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An illustrative example for the Perron condition

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Abstract. The purpose of this note is to give an example of a system of linear differential equations of the form $\dot{x}(t) = A(t)x(t)+f(t)$ with the function $f(t)$ bounded and continuous, the solution of which with an initial condition $x(0) = 0$ is unbounded. At the same time for any $u \in R^n$ the solution of the system $\dot{x}(t) = A(t)x(t)+u$, $x(0) = 0$ is bounded.

In stability theory for ordinary differential equations the Perron condition is known. For certain systems of equations this condition is equivalent to the uniform asymptotic stability of the zero solution. The example given below shows that the assumptions of the Perron condition cannot be weakened.

Before giving the example, we are going to recall some definitions of stability theory. Let us consider the system of equations

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in R^n$, and $A(t)$ is a real, measurable, $n \times n$ matrix.

DEFINITION 1. We will say that the zero solution $x(t) \equiv 0$ is *stable* if for every $\varepsilon > 0$ there exists a $\delta(\varepsilon, t_0)$ such that if $|x_0| < \delta(\varepsilon, t_0)$, then $|x(t)| < \varepsilon$ for $t \geqslant t_0$.

DEFINITION 2. The solution $x(t) \equiv 0$ is said to be *uniformly asymptotically stable* if for any $\varepsilon > 0$ there exists a $\delta_0 > 0$ and functions $\delta(\varepsilon)$ and $T(\varepsilon)$ with the property that $|x_0| < \delta(\varepsilon)$ implies that $|x(t)| < \varepsilon$ for $t \geqslant t_0$ and if $|x_0| < \delta_0$, then $|x(t)| < \varepsilon$ for $t \geqslant t_0+T(\varepsilon)$.

DEFINITION 3. We will say that system (1) *satisfies the Perron condition* (condition P) if for every continuous and bounded function $f(t)$ defined for $t \geqslant 0$ the solution of the system

$$\dot{x}(t) = A(t)x(t)+f(t), \quad x(0) = 0, \tag{2}$$

is bounded for $t \geqslant 0$.

The main theorem concerning the Perron condition is the following:

THEOREM. *If $|A(t)| \leqslant A$ and system (1) satisfies condition P, then the zero solution of the system is uniformly asymptotically stable.*

We assume $|A| = \sup\{|Ax|: |x| \leqslant 1\}$. Let us denote by $X(t, t_0)$ the fundamental matrix of system (1). We know that the uniform asymptotic stability of the zero solution implies, for linear systems, exponential stability, i.e., there are constans $M, \alpha > 0$ such that

$$|X(t, t_0)| \leqslant Me^{-\alpha(t-t_0)}.$$

Using this inequality, one can prove that uniform asymptotic stability implies the Perron condition. In fact, any solution of system (2) can be written in the form

$$x(t) = \int_0^t X(t, s)f(s)\,ds;$$

hence

$$|x(t)| \leqslant \sup_{s\geqslant 0}|f(s)| \int_0^t Me^{-\alpha(t-s)}\,ds = \frac{M}{\alpha}\sup_{s\geqslant 0}|f(s)|(1-e^{\alpha t}) \leqslant \frac{M}{\alpha}\sup_{s\geqslant 0}|f(s)|.$$

DEFINITION 4. System (1) is said to *satisfy condition P'* if for each $u \in R^n$ the solution of the system

$$\dot{x}(t) = A(t)x(t)+u, \qquad x(0) = 0,$$

is bounded for $t \geqslant 0$.

For $n = 1$ condition P is equivalent to condition P'. This follows from the fact that for $n = 1$ we have $X(t, t_0) > 0$ and then

$$|x(t)| \leqslant \int_0^t X(t, s)|f(s)|\,ds \leqslant \sup_{s\geqslant 0}|f(s)| \int_0^t X(t, s)\,ds.$$

The example presented below should convince us that the theorem will not be true if we replace condition P by P'. It will show also that conditions P and P' are not equivalent for $n > 1$.

EXAMPLE. Let us consider system (1) with the matrix $A(t)$ of the form

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & -\dfrac{2}{t+1} \end{pmatrix}.$$

It is easy to prove by calculation that the fundamental matrix of this system if of the form

$$X(t, t_0) = \begin{pmatrix} \dfrac{(t+1)\cos(t-t_0)+\sin(t-t_0)}{t+1}, & \dfrac{(t_0+1)\sin(t-t_0)}{t+1} \\ \dfrac{(t-t_0)\cos(t-t_0)-[(t+1)(t_0+1)+1]\sin(t-t_0)}{(t+1)^2}, & \dfrac{(t_0+1)[(t+1)\cos(t-t_0)-\sin(t-t_0)]}{(t+1)^2} \end{pmatrix}.$$

Next we will prove that the foregoing system satisfies condition P' and that the theorem will not be true if we replace condition P by P'. To get the first fact it is enough to prove that there is a $M < \infty$ such that

$$\left|\int_0^t X(t, s)\,ds\right| \leqslant M.$$

In the case considered this inequality is a simple consequence of the following two inequalities:

$$\left|\int_0^t \frac{s\sin(t-s)}{t+1}\right| ds \leqslant M \quad \text{and} \quad \left|\int_0^t \frac{s\cos(t-s)}{t+1}\right| ds \leqslant M.$$

Integrating for instance the first of them we obtain

$$\int_0^t \frac{s\sin(t-s)}{t+1}\,ds = \frac{1}{t+1}[s\cos(t-s)+\sin(t-s)]_0^t$$

and thus

$$\left|\int_0^t \frac{s\sin(t+s)}{t+1}\right| ds = \left|\frac{t-\sin t}{t+1}\right| \leqslant 1.$$

Therefore the system in question satisfies condition P'. Since some components of the fundamental matrix are of the form

$$\frac{(t_0+1)\sin(t-t_0)}{t+1},$$

the zero solution of this system is not exponentially stable. Consequently, the theorem is not true for this system.

Finally I want to state the following problem: Let us suppose that $|A(t)| \leqslant A$ and that system (1) satisfies condition P'. Does these assumptions imply that the zero solution of system (1) is stable?

References

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[2] O. Perron, *Die Stabilitätfrage bei Differentialgleichungen*, Math. Z. 32 (1830), p. 703–728.

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