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Asymptotic behavior of non-linear differential equations via non-standard analysis

Part III. Boundedness and monotone behavior of the equation

$$\big(a(t)\varphi(x)x'\big)' + c(t)f(x) = q(t)$$

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Abstract. Using the techniques of non-standard analysis this paper continues the study of the equation stated in the title. Some boundedness and monotonicity theorems are proved, following assumptions concerning the coefficients $a(t)$, $c(t)$, the forcing term $q(t)$, and properties of the functions $\varphi(x), f(x)$. The notation and basic properties of the non-standard model $^*R$ are the same as described in parts I and II.

1. Introductory remarks. Using the techniques of non-standard analysis this paper continues the study of the asymptotic behavior of the equation

$$\big(a(t)\varphi(x)x'\big)' + c(t)f(x) = q(t) \tag{1}$$

and of the homogeneous equation

$$\big(a(t)\varphi(x)x'\big)' + c(t)f(x) = 0, \qquad t \in [0, \infty), \tag{1^H}$$

where throughout this paper we assume that

$$a(t) \in C^1[0, \infty), \qquad c(t) \in C[0, \infty), \qquad q(t) \in L_{1(\text{locally})}[0, \infty).$$

$a(t)$ is eventually positive ($c(t)$ could change its sign). $f(\xi)$, $\varphi(\xi) \in C(-\infty, +\infty)$,

$$\limsup_{\xi\to 0} \left|\frac{\varphi(\xi)}{\xi}\right| < \infty.$$

Equation (1) occurs in celestial mechanics as the law of conservation of angular momentum. (See a brief explanation in [3], or read Poincaré [6], Chapter 7.) We assume that solutions are continuable. For exact necessary and sufficient conditions for continuability of equation (1^H) see [3], Theorems 1 and 2.

No exposition of the foundations of non-standard analysis will be attempted in this paper. We refer the reader to [5], [7], and [8].

For a discussion of oscillatory behavior of equation (1) and (1^H) we refer to [3] and [4]. For higher order equations see [3], [9] and [10]. This paper continues the basic ideas of [4] and [3] in the study of equations of type (1), (1^H).

2. Notation. As in part I and II of this work the notation follows closely that given by A. Robinson in [8]. ${}^*\boldsymbol{R}$ denotes a non-standard model of the real number system $\boldsymbol{R}$, which is an enlargement of $\boldsymbol{R}$. The notation $x \approx y$ $(x \in {}^*\boldsymbol{R}, y \in {}^*\boldsymbol{R})$ implies that $(x-y)$ is in the monad of zero, i.e. $(x-y)$ is an infinitesimal. ${}^*\boldsymbol{R}_{bd}$ denotes the elements of ${}^*\boldsymbol{R}$ which are bounded in absolute value by some standard number which ${}^*\boldsymbol{R}_\infty$ $({}^*\boldsymbol{R}_\infty = {}^*\boldsymbol{R}/{}^*\boldsymbol{R}_{bd})$ denotes the infinite elements of ${}^*\boldsymbol{R}$, with ${}^*\boldsymbol{R}_{+\infty}$, ${}^*\boldsymbol{R}_{-\infty}$ denoting respectively the positive and the negative infinite numbers.

Note in the non-standard arguments concerning asymptotic behavior of differential equations we can utilize the results of Bernstein [1], Loeb and Robinson to extend the Lebesgue measure to power set of ${}^*\boldsymbol{R}$, i.e., ${}^*\mu$: $\mathscr{P}({}^*\boldsymbol{R}) \to {}^*\boldsymbol{R}$. $\mathrm{std}({}^*\mu)$ coinciding with the Lebesgue measure on $\boldsymbol{R}$. In this context we introduce essential asymptotic concepts. We shall say that a function f: ${}^*\boldsymbol{R} \to {}^*\boldsymbol{R}$ has essential asymptotic property P if for $[t_1, t_2] \subset {}^*\boldsymbol{R}$ the measure ${}^*\mu$ of the set S on which P is not true is infinitesimal compared to (t_2-t_1), i.e. $\dfrac{{}^*\mu(s)}{t_2-t_1} \approx 0$. Essential limits will be understood in this sense.

3. Some boundedness theorems.

THEOREM 1. *Suppose that the following hypothesis are true:*

For Sufficiently large values of $|x|$ the function $\varphi(x)$ is of constant sign, the function $q(t)$ is bounded, and

$$\text{either } \lim_{|\xi|\to\infty} \frac{f(\xi)}{\xi} = 0 \text{ while } \limsup |c(t)| < \infty,$$

$$\text{or } \limsup_{|\xi|\to\infty} \left|\frac{f(\xi)}{\xi}\right| < \infty \text{ while } \lim_{t\to\infty} c(t) = 0,$$

while $\liminf\limits_{|\xi|\to\infty} |\varphi(\xi)| > 0$, $\liminf\limits_{t\to\infty} |a(t)| > 0$.

Than any solution $\hat{x}(t)$ of (1) *which fails to be eventually monotone, and such that $\lim\limits_{t\to\infty} |\hat{x}(t)| = \infty$ will have the property* $\operatorname{ess\,lim} \dfrac{\hat{x}'(t)}{\hat{x}(t)} = 0$.

Note. We observe that any equation of the form $x'' + c(t)x^a = q(t)$, where $0 < a < 1$, and $c(t)$ and $q(t)$ are bounded, satisfies our hypothesis.

Also the pendulum equation $x''+c(t)\cos(kx) = q(t)$, with the k a constant with the same assumptions concerning $c(t)$, $q(t)$, fits these hypothesis.

THEOREM 2. *Suppose that $\varphi(\xi)$ is of constant sign for large values of $|\xi|$, and $\xi f(\xi) > 0$ if $\xi \neq 0$. Then if in addition to the hypothesis of Theorem 1 we also assume that $c(t)$ is eventually positive and $q(t)$ is eventually non-positive, then any solution $\hat{x}(t)$ of (1) such that $\lim_{t\to\infty}|\hat{x}(t)| = +\infty$ must be eventually monotone.*

Proof of Theorem 1. Suppose that to the contrary there exists a solution $\hat{x}(t)$ of (1) such that $\lim_{t\to\infty}|\hat{x}(t)| = \infty$, but it fails to be eventually monotone. Then it is easily shown that there exists a sequence of points $\{t_i\} \in {}^*\boldsymbol{R}_{+\infty}$ such that $\hat{x}(t_i) \in {}^*\boldsymbol{R}_\infty$ and $x'(t_i) = 0, i = 1, 2, \ldots$ Let us select such pair of points $t_i,\ t_j \in {}^*\boldsymbol{R}_{+\infty}$. Since $\lim_{t\to\infty}|x(t)| = \infty$, $\hat{x}(t) \in {}^*\boldsymbol{R}_\infty$ $\forall t \in [t_i, t_j]$. We examine the behavior of the integral $\int_{t_i}^{t_j}\left(\frac{a(t)\varphi(\hat{x})\hat{x}'(t)}{x(t)}\right)' dt$. Clearly the value of this integral is zero since $\hat{x}'(t_i) = \hat{x}'(t_j) = 0$, while $\hat{x}(t) \neq 0$ on $[t_i, t_j]$. However, $\hat{x}(t)$ is a solution of (1), hence $\big(a(t)\varphi(\hat{x})\hat{x}'\big)' = -c(t)f(x)+q(t)$. Therefore

$$\text{(a)}\qquad 0 = \int_{t_i}^{t_j}\left(\frac{a(t)\varphi(\hat{x})\hat{x}'}{\hat{x}(t)}\right)' dt$$

$$= \int_{t_i}^{t_j}\left\{-c(t)\frac{f(\hat{x})}{\hat{x}} + \frac{q(t)}{\hat{x}(t)} - \left(\frac{a(t)\varphi(\hat{x})(\hat{x}')^2}{\hat{x}^2}\right)\right\} dt.$$

The hypothesis imply that $c(t)f(\hat{x})/\hat{x}$ and $q(t)/\hat{x}$ are infinitesimal for all $t \in {}^*\boldsymbol{R}_{+\infty}$. The last term in the integrand of (a) is of constant sign. Suppose that, on a subset $S \subset [t_i, t_j]$, $\eta(t) = \dfrac{a(t)\varphi\big(\hat{x}(t)\big)\big(\hat{x}'(t)\big)^2}{\big(\hat{x}(t)\big)^2}$ is not an infinitesimal. Then there exists a standard number $m > 0$ such that, for all $t \in S$, $|\eta(t)| > 2m$. Without any loss of generality let us assume $\eta(t) > 0$ on S.

Then

$$\int_S\left(\frac{a(t)\varphi(\hat{x})\hat{x}'}{\hat{x}}\right)' dt > m\cdot\mu(S)$$

on $[t_i, t_j]/S$, $\eta(t)$ is an infinitesimal, and

$$\int_{[t_i,t_j]/S}\left(\frac{a(t)\varphi(x)x'}{\hat{x}}\right)' dt = \int_{[t_i,t_j]/S} |k(t)|\, dt,$$

where $k(t) \approx 0\ \forall t \in [t_i, t_j]/S$. Then, for any standard $\varepsilon > 0$, $|k(t)| < \varepsilon$, and

$$\int_{[t_i,t_j]/S} |k(t)|\,dt < \varepsilon \cdot \mu([t_i, t_j]/S).$$

Hence it follows easily that

$$\varepsilon \cdot \mu\{[t_i, t_j]/S\} > m\mu(S) \quad \text{and} \quad \frac{\mu(S)}{\mu([t_i, t_j]/S)} < \frac{\varepsilon}{m}.$$

Since ε was an arbitrary standard number if follows that

$$\frac{\mu(S)}{\mu([t_i, t_j]/S)} \approx 0,$$

proving that $\eta(t)$ is essentially an infinitesimal on $[t_i, t_j]$. Since $\lim_{|\xi|\to\infty} \inf \varphi(\xi) \neq 0$, $\liminf_{t\to\infty} |a(t)| \neq 0$, it follows that $(\hat{x}'/\hat{x})$ is essentially an infinitesimal on $[t_i, t_j]$, completing the proof.

COROLLARY 1.1. *If hypothesis* $(*)$ $\lim_{|\xi|\to\infty} \inf |\varphi(\xi)| > 0$ *is replaced by the condition* $\liminf_{\xi\to\infty} |\varphi(\xi)/\xi| > 0$, *the conclusion of the theorem is changed to* $\operatorname{ess}\lim_{t\to\infty} (\hat{x}')^2/(\hat{x}) = 0$, *which is a stronger result.*

Similarily if $(*)$ is replaced by $\lim_{\xi\to\infty} \inf |\varphi(\xi)/\xi^2| > 0$, then the conclusion Theorem 1 is changed to $\operatorname{ess}\lim_{t\to\infty} x'(t) = 0$.

Proof of Theorem 2. The proof follows in a routine manner from the argument of Theorem 1. Suppose as before that there exists points $t_i, t_j \in {}^*\boldsymbol{R}_{+\infty}$ such that $\hat{x}'(t_i) = \hat{x}'(t_j) = 0$. Without any loss of generality it is assumed that $\hat{x}(t) > 0$ on $[t_i, t_j] \subset {}^*\boldsymbol{R}_{+\infty}$. Since $q(t) \leqslant 0$, $c(t) > 0\ \forall t \in [t_i, t_j]$ the integrand in formula (a) is of constant sign, and equality (a) is impossible. Consequently no points t_i, t_j can exist in ${}^*\boldsymbol{R}_{+\infty}$ such that $x'(t_i) = x'(t_j) = 0$. This easily implies that $x'(t)$ has no zeros on ${}^*\boldsymbol{R}_{+\infty}$, and $x'(t)$ is of constant sign $\forall\, t \in {}^*\boldsymbol{R}_{+\infty}$. Hence $\hat{x}(t)$ is eventually monotone.

THEOREM 3. *Suppose that* $\lim_{|\xi|\to\infty} \inf |f(\xi)/\xi| > 0$, $\varphi(\xi) > 0$ *if* $|\xi|$ *is sufficiently large* $\lim_{|\xi|\to\infty} \sup |\varphi(\xi)/\xi| < \infty$, *and* $a(t)$, $q(t)$ *are bounded functions. Let* $\varepsilon > 0$ *be chosen. Then any solution* $\hat{x}(t)$ *of* (1) *with the property* $\lim_{t\to\infty} |\hat{x}(t)| = \infty$ *will be eventually monotone on all intervals on which* $c(t)$ *is positive, and* $|c(t)| > \varepsilon$. *Moreover, for some* $M > 0$, *there exists* $T > \varepsilon$ *such that on any interval* $[t_1, t_2]$, $t_1 > T$ *on which* $c(t)$ *is negative it is possible to find* $\tilde{t} \in [t_1, t_2]$ *such that* $\left|(x'(\tilde{t}))^2/x(\tilde{t})\right| > M$ *(uniformly on all such intervals).*

Note. The choosing of (an arbitrary) $\varepsilon > 0$ was a device to avoid a non-standard theorem. What was needed was a statement $c(t) \not\approx 0$ on the interval in question when such interval lies in ${}^*\boldsymbol{R}_{\infty}$.

Proof. Proceeding as in Theorem 1 we derive equality (a). As before t_i, t_j are points in ${}^*\boldsymbol{R}_{+\infty}$ such that $\hat{x}'(t_i) = \hat{x}'(t_j)$. If $c(t)$ is positive, then $-c(t)f(\hat{x})/x$ is a negative non-infinitesimal number, $(q(t)/\hat{x})$ is an infinitesimal, and $(-c(t)f(\hat{x})/\hat{x}) \not\approx 0$ is a negative number, while $\dfrac{-a(t)\varphi(x)(x')^2}{x^2}$ is also negative. Since a negative non-infinitesimal number + an infinitesimal + a negative number = a negative (non-infinitesimal) number, and equality (a) can not be satisfied. This completes the proof of the first assertion of the theorem.

If $c(t) < 0$, then by a similar reasoning we have: $\int_{t_i}^{t_j} \Big(\text{negative non-infinitesimal} + \text{infinitesimal} - \Big(\dfrac{a(t)\varphi(\hat{x})(\hat{x}')^2}{\hat{x}}\Big)\Big)dt = 0$. Hence the mean value of $\dfrac{a(t)\varphi(\hat{x})(\hat{x}')^2}{\hat{x}}$ on (t_i, t_j) is a negative non-infinitesimal number. But $a(t) \in {}^*\boldsymbol{R}_{bd}$, and $(\varphi(\hat{x})/\hat{x}) \in {}^*\boldsymbol{R}_{bd}$. Hence $(\hat{x}')^2/(\hat{x}) \not\approx 0$ for some values of $t \in [t_i, t_j]$, which implies the following statement: "$\exists\ M > 0\ (\varepsilon{}^*\boldsymbol{R})$ such that $|(\hat{x}')^2/\hat{x}| > M$ an each such interval $[t_i, t_j]$" (with the properties described above which can be stated in our language $\mathscr{L}$). Since this statement is in $\mathscr{L}$ an identical statement is true in $\boldsymbol{R}$: "$\exists M\ (\in \boldsymbol{R}) \ldots$".

COROLLARY 3.1. *If the hypothesis* $\liminf_{|\xi|\to\infty} |f(\xi)/\xi| > 0$ *is replaced by* $\lim_{\xi\to\infty} f(\xi)/\xi = +\infty$, *then we can replace in the conclusion of the theorem* $\exists M > 0$ *by* "$\forall M > 0$", *i.e.* $\limsup_{t\to\infty} |(\hat{x}')^2/\hat{x}| = +\infty$.

COROLLARY 3.2. *If in the hypothesis of Theorem* 3 $\limsup_{|\xi|\to\infty} |\varphi(\xi)/\xi| < \infty$ *is replaced by* $\limsup_{|\xi|\to\infty} |\varphi(\xi)| < \infty$, *then we can assert a stronger conclusion*: "$\exists M > 0 \ni \liminf_{t\to\infty} |x'(t)/x(t)| > M$".

THEOREM 4. *Consider equation* (1) *with additional hypothesis*: $a(t)$ *is bounded*, $c(t)$ *is eventually positive and bounded away from zero*, $\lim_{t\to\infty} q(t) = 0$, $\xi f(\xi) > 0$, & $\xi\varphi(\xi) > 0\ \forall\ \xi \neq 0$, $\liminf_{|\xi|\to\infty} |f(\xi)/\xi| \neq 0$, *then any solution* $\hat{x}(t)$ *of* (1) *which is eventually bounded away from zero must be unbounded.*

Proof. We make the following observation. If $\hat{x}(t)\colon \boldsymbol{R}_+ \to \boldsymbol{R}$ is bounded away from zero, but is bounded on any ray $[T, \infty)$, then $\hat{x}(t)\colon {}^*\boldsymbol{R}_+ \to {}^*\boldsymbol{R}$ has the property $[\forall t \in {}^*\boldsymbol{R}_{+\infty}\, \hat{x}(t) \not\approx 0\ \&\ \hat{x}(t) \in {}^*\boldsymbol{R}_{bd}]\ \&\ [\exists\, t_i \in {}^*\boldsymbol{R}_{+\infty}$, $i = 1, 2, \ldots, x'(t_i) \approx 0]$. Suppose by the way of contradiciton that indeed $\hat{x}(t) \not\approx 0\ \forall t \in {}^*\boldsymbol{R}_{+\infty}$ but $x'(t_i) \approx 0$ for a sequence of points $\{t_i\}$, $i = 1, 2, \ldots$, when $t_j - t_i$ are arbitrarily large (in ${}^*\boldsymbol{R}$) for suitable choice of indices i, j. (Note. Consider the sentences describing this in $\boldsymbol{R}$.)

We repeat the device used in Theorem 1 and 2 and integrate along the trajectory of the solution $\hat{x}(t)$

$$\int_{t_i}^{t_j} \left(\frac{a(t)\varphi(\hat{x})\hat{x}'(t)}{\hat{x}(t)}\right)' dt = \frac{a(t)\varphi(\hat{x})x'(t)}{\hat{x}(t)}\bigg|_{t_i}^{t_j} \approx 0,$$

since

$$\frac{a(t)\varphi(\hat{x})}{\hat{x}} \in {}^*\boldsymbol{R}_{bd} \begin{cases} t = t_i, \\ t = t_j, \end{cases}$$

while $\hat{x}'(t_i) \approx \hat{x}'(t_j) \approx 0$.

We use again formula (a) (in the proof of Theorem 1) and observe that $-c(t)f(\hat{x})/\hat{x}$ is negative and is not an infinitesimal, $q(t)/\hat{x}(t)$ is infinitesimal, while the remaining term in the integrand of (a) is negative. Hence the integrand is bounded away from zero by some standard number. But then

$$\int_{t_i}^{t_j} \left(\frac{a(t)\varphi(\hat{x})\hat{x}'}{\hat{x}}\right)' dt \approx 0$$

is possible only if $t_j - t_i \approx 0$. This is a contradiction, which proves the theorem.

We comment that so far all theorems proved here and in parts I and II were standard theorems, i.e. the statement of each theorem referred only to $\boldsymbol{R}$ and to functions from $\boldsymbol{R}$ to $\boldsymbol{R}$. However, some asymptotic properties of solutions of differential equations are stated easier in non-standard terms. Some problems concerning oscillation, boundedness or monotone properties of solutions lead naturally to conditions which are complex when stated (in $\boldsymbol{R}$) in terms of limits. Restated in ${}^*\boldsymbol{R}$ they sometimes become quite simple and easy to interpret.

For this reason we shall offer two theorems stated in both the standard and non-standard versions.

THEOREM 5. *Consider the solution* $\hat{x}(t)$ *of* (1) $\hat{x}\colon {}^*\boldsymbol{R}_+ \to {}^*\boldsymbol{R}$. *Suppose that, for any* $t \in {}^*\boldsymbol{R}_{+\infty}$, $a(t) \in {}^*\boldsymbol{R}_{bd}$, $q(t) \in {}^*\boldsymbol{R}_{bd}$, $c(t) \not\approx 0$, *and for any* $\xi \in {}^*\boldsymbol{R}_\infty$, $f(\xi) \in {}^*\boldsymbol{R}_\infty$, *and* $a(t)c(t) \neq 0$. *We also assume that* $\varphi(\xi)f'(\xi)$ *is a positive function* $\big(\forall \xi \in (-\infty, +\infty)\big)$ *if* $\xi \neq 0$. *Suppose that, on some interval* $[t_1, t_2) \subset {}^*\boldsymbol{R}_{+\infty}$, $\hat{x}(t) \in {}^*\boldsymbol{R}_\infty$ *while* $\varphi(\hat{x})\hat{x}'(t) \in {}^*\boldsymbol{R}_{bd}$. *Then* $t_2 - t_1 \approx 0$.

STANDARD VERSION. *Suppose that for sufficiently large values of* t *the following conditions hold:* $\limsup\limits_{t\to\infty} |a(t)| < \infty$, $\limsup\limits_{t\to\infty} |q(t)| < \infty$, $\liminf\limits_{t\to\infty} |c(t)| > 0$, $\lim\limits_{|\xi|\to\infty} |f(\xi)| = \infty$, *and there exists* $\tilde{T} > 0$ *such that* $a(t)c(t) \neq 0$ *for all* $t > \tilde{T}$; $\varphi(\xi)f'(\xi) > 0$ *if* $\xi \neq 0$. *Suppose that given any* $T > 0$, *and given any* $M > 0$, $\varepsilon > 0$, *there exists an interval* $[t_i, t_j]$, $t_j > t_i > T$, *such* *that* $x(t) > M$ *for all* $t \in [t_i, t_j]$, *while* $|\varphi(x(t))x'(t)| < \varepsilon$ (*for all* $t \in [t_i, t_j]$). *Then* $\lim\limits_{T\to\infty}(t_j - t_i) = 0$.

Proof. Let $[t_1, t_2]$ be an interval (in $^*\boldsymbol{R}_{+\infty}$) with the properties stated above.

Consider the integral

$$\text{(b)} \qquad \int_{t_1}^{t_2} \left(\frac{a(t)\varphi(\hat{x})\hat{x}'}{f(\hat{x})}\right)' dt = \frac{a(t)\varphi(\hat{x}(t))\hat{x}'(t)}{f(\hat{x}(t))}\Bigg|_{t_1}^{t_2} \approx 0,$$

since $a(t) \in {}^*\boldsymbol{R}_{bd}$, $\varphi(x(t))x'(t) \in {}^*\boldsymbol{R}_{bd}$ but $f(\hat{x}(t)) \in {}^*\boldsymbol{R}_{\infty}$. However,

$$\int_{t_1}^{t_2} \left(\frac{a(t)\varphi(\hat{x})\hat{x}'}{f(\hat{x})}\right)' dt = \int_{t_1}^{t_2} \left[-c(t) + \frac{q(t)}{f(\hat{x})} - \frac{a(t)\varphi(\hat{x})\cdot f'(\hat{x})\cdot(\hat{x}')^2}{f^2(\hat{x})}\right] dt.$$

We observe that $c(t)$ is not an infinitesimal, $q(t)/f(\hat{x})$ is an infinitesimal, and $\dfrac{-a(t)\varphi(\hat{x})(\hat{x}')^2 f'(\hat{x})}{f^2(\hat{x})}$ has the same sign as $-c(t)$. Hence the integrand is bounded away from zero by a standard number. The only possibility which remains for the integral to have infinitesimal value is $t_2 - t_1 \approx 0$. This completes the proof. A very similar line of argument leads to the proof of the following

THEOREM 6. *Suppose that*

$$\limsup_{\xi\to\infty}\left|\frac{\varphi(\xi)f'(\xi)}{f^2(\xi)}\right| < \infty, \quad \lim_{\xi\to\infty}\frac{\varphi(\xi)}{f(\xi)} = 0, \quad \limsup_{t\to\infty}|a(t)| < \infty,$$

$$\lim_{\xi\to 0} f(\xi) = 0, \quad \limsup_{t\to\infty}|c(t)| < \infty, \quad \liminf_{t\to\infty}|q(t)| > 0.$$

Then for any solution $\hat{x}(t)$ *of* (1) *if* $\hat{x}(t) \neq 0$, *but* $\hat{x}(t) \approx 0$ $\forall t \in [t_1, t_2] \Rightarrow t_2 - t_1 \approx 0$.

Comment on the proof. We make an observation that if $x'(t_1)$ is infinite it must assume a near standard value in an infinitesimal neighborhood of t_1 if $x(t)$ is not infinite in some non-infinitesimal neighborhood of t_1.

Hence if the interval $[t_1, t_2]$ stated in the conclusion of the theorem was of length $t_2 - t_1 \not\approx 0$, while $x'(t)$ was infinite at t_1 and/or t_2, then $[t_1, t_2]$ can be replaced by an interval $[t_1', t_2']$ with identical properties (as stated in the hypothesis) but such that $\hat{x}'(t_1) \in {}^*\boldsymbol{R}_{bd}$, and $\hat{x}'(t_2) \in {}^*\boldsymbol{R}_{bd}$. Now an argument similar to the one given in the proof of Theorems 5 and 6 completes this proof. The details are omitted.

During the writing of this paper I was told of the death of Professor Abraham Robinson the originator of the non-standard analysis, a gentle man with a sense of humor and a great mathematician. Professor Robinson offered me his advice and encouraged me to work on the subject of this paper. All of us working in this field will miss his leadership.

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