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Existence of solutions for non-linear systems of differential-functional equations of parabolic type in an arbitrary domain

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Abstract. Let us consider a system of differential-functional equations of the type

(1) $$F^i[z^i] = f^i(t, x, z(t, x), z), \quad i \in J = \{1, 2, \ldots, m\},$$

where

$$F^i = \frac{\partial}{\partial t} - \sum_{\alpha,\beta=1}^{m} a^i_{\alpha\beta}(t, x) \frac{\partial^2}{\partial x_\alpha \partial x_\beta},$$

$(t, x) = (t, x_1, \ldots, x_n) \in D \subset \mathbf{R}^{n+1}$ and D is an arbitrary open domain whose projection on the t-axis is the interval $(0, T)$, $z(t, x) = (z^1(t, x), \ldots, z^m(t, x))$ and z denotes the mapping

$$z\colon D \ni (t, x) \to (z(t, x)\colon J \ni i \to z^i(t, x) \in \mathbf{R}).$$

For this system we shall consider Fourier's first boundary value problem.

As a special case of equation (1) we may consider the differential-integral equation

$$F^i[z] = \int_{t_0}^{t} \int_E z(\tau, \xi)\, d\tau d\xi$$

in which the integral is a functional of Volterra's type.

Using the iterative method of successive approximations, we shall prove that there is a solution of the above problem in a function class E_2. A similar problem for a cylindrical domain has been considered in [3].

1. Notation, definitions and assumptions. Let $D \subset \mathbf{R}^{n+1}$ be an unbounded domain with respect to x, lying between the hyperplanes $t = 0$ and $t = T < h_0$ (where h_0 is a constant depending on system (1) and defined as in [2]) whose boundary ∂D consists of two m-dimensional unbounded domains S^0 and S^T lying on the planes $t = 0$ and $= T$, respectively, as well as of a certain surface σ which is not tangent to any hyperplane $t = \text{const}$.

We shall assume that the surface σ is of class C^1_*, i.e., it consists of a finite number of surfaces of class C^1, not overlapping but having common boundary points.

Let us write: $\Sigma = S^0 \cup \sigma$, $\bar{D} = D \cup \Sigma \cup S^T$, $D_R = D \cap \{(t, x)\colon |x| < R, R > 0\}$, $\Sigma_R = \partial D_R - S^T$.

E_2 denotes the class of functions $\varphi(t, x)$ defined in an unbounded domain $\Delta \subset R^{n+1}$ for which there exist positive constants M and K such that

$$|\varphi(t, x)| \leqslant M\exp(K|x|^2) \quad \text{for } (t, x)\in\Delta.$$

We write shortly $\varphi(t, x)\in E_2(M, K; \Delta)$.

A function $z(t, x) = (z^1(t, x), \ldots, z^m(t, x))$ will be called *regular in D* if it is continuous in $\bar{D}$ and has continuous derivatives $\partial/\partial t$, $\partial/\partial x_\alpha$, $\partial^2/\partial x_\alpha \partial x_\beta$ in D.

The space of continuous functions $z(t, x) = (z^1(t, x), \ldots, z^m(t, x))$ defined in D with values in R^m is denoted by $C_m(D)$. For the subspace of those z which are bounded in D we introduce the norm

$$\|z\|_t = \sup\{|z^i(\tilde{t}, x)|:\ i\in J,\ \tilde{t}\leqslant t,\ (\tilde{t}, x)\in D\}.$$

A partial order is given in $C_m(D)$: for z, $\tilde{z}\in C_m(D)$ the inequality $z\leqslant\tilde{z}$ means that

$$z^i(t, x)\leqslant\tilde{z}^i(t, x) \quad \text{for } (t, x)\in D\ (i\in J).$$

We assume that the quadratic forms

$$\sum_{\alpha,\beta=1}^{m} a^i_{\alpha\beta}(t, x)\lambda_\alpha\lambda_\beta \qquad (i\in J)$$

are positive-definite in D.

For system (1) we consider Fourier's first boundary value problem: Find the regular solution $z(t, x)$ of system (1) in D fulfilling the boundary condition

$$z(t, x) = 0 \quad \text{for } (t, x)\in\Sigma. \tag{2}$$

Let the functions $f^i(t, x, p, s)$ $(i\in J)$ be defined for $(t, x)\in D$, arbitrary p and $s\in C_m(D)$ and let the coefficients $a^i_{\alpha\beta}(t, x)$ $(i\in J;\ \alpha,\ \beta = 1, \ldots, m)$ be defined for $(t, x)\in D$.

We introduce the following assumptions:

H_f: the functions $f^i(t, x, p, s)$ $(i\in J)$ are continuous with respect to t, x and locally Hölder continuous with respect to x:

L: the functions $f^i(t, x, p, s)$ $(i\in J)$ fulfil the Lipschitz condition with respect to p and s: for arbitrary p, $\tilde{p}$, s, $\tilde{s}$ we have the inequality

$$|f^i(t, x, p, s)-f^i(t, x, \tilde{p}, \tilde{s})| \leqslant L_1\sum_{j=1}^{m}|p^j-\tilde{p}^j| + L_2\sum_{j=1}^{m}\|s^j-\tilde{s}^j\|_t,$$

where L_1, L_2 are positive constants;

W: the function $f^i(t, x, p, s)$ is increasing with respect to $p^1, \ldots, p^{i-1}$, $p^{i+1}, \ldots, p^m, z$;

E_f: the functions $f^i(t, x, 0, 0)$ belong to $E_2(M_f, K_f; D)$ $(i\in J)$;

H_a: the coefficients $a^i_{\alpha\beta}(t, x)$ $(i \in J;\ \alpha,\ \beta = 1, \ldots, m)$ are continuous with respect to t and x, bounded in D and locally Hölder continuous with respect to x.

Functions $u(t, x) = (u^1(t, x), \ldots, u^m(t, x))$ and $v(t, x) = (v^1(t, x), \ldots, v^m(t, x))$ regular in D and satisfying the differential inequalities

$$F^i[u^i] \leqslant f^i(t, x, u(t, x), u) \qquad (i \in J) \tag{3}$$

$$F^i[v^i] \geqslant f^i(t, x, v(t, x), v) \tag{4}$$

in D and the boundary condition (2) will be called the *lower* and the *upper function* for problem (1), (2) in D, respectively.

Assumption A. We assume that there exists at least one pair $u_0(t, x)$, $v_0(t, x)$ of lower and upper functions of class $E_2(M_0, K_0; D)$ for problem (1), (2) in D.

2. Remarks and lemmas. The following lemma is a consequence of Szarski's theorem on weak differential-functional inequalities [11]:

Lemma 1. *Let us assume that*

(i) *the functions* $f^i(t, x, p, s)$ $(i \in J)$ *satisfy conditions* L *and* W,

(ii) *the differential inequalities*

$$\begin{aligned} F^i[u^i] &\leqslant f^i(t, x, u(t, x), u) \\ F^i[v^i] &\geqslant f^i(t, x, v(t, x), v) \end{aligned} \qquad (i \in J)$$

hold for $(t, x) \in D$,

(iii) $u(t, x) \leqslant v(t, x)$ *for* $(t, x) \in \Sigma$.

Then

$$u(t, x) \leqslant v(t, x) \quad \textit{for } (t, x) \in D.$$

It follows from this lemma that if $u(t, x)$ and $v(t, x)$ are lower and upper functions for problem (1), (2) in D, $z(t, x)$ is a regular solution of problem (1), (2) in D and conditions L and W are satisfied, then

$$u_0(t, x) \leqslant z(t, x) \leqslant v_0(t, x) \quad \text{for } (t, x) \in \Sigma. \tag{5}$$

We have in particular

$$u_0(t, x) \leqslant z(t, x) \leqslant v_0(t, x) \quad \text{for } (t, x) \in D. \tag{6}$$

Let $Z(D)$ be the set of regular functions of class E_2 in D satisfying condition (2) on Σ. Denote by P the transformation

$$P\colon Z(D) \ni u(t, x) \to s(t, x) = P[u(t, x)], \tag{7}$$

where $s(t, x)$ is the solution of the system of equations

(8) $$F^i[s^i] = f^i(t, x, u(t, x), u) \quad (i \in J)$$

in D with the boundary condition (2).

The following lemma holds:

LEMMA 2. *Let*

(i) *the functions* $f^i(t, x, p, s)$ $(i \in J)$ *satisfy conditions* H_f, L, E_f;

(ii) *the coefficients* $a^i_{\alpha\beta}(t, x)$ $(i \in J;\ \alpha, \beta = 1, \ldots, n)$ *satisfy condition* H_a;

(iii) $u(t, x) \in Z(D)$.

Then the function $s(t, x) = P[u(t, x)]$ *is uniquely determined and* $s(t, x) \in Z(D)$.

Proof. From assumptions H_f, L, H_a and $u(t, x) \in Z(D)$ it follows that the right-hand sides of system (8) are continuous in D and locally Hölder continuous with respect to x. This fact and the assumption on the domain D imply M. Krzyżański's Hypothesis (A) from [5]–[7]. It means that for any $R > 0$ the system of equations (8) considered in D_R with the boundary condition

$$s(t, x) = \varphi_R(t, x) \quad \text{for } (t, x) \in \Sigma_R$$

has a regular solution in D_R for any continuous function $\varphi_R(t, x)$. Thus, by using the above and assumption E_f, P. Besala's theorem [1], [2] can be applied which implies that problem (8), (2) has exactly one regular solution of class E_2 in D provided $T < h_0$.

LEMMA 3. *If the functions* f^i $(i \in J)$ *satisfy condition* W, *then the operator* $P\colon Z \to Z$ *is increasing.*

Proof. Let $\tilde{u}(t, x), \tilde{\tilde{u}}(t, x) \in Z(D)$ and $\tilde{u}(t, x) \leqslant \tilde{\tilde{u}}(t, x)$ in D. Using (7), (8) and W we have

$$F^i[\tilde{s}^i - \tilde{\tilde{s}}^i] = f^i(t, x, \tilde{u}(t, x), \tilde{u}) - f^i(t, x, \tilde{\tilde{u}}(t, x), \tilde{\tilde{u}}) \leqslant 0 \quad (i \in J) \text{ in } D.$$

From this and from a theorem on differential inequalities [2] it follows that

$$\tilde{s}(t, x) \leqslant \tilde{\tilde{s}}(t, x) \quad \text{in } D$$

which means that

$$P[\tilde{u}(t, x)] \leqslant P[\tilde{\tilde{u}}(t, x)].$$

3. Existence theorem. Let us assume that Assumption A is satisfied, i.e., we are given a lower function $u_0(t, x)$ and an upper function $v_0(t, x)$. Using these functions, we shall construct two sequences of functions $\{u_n(t, x)\}$ and $\{v_n(t, x)\}$

(9) $$u_n(t, x) = P[u_{n-1}(t, x)],$$

(10) $$v_n(t, x) = P[v_{n-1}(t, x)], \qquad n = 1, 2, \ldots$$

We shall prove the following theorem:

THEOREM. *Let*

(i) *the functions* $f^i(t, x, p, s)$ $(i \in J)$ *satisfy conditions* H_f, L, W, E_f,

(ii) *the coefficients* $a^i_{\alpha\beta}(t, x)$ $(i \in J;\ \alpha, \beta = 1, \ldots, n)$ *satisfy condition* H_a,

(iii) *Assumption* A *be satisfied and*

(11) $$N = \max\sup[v^i_0(t, x) - u^i_0(t, x)] < +\infty.$$

Under these assumptions

1° *the successive terms of the iteration sequences* $\{u_n(t, x)\}$, $\{v_n(t, x)\}$, $n = 1, 2, \ldots,$ *are uniquely determined in* D *and* $u_n(t, x)$, $v_n(t, x) \in Z(D)$, $n = 1, 2, \ldots;$

2° *the following inequalities hold*:

(12) $$u_0(t, x) \leqslant u_1(t, x) \leqslant u_2(t, x) \leqslant \ldots$$

(13) $$v_0(t, x) \geqslant v_1(t, x) \geqslant v_2(t, x) \geqslant \ldots \qquad \text{in } D;$$

3° *the functions* $u_n(t, x)$ *and* $v_n(t, x)$, $n = 1, 2, \ldots$ *are the lower and upper functions of class* $E_2(M_0, K_0; D)$ *for problem* (1), (2) *in* D, *respectively*;

(14) 4° $\lim_{n\to\infty} [v_n(t, x) - u_n(t, x)] = 0$ *uniformly in* D;

5° *the function*

$$z(t, x) = \lim_{n\to\infty} u_n(t, x)$$

is the unique regular solution of the class $E_2(M_0 K_0, D)$ *of problem* (1), (2) *in* D.

Proof. The first part of the conclusion is a direct consequence of Lemma 2 and (3), (8), (10) (at this moment the problem whether h_0 is decreased during the iteration process is open — it will be studied later).

In order to prove inequalities (12) and (13) by Lemma 3 it is enough to show that

$$u_0(t, x) \leqslant P[v_0(t, x)] \quad \text{and} \quad v_0(t, x) \geqslant P[v_0(t, x)] \quad \text{in } D.$$

It follows from (9), (8) and (3) that

$$F^i[u^i_0 - u^i_1] \leqslant f^i(t, x, u_0(t, x), u_0) - f^i(t, x, u_0(t, x), u_0) = 0 \quad (i \in J)$$

in D and $u_0 - u_1 = 0$ or Σ. Therefore according to a theorem on differential inequalities [2] we obtain

$$u_0(t, x) \leqslant u_1(t, x) = P[u_0(t, x)] \quad \text{in } D.$$

The proof of the second inequality is analogous.

Starting with determining $u_1(t, x)$ and applying condition W and inequality (12) we have

$$F^i[u_1^i]-f^i(t, x, u_1(t, x), u_1)$$
$$=f^i(t, x, u_0(t, x), u_0)-f^i(t, x, u_1(t, x), u_1) \leqslant 0 \quad (i \in J) \quad \text{in } D.$$

It now follows from Lemma 1 that the function $u_1(t, x)$ is a lower function for problem (1), (2) in D.

Repeating the above reasoning in the next induction step we prove the first part of assertion 3°.

Denoting by $z(t, x)$ the solution to be found of the considered problem (1), (2) in D and applying Lemma 1 and (12), (13) we obtain the inequalities

$$(15) \quad u_0(t, x) \leqslant u_1(t, x) \leqslant \ldots \leqslant u_n(t, x) \leqslant \ldots \leqslant z(t, x) \leqslant \ldots$$
$$\ldots \leqslant v_n(t, x) \leqslant \ldots \leqslant v_1(t, x) \leqslant v_0(t, x) \quad \text{in } D.$$

From (15) and Assumptions A it follows that all functions considered belong to some class E_2:

$$(16) \qquad u_n(t, x),\ v_n(t, x) \in E_2(M_0, K_0; D) \quad \text{for } n = 1, 2, \ldots$$

The admissible height h_0 of the domain D depends on the constants K_0 and L, $m, \ldots$ and can be specified using P. Besala's formula from paper [2]. Therefore h_0 does not decrease during the considered iteration process.

In order to prove part 4° of the conclusion we shall first prove inductively the inequality

$$(17) \qquad \omega_n^i(t, x) \leqslant N\frac{[m(L_1+L_2)t]^n}{n!} \quad (i \in J),\ n = 1, 2, \ldots \text{ in } D,$$

where

$$\omega_n(t, x) = v_n(t, x) - u_n(t, x).$$

Using (8)–(10) and condition L we obtain the inequality

$$(18) \quad F^i[\omega_n^i] = f^i(t, x, v_{n-1}(t, x), v_{n-1}) - f^i(t, x, u_{n-1}(t, x), u_{n-1})$$
$$\leqslant L_1 \sum_{j=1}^{m} \omega_{n-1}^j(t, x) + mL_2 \|\omega_{n-1}\|_t$$

and

$$(19) \qquad \omega_n(t, x) = 0 \quad \text{for } (t, x) \in \Sigma.$$

Assuming inductively that inequality (17) holds for $n-1$ and using the definition of the norm we have

$$(20) \qquad \|\omega_{m-1}\|_t \leqslant N\frac{[m(L_1+L_2)t]^{n-1}}{(n-1)!}.$$

From (18) and (20) we conclude that the function $\omega_n(t, x)$ is a solution of the system of inequalities

$$F^i[\omega_n^i] \leqslant N\frac{[m(L_1+L_2)]^n t^{n-1}}{(n-1)!} \quad (i \in J) \text{ for } (t, x) \in D \tag{21}$$

with the boundary condition (19).

If we now consider the comparison problem for problem (21), (19)

$$F^i[M_n^i] = N\frac{[m(L_1+L_2)]^n t^{n-1}}{(n-1)!} \quad (i \in J) \text{ for } (t, x) \in D \tag{22}$$

with the boundary condition

$$M_n(t, x) \geqslant 0 \quad \text{on } \Sigma, \tag{23}$$

then the functions

$$M_n^i(t, x) = N\frac{[m(L_1+L_2)t]^n}{n!} \quad (i \in J),\ n = 1, 2, \ldots,$$

are its regular solutions in D.

Applying the theorem on differential inequalities in an unbounded domain [1], [2] to systems (21) and (22) we get

$$\omega_n^i(t, x) \leqslant M_n^i(t, x) = N\frac{[m(L_1+L_2)t]^n}{n!} \quad (i \in J) \text{ in } D$$

which, using the principle of mathematical induction, ends the proof of (17).

Assertion 4° follows directly from inequalities (17) and $0 \leqslant t \leqslant T < h_0$.

The iteration sequences $\{u_n\}$ and $\{v_n\}$ are monotonous and (14) holds so there is a function $U(t, x)$ continuous in D such that

$$\begin{aligned} &u_n(t, x) \underset{n\to\infty}{\Rightarrow} U(t, x) \quad \text{uniformly in } D,\\ &v_n(t, x) \underset{n\to\infty}{\Rightarrow} U(t, x) \quad \text{uniformly in } D, \end{aligned} \tag{24}$$

and this function satisfies the boundary condition (2).

To prove that the function $U(t, x)$ defined by (24) is a regular solution of system (1) in D it is enough to show that it fulfils (1) in any compact set contained in D. Consequently, because of the definition of D_R, we only need to prove that it is a regular solution in D_R for any $R > 0$.

It follows from (12), (15) and condition W that the functions $f^i(t, x, u_n(t, x), u_n)$ $(i \in J)$ are uniformly bounded in D_R (with respect to n). On the basis of W. Pogorzelski's results [8] concerning the properties of weak singular integrals by means of which the solution of the system

$$F^i[u_n^i] = f^i(t, x, u_{n-1}(t, x), u_{n-1}) \quad (i \in J) \text{ for } (t, x) \in D_R \tag{25}$$

is expressed we deduce that the functions $u_n(t, x)$ satisfy locally the Lipschitz condition with respect to x, with a constant independent of n. Hence we conclude by (24) that the boundary function $U(t, x)$ satisfies locally the Lipschitz condition with respect to x.

If we now take the system of equations

$$F^i[z^i] = f^i(t, x, U(t, x), U) \quad (i \in J) \text{ for } (t, x) \in D_R \tag{26}$$

with the boundary condition

$$z(t, x) = U(t, x) \quad \text{for } (t, x) \in \Sigma_R, \tag{27}$$

then the last property of $U(t, x)$ together with conditions H_f and L implies that the right-hand sides of system (26) are continuous with respect to t, x in D_R and locally Hölder continuous with respect to x. Hence it follows from condition H_a and from a well-known theorem (cf. [4]) that there exists a regular solution $z(t, x)$ of problem (26), (27) in D_R.

On the other hand, using (24) we conclude that the right-hand sides of (25) converge uniformly in D_R to the right-hand sides of (26)

$$\lim_{n\to\infty} f^i(t, x, u_n(t, x), u_n) = f^i(t, x, U(t, x), U) \quad \text{uniformly in } D_R, \tag{28}$$

and the boundary values of $u_n(t, x)$ converge uniformly on Σ_R to the respective values of $U(t, x)$. Hence using the theorem on the continuous dependence of the solution on the right-hand sides of the system and on the boundary condition [12] (see also [9], [10]) we obtain

$$\lim u_n(t, x) = z(t, x) \quad \text{uniformly in } D_R. \tag{29}$$

Relations (24) and (28) imply that

$$z(t, x) = U(t, x) \quad \text{for } (t, x) \in D_R$$

for any arbitrary R, which means that

$$u(t, x) = U(t, x) \quad \text{for } (t, x) \in D,$$

i.e., $z(t, x)$ is a regular solution of problem (1), (2) in D.

The uniqueness of the solution of this problem follows directly from J. Szarski's results [12].

Moreover, by (15), (16) we have

$$z(t, x) \in E_2(M_0, K_0; D)$$

which ends the proof.

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