a_{ij} v_{mx_i} \varphi_{x_j}\Big) dx\, dt = 0$$

for all smooth φ with compact support in $|x_i| < \varrho_1$. By Theorem 3 and the Corollary to Theorem 1 of [5], we conclude from (4) and (5) that the sequence $\{v_m\}$ is uniformly bounded and equi-continuous in $\bar{\Sigma}_0$. Since $v_m \to v$ weakly in $\mathfrak{L}$ and $v \equiv u$ on $\bar{\Sigma}_0$, every convergent subsequence of $\{v_m\}$ converges to u on $\bar{\Sigma}_0$. Hence $v_m \to u$ uniformly on $\bar{\Sigma}_0$.

Given arbitrary $\varepsilon > 0$ there exists an $m_0 = m_0(\varepsilon)$ such that $u(x,t) - \varepsilon < v_m(x,t) < u(x,t)+\varepsilon$ on $\bar{\Sigma}_0$ for all $m > m_0$. Since $u \geqslant 0$ on $\bar{\Sigma}_0$, $v_m+\varepsilon > 0$ on $\bar{\Sigma}_0$ for all $m > m_0$. Now $v_m+\varepsilon$ is a non-negative weak solution of (1.1) in Σ_0 in the sense of (1). Hence we may apply Theorem 1 to obtain

$$\max_{R^-}(v_m+\varepsilon) \leqslant \gamma \min_{R^+}(v_m+\varepsilon) \tag{6}$$

for all $m > m_0$, where γ depends only on n, ν and the geometry. For $(x,t) \in R^-$ we have

$$\max_{R^-}(v_m+\varepsilon) \geqslant v_m(x,t)+\varepsilon > u(x,t)$$

and for $(x,t) \in R^+$

$$\min_{R^+}(v_m+\varepsilon) \leqslant v_m(x,t)+\varepsilon < u(x,t)+2\varepsilon$$

for all $m > m_0$. Thus (5) implies that

$$\max_{R^-} u < \gamma \min_{R^+} u + 2\gamma\varepsilon .$$

Since $\varepsilon > 0$ is arbitrary the assertion follows.

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