ANNALES
POLONICI MATHEMATICI
XXXI (1976)

Uniqueness of regular similarity functions

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Abstract. A *similarity function* is defined as a function F satisfying (1) $F\circ H = G\circ F$, where G and H are known, real-valued functions of a real variable. It is remarked that this equation includes the question of commuting functions and fractional iteration. A uniqueness theorem for the regular case is proven. Some directions for further work are indicated.

I. Introduction. Two functions G and H are said to be *similar* iff there exists a *similarity function* F satisfying the relation

$$G\circ F = F\circ H. \tag{1}$$

This equation is a general one, for it includes as a special case the question of commuting functions:

$$G\circ F = F\circ G. \tag{2}$$

Furthermore, the equation of partial iteration,

$$F^N = G, \tag{3}$$

implies relation (2).

Kuczma and Smajdor [2], and Reznick [3], have proven certain uniqueness theorems regarding (3). We shall give conditions similar to Reznick's so that uniqueness applies also to (1), where F is regarded as the unknown.

II. Hypotheses and Definitions. Throughout the paper, I denotes the interval $[\xi, A)$, where A may be infinite. F will be called *restricted* iff $F\in S^1_\xi[I]$ [1].

The following hypotheses will suffice for the main theorem:

(H1) G, $H\in R^1_\xi[I]$ [1].

(H2) H' *and* $G'>0$ *on* I.

(H3) G' *is increasing not necessarily strictly on a neighborhood* (¹) Γ *of* ξ.

(¹) In I, of course.

III. Lemma.

LEMMA. *Suppose F and E are restricted solutions of* (1), *where G and H satisfy* (H1). *If K is a neighborhood of ξ, then $F = E$ on K iff $F = E$ on all of I.*

Proof. If $F = E$ on K, (1) yields $F \circ H = G \circ F = G \circ E = E \circ H$ on K. Hence, $F = E$ on $H(K)$. Similarly, $F = E$ on $H^{-1}(K)$, wherever that is defined. (H1) completes the proof.

IV. Main Theorem.

THEOREM. *If G and H satisfy* (H1)–(H3), *then there is no more than one restricted solution of* (1) *having a given derivative at ξ.*

Proof. Suppose F is a restricted solution of (1). By (1) and (H1), $F(\xi) = \xi$. Now, differentiating the identity $F \circ H^a = G^a \circ F$ yields

$$(F' \circ H^a)\Big(\prod_{i=0}^{a-1} H' \circ H^i\Big) = (F')\Big(\prod_{i=0}^{a-l} G' \circ G^i \circ F\Big). \tag{4}$$

Calculating hence F', and letting $a \to \infty$ yields,

$$F' = F'(\xi)\Big(\prod_{i=0}^{\infty} \frac{H' \circ H^i}{G' \circ G^i \circ F}\Big). \tag{5}$$

If $F'(Z) = 0$ for some $Z \epsilon I$, (4) implies $F'\big(H^a(Z)\big) = 0$, and thus $F'(\xi) = 0$. In this case, any other restricted solution E with the same derivative at ξ is, with F, identically ξ, by (5). Otherwise, we can divide the expressions given by (5) for E' and F' to obtain

$$\frac{E'}{F'} = \prod_{i=0}^{\infty} \frac{G' \circ G^i \circ F}{G' \circ G^i \circ E}. \tag{6}$$

Because $E(\xi) = F(\xi)$, either $E = F$ on Γ, in which case the lemma yields the theorem, or there exists an interval $[L, C)$ in Γ such that $E(L) = F(L)$ and $E \neq F$ elsewhere in (L, C). In the latter case, suppose without loss of generality that $E > F$ on (L, C). By the mean value theorem of elementary calculus, there exists a $w \epsilon (L, C)$ such that $E'(w) > F'(w)$. In view of (H3), this yields a contradiction in (6).

V. Comments.

1. It appears likely that (H3) can be weakened to

(H3)$'$ *G' is monotonic in Γ and $G'(\xi) \neq 1$.*

References

[1] M. Kuczma, *Functional equations in a single variable*, Warszawa 1968.
[2] — and A. Smajdor, *Regular fractional iteration*, Bull. Acad. Polon. Sci. 19 (1971), p. 203–207.
[3] B. A. Reznick, *A uniqueness criterion for fractional iteration*, Ann. Polon. Math. 30 (1974), p. 219–224.

Reçu par la Rédaction le 30. 10. 1973