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On fixed points of continuous real functions

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Abstract. The paper contains some conditions for attractive and repulsive fixed points of continuous real functions. Its main theorem is the following sufficient condition: If in a certain neighbourhood of a point ξ a continuous function f fulfils the following conditions:

$$f_2(x) \neq x \quad \text{for } x \neq \xi,$$
$$x < f(x) \leqslant k(x-\xi)+\xi \quad \text{for } x < \xi,$$
$$\frac{1}{k}(x-\xi)+\xi < f(x) < x \quad \text{for } x > \xi,$$

where k is a negative constant, then ξ is an attractive fixed point of the function f. The analogical theorem for repulsive points is also proved.

In this paper we study some properties of attractive and repulsive fixed points of continuous real functions. Fixed points and their properties have been investigated by many authors because of their importance. Every result in this direction finds immediately applications in the theory of difference equations and in numerical analysis. A fairly complete bibliography of the subject may be found in [2], our researches are based mainly on [1] and [3].

1. Notations and definitions. To start with we fix some notions. Let E be a connected, non-degenerated to one point, subset of the space of real numbers. All topological notions will be understood relatively to the set E. Let f be a continuous function defined in E and fulfilling the condition

$$f(E) \subset E. \tag{1.1}$$

The class of all such functions will be denoted by $C[E]$. In the sequel we shall always assume that f belongs to $C[E]$. The successive iterates of f are defined by the following recurrence relation:

$$f_{n+1}(x) = f\big(f_n(x)\big), \quad f_0(x) = x, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

The sequence $\{x_n\}$ defined by

$$(1.3) \qquad x_n = f_n(x_0), \quad n = 0, 1, 2, \ldots$$

is called the *iterative sequence* of the point x_0. Every point x fulfilling the equation

$$(1.4) \qquad f(x) = x$$

is called a *fixed point* of the function f. Fixed points of the function f_k will be called *fixed points* of order k, and the set of points $\{x_1, x_2, \ldots, x_k\}$ which fulfil the relation

$$(1.5) \qquad f(x_1) = x_2, \quad f(x_2) = x_3, \quad \ldots, \quad f(x_{k-1}) = x_k, \quad f(x_k) = x_1$$

will be called a *cycle of order k*. A cycle of order one is simply a fixed point. Every point y fulfilling

$$(1.6) \qquad f_p(y) = x$$

is called the *p-th antecedent of x*, the point $v = f_q(x)$ is called the *q-th consequent of x*.

Now we recall the well-known equivalence relation (cf. [2])

$$(1.7) \qquad x \sim y \Leftrightarrow \bigvee_{p,q} f_p(x) = f_q(y).$$

In all the above definitions k, p, q denote non-negative integers. The set of all points which are equivalent to a given x_0 will be called the *orbit of x_0* under f. The orbit of an x will be denoted by $C_f(x)$ or, if it is clear which function is involved, by $C(x)$.

It follows from this definition that the orbit contains the antecedents and iterative sequences of all its elements. It may happen that some of those sequences are convergent. Let us write

$$(1.8) \qquad A_f(\xi) = \{x \in E\colon \lim_{n\to\infty} f_n(x) = \xi\}.$$

The set $A_f(\xi)$ will be called the *total domain of attraction of the point* ξ with respect to f.

Definition 1.1. (a) A fixed point ξ of a function f will be called *attractive* if it is an inner point of $A_f(\xi)$.

(b) A fixed point ξ of a function f will be called *repulsive* if there exists a neighbourhood U of ξ such that

$$(1.9) \qquad x \in U \setminus C_f(\xi) \Rightarrow x \notin A_f(\xi).$$

(c) All the remaining fixed points of the function f will be called *mixed fixed points* (cf. also [1], [2]).

It is clear that an attractive fixed point must be an isolated fixed point.

2. Attractive fixed points. Now we are going to give some conditions which are necessary or sufficient for the attractive character of a given fixed point. In the proof of one of them we shall make use of the following result:

LEMMA 2.1. *If an iterative sequence has exactly k cluster points, then these points form a cycle of order k.*

The proof of this fact may be found in [1].

THEOREM 2.1. *If there exists a left neighbourhood U_ξ^- of ξ in which*

$$x < f(x) < k(x-\xi)+\xi \quad \text{for all } x \neq \xi, \tag{2.1}$$

and if there exists a right neighbourhood U_ξ^+ of ξ in which

$$\frac{1}{k}(x-\xi)+\xi < f(x) < x \quad \text{for all } x \neq \xi, \tag{2.2}$$

where k is a negative constant, then ξ is an attractive fixed point.

Proof. It follows from each of inequalities (2.1), (2.2), that ξ is a fixed point. We shall find an open interval containing ξ and contained in $A_f(\xi)$. Put

$$U_\xi^- = (\xi-\varepsilon_1, \xi], \qquad U_\xi^+ = [\xi, \xi+\varepsilon_2), \qquad \varepsilon = \min(\varepsilon_1, \varepsilon_2).$$

We shall examine two cases:

1° $|k| \geqslant 1$. Put $U = \left(\xi - \frac{\varepsilon}{|k|}, \xi+\varepsilon\right)$. For $x \in U$, $x < \xi$, we have

$$\xi - \frac{\varepsilon}{|k|} < x < f(x) < k(x-\xi)+\xi,$$

and consequently

$$\xi - \frac{\varepsilon}{|k|} < f(x) < \xi+\varepsilon. \tag{2.3}$$

Similarly, for $x \in U$, $x > \xi$, we obtain

$$\frac{1}{k}(x-\xi)+\xi < f(x) < x < \xi+\varepsilon,$$

whence

$$\xi - \frac{\varepsilon}{|k|} < f(x) < \xi+\varepsilon. \tag{2.4}$$

Inequalities (2.3) and (2.4) yield $f(U) \subset U$.

2° $|k| < 1$. Now we define $U = \left(\xi-\varepsilon, \xi + \frac{\varepsilon}{|k|}\right)$ and proceeding similarly as in case 1° we obtain $f(U) \subset U$.

Now we are going to prove that $U \subset A_f(\xi)$. We take an arbitrary $x_0 \in U$ and form its iterative sequence $x_n = f_n(x_0)$. Let us consider the sets $(-\infty, \xi] \cap \{x_n\}_0^\infty$ and $[\xi, \infty) \cap \{x_n\}_0^\infty$. If one of these sets is empty, then, since $f(x) > x$ for $x \in U_\xi^-$ and $f(x) < x$ for $x \in U_\xi^+$, the sequence $\{x_n\}$ is strictly monotonic. Consequently, it must converge to ξ. Thus we may restrict attention to the case where neither of the sets $(-\infty, \xi] \cap \{x_n\}_0^\infty$, $[\xi, \infty) \cap \{x_n\}_0^\infty$ is empty. The former is the sequence of those terms of $\{x_n\}$ which lie on the left-hand side of ξ. Let us denote them as follows

$$x_{p_0}, x_{p_1}, x_{p_2}, \ldots, \quad p_0 < p_1 < p_2 < \ldots \tag{2.5}$$

We shall prove that this sequence is increasing. Consider one of its terms, for example x_{pl}.

If $f(x_{p_l}) < \xi$, then $f(x_{p_l}) = x_{p_{l+1}}$ and inequality (2.1) gives

$$x_{p_l} < x_{p_{l+1}}.$$

If $f(x_{p_l}) = \xi$, then evidently $\{x_n\}$ converges to ξ.

If $f(x_{p_l}) > \xi$, then there exists such an $m > 1$ that

$$x_{p_{l+1}} = f_m(x_{p_l}) = x_{p_l+m},$$

whereas the terms

$$x_{p_l+1}, x_{p_l+2}, \ldots, x_{p_l+m-1}$$

lie on the right-hand side of ξ. We have for them

$$\xi < x_{p_l+m-1} < x_{p_l+m-2} < \ldots < x_{p_l+1},$$

so in virtue of (2.1)

$$x_{p_l+m-1} - \xi < x_{p_l+1} - \xi < k(x_{p_l} - \xi). \tag{2.6}$$

Inequality (2.2) gives

$$\frac{1}{k}(x_{p_l+m-1} - \xi) < x_{p_l+m} - \xi = x_{p_{l+1}} - \xi,$$

whence

$$k(x_{p_{l+1}} - \xi) < x_{p_l+m-1} - \xi. \tag{2.7}$$

From (2.6) and (2.7) we obtain

$$k(x_{p_{l+1}} - \xi) < k(x_{p_l} - \xi),$$

which gives

$$x_{p_l} < x_{p_{l+1}}.$$

The monotonicity of the subsequence $\{x_{q_j}\}$ lying in U_ξ^+ can be proved analogically. Finally we obtain

$$\lim x_{p_i} = a \leqslant \xi, \quad \lim x_{q_j} = b \geqslant \xi.$$

But, as was remarked before (Lemma 2.1), the partial limits of an iterative sequence form a cycle. Consequently, we must have

$$f(a) = b, \quad f(b) = a.$$

If we had e. g. $a \neq \xi$, i.e. $a < \xi$, so from the fact that $a \in U$ and in view of inequalities (2.1) and (2.2) we conclude

$$b-\xi < k(a-\xi) \quad \text{and} \quad b-\xi > k(a-\xi).$$

This obvious contradiction completes the proof of our theorem.

As an immediate consequence of this theorem we have the following

COROLLARY 2.1. *If there exists a neighbourhood U of ξ such that*

$$|f(x)-\xi| < |x-\xi| \tag{2.8}$$

for $x \in U$, $x \neq \xi$, then ξ is an attractive fixed point of f (cf. [4]).

Inequality (2.8) follows from (2.1), (2.2) by setting $k = -1$.

The assumptions of Theorem 2.1 can be weakened, namely in one of the neighbourhoods U_ξ^-, U_ξ^+ we can admit the weak inequality. If we admitted the weak inequalities in both U_ξ^-, U_ξ^+, then we should have to assume additionally that $f_2(x) \neq x$ in a certain neighbourhood of ξ. In fact, the proof of Theorem 2.1 is based on this very fact. From the above considerations we obtain the following

THEOREM 2.2. *If there exists a neighbourhood U of the point ξ such that*

1° $f_2(x) \neq x$ *for* $x \in U$, $x \neq \xi$,

2° $x < f(x) \leqslant k(x-\xi)+\xi$ *for* $x \in U$, $x < \xi$,

3° $\frac{1}{k}(x-\xi)+\xi \leqslant f(x) < x$ *for* $x \in U$, $x > \xi$,

where k is a negative constant, then ξ is an attractive fixed point.

The following two theorems concern the case where $k \to -\infty$ and $k \to 0$.

THEOREM 2.3. *If there exists a left neighbourhood U_ξ^- of the point ξ such that*

$$x < f(x) \quad \text{for } x \in U_\xi^-,\ x \neq \xi, \tag{2.9}$$

and if there exists a right neighbourhood U_ξ^+ of the point ξ such that

$$\xi \leqslant f(x) < x \quad \text{for } x \in U_\xi^+,\ x \neq \xi, \tag{2.10}$$

then ξ is an attractive fixed point.

Proof. It follows from (2.10) that $f(\xi) = \xi$. Put

$$U_\xi^- = (\xi-\varepsilon_1, \xi], \quad U_\xi^+ = [\xi, \xi+\varepsilon_2), \quad \varepsilon = \min(\varepsilon_1, \varepsilon_2).$$

The continuity of f implies the existence of a $\delta > 0$ such that

$$f(x) < \xi + \varepsilon \quad \text{if } \xi - \delta < x < \xi.$$

Consider the interval

$$V = \begin{cases} (\xi - \varepsilon, \xi + \varepsilon) & \text{if } \delta \geqslant \varepsilon, \\ (\xi - \delta, \xi + \varepsilon) & \text{if } \delta < \varepsilon. \end{cases}$$

It is easy to check that $f(V) \subset V$. Take an arbitrary $x_0 \in V$, $x_0 \neq \xi$. If $x_0 > \xi$, then $f_n(x_0)$ is a decreasing sequence, so it must be convergent to ξ. If $x_0 < \xi$ and all the terms $f_n(x_0)$ lie on the left of ξ, then they form an increasing sequence convergent to ξ, and if, for some $l > 0$, $f_l(x_0) > \xi$, then all the consequents of $f_l(x_0)$ lie on the right of ξ. Hence it follows that $\lim f_n(x_0) = \xi$.

The proof of the next theorem is similar.

THEOREM 2.4. *If there exists a left neighbourhood U_ξ^- of the point ξ such that*

$$x < f(x) \leqslant \xi \quad \textit{for } x \in U_\xi^-,\ x \neq \xi, \tag{2.11}$$

and if there exists a right neighbourhood U_ξ^+ of the point ξ such that

$$f(x) < x \quad \textit{for } x \in U_\xi^+,\ x \neq \xi, \tag{2.12}$$

then ξ is an attractive fixed point.

Those were sufficient conditions for attractive fixed points, now we are going to prove a necessary condition, but first we shall prove the following auxiliary theorem.

LEMMA 2.2. *If there exists a left (right) neighbourhood of the fixed point ξ in which $f(x) < x$ $(f(x) > x)$ holds for $x \neq \xi$, then ξ is not attractive.*

Proof. Let us consider a left neighbourhood of ξ (for a right neighbourhood the proof is the same) and assume that

$$f(x) < x \quad \text{for } x \in (\xi - \varepsilon, \xi),\ \varepsilon > 0. \tag{2.13}$$

Two cases may occur:

1° $f(x) < x$ for all $x < \xi$, $x \in E$. Then the iterative sequence of an arbitrary $x_0 < \xi$ is convergent to an a, $-\infty \leqslant a < \xi$, which means that ξ cannot be attractive.

2° There exist in the set $E \cap (-\infty, \xi - \varepsilon]$ points for which $f(x) = x$. Write

$$\eta = \max_{x \leqslant \xi - \varepsilon} \{x : f(x) = x\}.$$

In the whole interval (η, ξ) we have $f(x) < x$. Again we shall consider two cases:

(a) $f(x) \geqslant \eta$ for $x \in (\eta, \xi)$. Then $[\eta, \xi) \subset A_f(\eta)$. So ξ is not attractive, since in every neighbourhood of ξ there exist points generating iterative sequences convergent to $\eta \neq \xi$.

(b) There exists an $x \in (\eta, \xi)$ for which $f(x) < \eta$. It follows from the continuity of f that there exists a $y \in (\eta, \xi)$ such that $f(y) = \eta$. Write

$$\eta_{-1} = \max_{x \in (\eta, \xi)}\{x\colon f(x) = \eta\}.$$

The existence of this maximum is guaranteed by the continuity of f. Further, put

$$\eta_{-2} = \max_{x \in (\eta_{-1}, \xi)}\{x\colon f(x) = \eta_{-1}\}.$$

Continuing this procedure we obtain the sequence of antecedents of η

$$\eta < \eta_{-1} < \eta_{-2} < \ldots < \xi.$$

This sequence is increasing and bounded, so it converges. Write

$$\lim \eta_{-n} = a.$$

We have

$$\eta < a \leqslant \xi$$

and we obtain

$$f(a) = f(\lim \eta_{-n}) = \lim f(\eta_{-n}) = \lim \eta_{-n+1} = a.$$

Thus we must have

$$a = \xi,$$

since in the interval $(\eta, \xi]$ there are no other fixed points of f. Hence, in every neighbourhood of ξ we can find antecedents of η, which means that ξ is not attractive. Now we can prove the announced theorem:

THEOREM 2.5. *If ξ is an attractive fixed point, then there exists a left neighbourhood of ξ in which $f(x) > x$ for $x \neq \xi$, and there exists a right neighbourhood of ξ in which $f(x) < x$ for $x \neq \xi$.*

Proof. Suppose e.g. that in every left neighbourhood of ξ there exist points for which $f(x) \leqslant x$. If in every left neighbourhood of ξ one may find points fulfilling $f(x) = x$, then ξ, as a cluster point of the set of fixed points, cannot be attractive. Now it is enough to consider the case where ξ has a left neighbourhood in which $f(x) < x$ for all $x \neq \xi$. But this case, in view of Lemma 2.2, also leads to contradiction with the assumption. The proof concerning right neighbourhoods is the same.

3. Repulsive fixed points. In this part we are going to prove some conditions analogous in a certain sense to the theorems of Section 2.

THEOREM 3.1. *If there exists a left neighbourhood U_ξ^- of the point ξ such that*

$$k(x-\xi)+\xi \leqslant f(x) \quad \textit{for } x \in U_\xi^-, \tag{3.1}$$

and if there exists a right neighbourhood U_ξ^+ of the point ξ such that

$$f(x) \leqslant \frac{1}{k}(x-\xi)+\xi \quad \textit{for } x \in U_\xi^+, \tag{3.2}$$

where k is a negative constant, then ξ is a repulsive fixed point.

Proof. It follows from (3.1) and (3.2) that $f(\xi) = \xi$. Let us put $U = U_\xi^- \cup U_\xi^+$ and take an arbitrary $x_0 \in U \setminus C_f(\xi)$. Thus, if the sequence $\{f_n(x_0)\}$ contains only a finite number of different terms, it converges to a fixed point different from ξ or contains a cycle. Suppose then that $\{f_n(x_0)\}$ is infinite. We shall consider two cases:

1° An infinite number of the terms of the sequence $f_n(x_0)$ lie outside U. Then evidently $f_n(x_0)$ cannot converge to ξ.

2° Almost all terms of the sequence $f_n(x_0)$ lie in U. For those terms we observe that $f_n(x_0) < \xi$ implies $f_{l+1}(x_0) > \xi$, and similarly $f_m(x_0) > \xi$ implies $f_{m+1}(x_0) < \xi$. So for n sufficiently large the terms of the sequence $f_n(x_0)$ lie alternatively on the left- and right-hand side of ξ. Suppose that e. g.

$$f_l(x_0) < \xi.$$

Then

$$k\big(f_l(x_0)-\xi\big) < f_{l+1}(x_0)-\xi,$$

and

$$f_{l+2}(x_0)-\xi \leqslant \frac{1}{k}\big(f_{l+1}(x_0)-\xi\big), \tag{3.3}$$

whence

$$k\big(f_l(x_0)-\xi\big) \leqslant k\,\big(f_{l+2}(x_0)-\xi\big)$$

and

$$f_{l+2}(x_0) \leqslant f_l(x_0).$$

Thus those terms of $f_n(x_0)$ which lie on the left-hand side of ξ form a decreasing sequence and hence they converge to a number different from ξ. This is enough to prove the theorem, although the similar argument shows that the terms of $\{f_n(x_0)\}$ lying on the right-hand side of ξ increase away from ξ.

COROLLARY 3.1. *If $f(\xi) = \xi$ and there exists a neighbourhood U of the point ξ in which*

$$|f(x)-\xi| \geqslant |x-\xi|, \tag{3.4}$$

then ξ is repulsive.

The proof of this fact is similar to the proof of Theorem 3.1.

THEOREM 3.2. *If there exists a left neighbourhood U_ξ^- of the fixed point ξ such that*

$$\xi < f(x) \quad \textit{for } x \in U_\xi^-,$$

and if there exists a right neighbourhood U_ξ^+ of ξ such that

$$x \leqslant f(x) \quad \textit{for } x \in U_\xi^+,$$

then ξ is repulsive.

Proof. First we note that $x \in U_\xi^-$ implies $f(x) > \xi$, so we can restrict our considerations to the points of U_ξ^+. But for those points we have

$$|f(x) - \xi| \geqslant |x - \xi|,$$

which in virtue of Corollary 3.1 proves the theorem.

Quite similar is the proof of the following

THEOREM 3.3. *If there exists a left neighbourhood U_ξ^- of the fixed point ξ such that*

$$f(x) \leqslant x \quad \textit{for } x \in U_\xi^-,$$

and if there exists a right neighbourhood U_ξ^+ of ξ such that

$$f(x) < \xi \quad \textit{for } x \in U_\xi^+,$$

then ξ is a repulsive fixed point.

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