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Unitary dilation and fixed point theorem

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Abstract. In this paper certain fixed point theorems for functions on groups $\boldsymbol{Z}^{\Omega}$ and $\boldsymbol{R}^{\Omega}$ are proved, using unitary dilations.

Let H be a Hilbert space. $L(H)$ denotes the space of all linear bounded operators in H. If H is a subspace of a Hilbert space K, then P_H denotes the (orthogonal) projection of K onto H. The set of all functions from a set Ω into set A with finite support is denoted by A^{Ω}. $\boldsymbol{R}$, $\boldsymbol{Z}$ stand for the sets of real integer numbers, respectively.

Let G be a group (we only consider groups $\boldsymbol{D}^{\Omega}$, $\boldsymbol{R}^{\Omega}$) with unit e. A functions $T(\cdot)$: $G \to L(H)$ is called *positive definite* iff $T(s^{-1}) = T(s)^*$ for all s in G and

$$\sum_{s\in G}\sum_{t\in G}(T(t^{-1}s)h(s),\, h(t)) \geq 0 \quad \text{for all } h \in H^G.$$

A function $U(\cdot)$: $G \to L(K)$ is called a *unitary representation of* G *in* K if $U(e) = I_K$, $U(s\cdot t) = U(s)\cdot U(t)$ for all $s, t \in G$ and $U(s)$ is a unitary operator in $L(K)$ for all $s \in G$. Let $T(\cdot)$: $G \to L(H)$ be a function; then a unitary representation $U(\cdot)$: $G \to L(K)$ is called a *unitary dilation* of $T(\cdot)$ if $H \subset K$ and $T(s) = P_H U(s)|_H$ for all $s \in G$.

Dash, in [1], has proved a fixed point theorem for a positive definite sequence of operators indexed by integer numbers. In the present paper we prove fixed point theorems for a positive definite function on groups $\boldsymbol{Z}^{\Omega}$ and $\boldsymbol{R}^{\Omega}$, using unitary dilations.

Let Ω be a set; for $\omega \in \Omega$ we define a function e_{ω} in $\boldsymbol{Z}^{\Omega}$ (or in $\boldsymbol{R}^{\Omega}$) by $e_{\omega}(\alpha) = 1$ if $\alpha = \omega$, and $e_{\omega}(\alpha) = 0$, otherwise. Now, we generalize the theorem from [1] to multi-parameter discrete functions.

Theorem 1. *Let* $T(\cdot)$: $\boldsymbol{Z}^{\Omega} \to L(H)$ *be a positive definite function on* $\boldsymbol{Z}^{\Omega}$ *(the group of all functions from a set* Ω *into* $\boldsymbol{Z}$ *with finite support) with* $T(0) = I$. *If for some* f *in* H *and all* $\omega \in \Omega$

$$T(e_{\omega})f = f,$$

then

$$T(n)f = f \quad \text{for all } n \in \boldsymbol{Z}^{\Omega}.$$

Proof. $T(\cdot)$ is a positive definite, thus [Theorem I.7.1,2] there is $U(\cdot)$: $\boldsymbol{Z}^{\Omega} \to L(K)$, a unitary dilation of $T(\cdot)$, so that $T(n)g = P_H U(n)g$ for all g in H and all n in $\boldsymbol{Z}^{\Omega}$.

Let $\omega \in \Omega$; then

$$\|f\| = \|T(e_{\omega})f\| = \|P_H U(e_{\omega})f\| \leqslant \|U(e_{\omega})f\| = \|f\|,$$

thus $U(e_{\omega}), f \in H$. For all $h \in H$

$$(h, f) = (h,\ T(e_{\omega})f) = (h,\ P_H U(e_{\omega})f) = (h,\ U(e_{\omega})f).$$

Hence $U(e_{\omega})f = f$ and also $U(-e_{\omega})f = U(e_{\omega})^{-1}f = f$. If $n \in \boldsymbol{Z}^{\Omega}$ then, because $U(\cdot)$ is a representation, $U(n)$ is a multiplication of a finite number of operators of the forms $U(e_{\omega})$ or $U(-e_{\omega})$ for some ω in Ω. Thus $U(n)f = f$ for all $n \in \boldsymbol{Z}^{\Omega}$. Hence

$$T(n)f = P_H U(n)f = P_H f = f \quad \text{for all } n \in \boldsymbol{Z}^{\Omega}. \ \blacksquare$$

Now, we present a fixed point theorem for a function on the group $\boldsymbol{R}^{\Phi}$, It can be observed that:

Remark. Let $d_k \neq 0$ be a sequence converging monotonically to 0 and let $r \in \boldsymbol{R}$. Then there is a sequence n_k of elements of $\boldsymbol{Z}$ such that $d_k n_k \to r$ whenever $k \to \infty$.

n_k can be defined as an integer number such that $|d_k n_k| \leqslant |r|$ and $|d_k n_k - r|$ has the smallest value.

Note that $\boldsymbol{R}^{\Omega}$ is a topological group with pointwise convergence topology and that each element of $\boldsymbol{R}^{\Omega}$ can be multiplied pointwise by any functions from Ω to $\boldsymbol{R}$.

THEOREM 2. *Let $T(\cdot)$: $\boldsymbol{R}^{\Omega} \to L(H)$ be a continuous in the strong operator topology in $L(H)$ positive definite function on $\boldsymbol{R}^{\Omega}$ (the group of all functions from a set Ω into $\boldsymbol{R}$ with finite support) such that $T(0) = I$ and let there be for some f in H a sequence $s_k(\cdot)$: $\Omega \to \boldsymbol{R}$, converging pointwisely to 0, of non--vanishing functions such that $T(s_k \cdot e_{\omega})f = f$ for all k and ω in Ω. Then*

$$T(s)f = f \quad \text{for all } s \text{ in } \boldsymbol{R}^{\Omega}.$$

Proof. By Theorem I.7.1 of [2] there is $U(\cdot)$: $\boldsymbol{R}^{\Omega} \to L(K)$ a unitary dilation of $T(\cdot)$, thus $T(s)g = P_H U(s)g$ for all g in H and all $s \in \boldsymbol{R}^{\Omega}$.

Like in Theorem 1, it can be proved that

$$U(s_k \cdot n)f = f \quad \text{for all } k \text{ and all } n \text{ in } \boldsymbol{Z}^{\Omega}.$$

Let $s \in \boldsymbol{R}^{\Omega}$ and denote by supp s the support of s. We can choose from the sequence $s_k(\cdot)$ a subsequence, also denoted by $s_k(\cdot)$, such that $s_k(\omega)$ converges monotonically to 0 for all $\omega \in \operatorname{supp} s$, because the support of s is finite. If $\omega \in \operatorname{supp} s$, then, by the remark above, there is a sequence $n_k^{s(\omega)}$ of elements of $\boldsymbol{Z}$ such that $s_k(\omega) \cdot n_k^{s(\omega)} \to s(\omega)$ whenever $k \to \infty$. We define a sequence of functions $n_k^s \in \boldsymbol{Z}^\Omega$ as follows: $n_k^s(\omega) = n_k^{s(\omega)}$ if $\omega \in \operatorname{supp} s$, and $n_k^s(\omega) = 0$ otherwise. Then $s_k \cdot n_k^s \in \boldsymbol{R}^\Omega$ and $s_k n_k^s \to s$ pointwise in $\boldsymbol{R}^\Omega$.

The representation $U(\cdot)$ is continuous in the strong operator topology in $L(K)$, because $T(\cdot)$ is such (Theorem I.7,1 [2]), thus $U(s)f = f$. Hence

$$T(s)f = P_H U(s)f = P_H f = f. \quad \blacksquare$$

References

[1] A. T. Dash, *Positive definite sequences of operators and a fixed point theorem*, Canad. Math. Bull. 15 (2) (1972), 295.

[2] B. Sz. Nagý, C. Foiaş, *Analyse harmonique des opérateurs de l'espace de Hilbert*, Masson et C^ie, Academiai Kiado, 1967.

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