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On expansions of Meijer's functions II

The method of the exponential factor

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§ 3. The method of the exponential factor. Let us introduce the notations of the first part of this paper (see [3]) and further let

$$(45)\quad B_h(s) = \Big\{\prod_{\substack{j=1\\ j\neq h}}^{m}\Gamma(s-b_j)\prod_{j=1}^{n}\Gamma(1-s+a_j)\Big\}\Big/\Big\{\prod_{j=m+1}^{q}\Gamma(1-s+b_j)\prod_{j=n+1}^{p}\Gamma(s-a_j)\Big\}\qquad (h=1,\ldots,m),$$

$$(46)\quad \beta(s) = \prod_{j=m+1}^{q}\Gamma(1-s+b_j),$$

$$(47)\quad D_h(s) = \Big\{\prod_{\substack{j=1\\ j\neq h}}^{\mu}\Gamma(d_j-s)\prod_{j=1}^{\nu}\Gamma(1+s-c_j)\Big\}\Big/\Big\{\prod_{j=\mu+1}^{\tau}\Gamma(1+s-d_j)\prod_{j=\nu+1}^{\sigma}\Gamma(c_j-s)\Big\}\qquad (h=1,\ldots,\mu),$$

$$(48)\quad \delta(s) = \prod_{j=\mu+1}^{\tau}\Gamma(1+s-d_j).$$

Finally let

$$(49)\quad G_{1A}(x) = \Big|\exp(-x/t)\,G_{\sigma,\tau}^{\mu,\nu}\Big(\omega x\Big|\begin{matrix}c_1,\ldots,c_\sigma\\ d_1,\ldots,d_\tau\end{matrix}\Big)\Big|\sum_{r=0}^{\infty}\Big|\sum_{h=1}^{m}(1/r!)B_h(b_h)(x/\eta)^{-b_h}\times$$
$$\times(x/t)^r{}_{p+1}F_{q-1}\Big(\begin{matrix}-r,1-b_h+a_1,\ldots,1-b_h+a_p;\\ 1-b_h+b_1,\ldots{*}\ldots,1-b_h+b_q;(-1)^{p-m-n+1}t/\eta\end{matrix}\Big)\Big|,$$

$$(50)\quad G_{1B}(x) = \Big|\exp(-tx)\,G_{p,q}^{m,n}\Big(x/\eta\Big|\begin{matrix}-a_1,\ldots,-a_p\\ -b_1,\ldots,-b_q\end{matrix}\Big)\Big|\sum_{r=0}^{\infty}\Big|\sum_{h=1}^{\mu}(1/r!)D_h(d_h)(\omega x)^{d_h}\times$$
$$\times(tx)^r{}_{\sigma+1}F_{\tau-1}\Big(\begin{matrix}-r,1+d_h-c_1,\ldots,1+d_h-c_\sigma;\\ 1+d_h-d_1,\ldots{*}\ldots,1+d_h-d_\tau;(-1)^{\sigma-\mu-\nu+1}\omega/t\end{matrix}\Big)\Big|,$$

where the asterisk * in the first formula denotes that the number $1-b_h+b_h$ is to be omitted in the sequence $1-b_h+b_1, \ldots, 1-b_h+b_q$ and, analogously, the asterisk * in the second formula denotes that the number $1+d_h-d_h$ is to be omitted in the sequence $1+d_h-d_1, \ldots, 1+d_h-d_\tau$. In formulae (49) and (50) we assume that $x>0$ and t fulfils the condition

$$t \neq 0, \qquad |\arg t| < \tfrac{1}{2}\pi . \tag{28}$$

The functions B_h and D_h exist if the Gamma functions appearing in the numerators have no poles at the given points. Analogously, the functions β and δ exist if the Gamma functions appearing in the respective formulae exist. The function G_{1A} exists in each of the cases (I), (II), (III), (IV), (V) (see [3]); this will be proved in Theorem 1A below. The function G_{1B} exists also in each of the cases (I), (II), (III), (IV), (V), where in (II) and (IV) we assume additionally (30); this will be proved in Theorem 1B below. In the case where some of the numbers

$$b_h - b_j \qquad (j = m+1, \ldots, q;\ h = 1, \ldots, m), \tag{51}$$

$$d_j - d_h \qquad (j = \mu+1, \ldots, \tau;\ h = 1, \ldots, \mu) \tag{52}$$

are positive integers, formulae (49) and (50) must be understood in the sense of Remarks 5 and 6, respectively.

Remark 5. In the case where some of the numbers (51) are natural, the respective coefficients $B_h(b_h)$ are to be replaced by the limit of the products $B_h(b_h^*)\beta(b_h^*)$ as $b_h^* \to b_h$, and the respective functions ${}_{p+1}F_{q-1}(1-b_h+a_1, \ldots)$ by the limit of the quotients

$${}_{p+1}F_{q-1}(1-b_h^*+a_1, \ldots)/\beta(b_h^*) \qquad \text{as} \qquad b_h^* \to b_h .$$

Remark 6. In the case where some of the numbers (52) are natural, the respective coefficients $D_h(d_h)$ are to be replaced by the limit of the products $D_h(d_h^*)\delta(d_h^*)$ as $d_h^* \to d_h$, and the respective functions ${}_{\sigma+1}F_{\tau-1}(1+d_h-c_1, \ldots)$ by the limit of the quotients

$${}_{\sigma+1}F_{\tau-1}(1+d_h^*-c_1, \ldots)/\delta(d_h^*) \qquad \text{as} \qquad d_h^* \to d_h .$$

We shall prove two theorems by the method of the exponential factor announced in § 1 (see [3], pp. 245-247).

THEOREM 1A. *Let $m, n, p, q, \mu, \nu, \sigma, \tau$ be integers, let t fulfil* (28) *and let one of the cases* (I), (II), (III), (IV), (V) *take place. If for large x_0*

$$\int_{x_0}^{\infty} G_{1A}(x)\,dx \qquad \textit{converges}, \tag{53}$$

then

$$(54)\qquad G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}\left(\eta\omega\,\middle|\,\begin{matrix} b_1, \ldots, b_m, c_1, \ldots, c_\sigma, b_{m+1}, \ldots, b_q \\ a_1, \ldots, a_n, d_1, \ldots, d_\tau, a_{n+1}, \ldots, a_p\end{matrix}\right)$$

$$= \sum_{r=0}^{\infty} \sum_{h=1}^{m} (1/r!) B_h(b_h) (t/\eta)^{1-b_h} \times$$

$$\times {}_{p+1}F_{q-1}\left(\begin{matrix} -r, 1-b_h+a_1, \ldots, 1-b_h+a_p; \\ 1-b_h+b_1, \ldots * \ldots, 1-b_h+b_q; \ (-1)^{p-m-n+1} t/\eta\end{matrix}\right) \times$$

$$\times G^{\mu,\nu+1}_{\sigma+1,\tau}\left(\omega t\,\middle|\,\begin{matrix} b_h - r, c_1, \ldots, c_\sigma \\ d_1, \ldots, d_\tau \end{matrix}\right),$$

*where the asterisk * denotes that the number $1-b_h+b_h$ is to be omitted in the sequence $1-b_h+b_1, \ldots, 1-b_h+b_q$, and in the case where some of the numbers* (51) *are natural, formulae* (54) *and* (49) *must be understood in the sense of Remark* 5. *The connection between the branches of $G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}$ and $G^{\mu,\nu+1}_{\sigma+1,\tau}$ is determined by Remark* 1.

Proof. Let J denote the left-hand side of (54) and let there occur one of the five cases (I), (II), (III), (IV), (V), where $m, n, p, q, \mu, \nu, \sigma, \tau$ are integers. By virtue of Lemma 1 we have

$$J = (1/\eta) \int_0^\infty G^{m,n}_{p,q}\left(x/\eta\,\middle|\,\begin{matrix} -a_1, \ldots, -a_p \\ -b_1, \ldots, -b_q\end{matrix}\right) G^{\mu,\nu}_{\sigma,\tau}\left(\omega x\,\middle|\,\begin{matrix} c_1, \ldots, c_\sigma \\ d_1, \ldots, d_\tau\end{matrix}\right) dx .$$

It is known (see [6], I, formula (18), p. 371 or [5], I, formula (1), p. 82) that the function $G^{m,n}_{p,q}$ can be expressed in the form

$$(55)\qquad G^{m,n}_{p,q}\left(x/\eta\,\middle|\,\begin{matrix} -a_1, \ldots, -a_p \\ -b_1, \ldots, -b_q\end{matrix}\right) = \sum_{h=1}^{m} B_h(b_h) (x/\eta)^{-b_h} \times$$

$$\times {}_{p}F_{q-1}\left(\begin{matrix} 1-b_h+a_1, \ldots, 1-b_h+a_p; \\ 1-b_h+b_1, \ldots * \ldots, 1-b_h+b_q; (-1)^{p-m-n} x/\eta\end{matrix}\right),$$

where the asterisk * denotes that the number $1-b_h+b_h$ is to be omitted in the sequence $1-b_h+b_1, \ldots, 1-b_h+b_q$ and in the case where some of the numbers (51) are natural, the formula must be understood in the sense of Remark 5. Hence

$$J = (1/\eta) \int_0^\infty \sum_{h=1}^{m} B_h(b_h) (x/\eta)^{-b_h} \times$$

$$\times {}_{p}F_{q-1}\left(\begin{matrix} 1-b_h+a_1, \ldots, 1-b_h+a_p; \\ 1-b_h+b_1, \ldots * \ldots, 1-b_h+b_q; \ (-1)^{p-m-n} x/\eta\end{matrix}\right) \times$$

$$\times G^{\mu,\nu}_{\sigma,\tau}\left(\omega x\,\middle|\,\begin{matrix} c_1, \ldots, c_\sigma \\ d_1, \ldots, d_\tau\end{matrix}\right) dx .$$

Now we use for $h=1,\ldots,m$ the formulae

$$(56)\quad {}_pF_{q-1}\left(\begin{matrix}1-b_h+a_1,\ldots,1-b_h+a_p;\\ 1-b_h+b_1,\ldots *\ldots,1-b_h+b_q;(-1)^{p-m-n}x/\eta\end{matrix}\right)$$
$$=\exp(-x/t)\sum_{r=0}^{\infty}(1/r!)(x/t)^r\times$$
$$\times{}_{p+1}F_{q-1}\left(\begin{matrix}-r,1-b_h+a_1,\ldots,1-b_h+a_p;\\ 1-b_h+b_1,\ldots *\ldots,1-b_h+b_q;\ (-1)^{p-m-n+1}t/\eta\end{matrix}\right)$$

(see [6], IX, formula (201), p. 244), which are valid only under the assumptions $t\neq 0$ and $p<q$. Then we obtain

$$(57)\quad J=(1/\eta)\int_0^{\infty}\sum_{h=1}^{m}B_h(b_h)(x/\eta)^{-b_h}\exp(-x/t)\sum_{r=0}^{\infty}(1/r!)(x/t)^r\times$$
$$\times{}_{p+1}F_{q-1}\left(\begin{matrix}-r,1-b_h+a_1,\ldots,1-b_h+a_p;\\ 1-b_h+b_1,\ldots *\ldots,1-b_h+b_q;\ (-1)^{p-m-n+1}t/\eta\end{matrix}\right)\times$$
$$\times G^{\mu,\nu}_{\sigma,\tau}\left(\omega x\,\middle|\,\begin{matrix}c_1,\ldots,c_\sigma\\ d_1,\ldots,d_\tau\end{matrix}\right)dx\,.$$

Let us write the integrand of (57) in the form

$$(x/\eta)^{-\delta_{1A}-1}\exp(-x/t)\,G^{\mu,\nu}_{\sigma,\tau}\left(\omega x\,\middle|\,\begin{matrix}c_1,\ldots,c_\sigma\\ d_1,\ldots,d_\tau\end{matrix}\right)\times$$
$$\times\sum_{h=1}^{m}\sum_{r=0}^{\infty}B_h(b_h)(1/r!)(x/\eta)^{\delta_{1A}-b_h+1}(x/t)^r\times$$
$$\times{}_{p+1}F_{q-1}\left(\begin{matrix}-r,1-b_h+a_1,\ldots,1-b_h+a_p;\\ 1-b_h+b_1,\ldots *\ldots,1-b_h+b_q;\ (-1)^{p-m-n+1}t/\eta\end{matrix}\right),$$

where

$$\delta_{1A}=\min_{j=1,\ldots,\mu}\operatorname{re}d_j-\varepsilon^*\qquad(\varepsilon^*>0\ \text{sufficiently small})$$

and let us introduce the notations

$$\Phi^{(1A)}(x)=(x/\eta)^{-\delta_{1A}-1}\exp(-x/t)\,G^{\mu,\nu}_{\sigma,\tau}\left(\omega x\,\middle|\,\begin{matrix}c_1,\ldots,c_\sigma\\ d_1,\ldots,d_\tau\end{matrix}\right),$$

$$f^{(1A)}_{r,h}(x)=(1/r!)B_h(b_h)(1/\eta)^{\delta_{1A}-b_h+1}(x/t)^r\times$$
$$\times{}_{p+1}F_{q-1}\left(\begin{matrix}-r,1-b_h+a_1,\ldots,1-b_h+a_p;\\ 1-b_h+b_1,\ldots *\ldots,1-b_h+b_q;\ (-1)^{p-m-n+1}t/\eta\end{matrix}\right)$$
$$(r=0,1,\ldots;\ h=1,\ldots,m),$$

$$f^{(1A)}_{r}(x)=\sum_{h=1}^{m}x^{\delta_{1A}-b_h+1}f^{(1A)}_{r,h}(x),$$

$$\hat{f}^{(1A)}_{r}(x)=\sum_{h=1}^{m}|x^{\delta_{1A}-b_h+1}|\,|f^{(1A)}_{r,h}(x)|\,.$$

We shall investigate, on the basis of Tests 1 and 2 given in § 1 (see [3]), under which conditions it is possible to perform integration in (57) term by term with the restriction that the integrals will be summed first with respect to h and only then with respect to r. We shall do it in three steps.

(i) Formulae (56) hold for any x from the interval $0 \leqslant x < \infty$; thus in view of a well-known theorem the power series

$$\sum_{r=0}^{\infty} f_{r,h}^{(1A)}(x) \qquad (h = 1, \ldots, m)$$

are absolutely and uniformly convergent in any fixed interval $0 \leqslant x \leqslant x_0$, where x_0 is arbitrary.

Let

$$f_{(h)}^{(1A)}(x) = \sum_{r=0}^{\infty} |f_{r,h}^{(1A)}(x)| \qquad (h = 1, \ldots, m), \qquad f^{(1A)}(x) = \sum_{r=0}^{\infty} \hat{f}_r^{(1A)}(x),$$

$$\varrho_1 = \min_{j=1,\ldots,\mu} \operatorname{re} d_j - \max_{j=1,\ldots,m} \operatorname{re} b_j + 1 - \varepsilon^*,$$

$$\varrho_2 = \max_{j=1,\ldots,\mu} \operatorname{re} d_j - \min_{j=1,\ldots,m} \operatorname{re} b_j + 1 - \varepsilon^*.$$

For any $\varepsilon > 0$ there exists such a number r_0 that ([5])

$$f_{(h)}^{(1A)}(x) - \sum_{s=0}^{r} |f_{s,h}^{(1A)}(x)| < (1/m)\,\varepsilon/\max\{1, x_0^{\varrho_2}\} \qquad (h = 1, \ldots, m)$$

for any x from the interval $0 \leqslant x \leqslant x_0$. Since for sufficiently small $\varepsilon^* > 0$ in view of (11) we have $\varrho_1 > 0$, then for $0 \leqslant x \leqslant 1$

$$f^{(1A)}(x) - \sum_{s=0}^{r} \hat{f}_s^{(1A)}(x) \leqslant \sum_{h=1}^{m} |x^{\delta_{1A} - b_h + 1}| \Big\{ f_{(h)}^{(1A)}(x) - \sum_{s=0}^{r} |f_{s,h}^{(1A)}(x)| \Big\}$$

$$< \sum_{h=1}^{m} x^{\varrho_1} (1/m)\,\varepsilon/\max\{1, x_0^{\varrho_2}\} \leqslant \varepsilon.$$

Similarly for $1 \leqslant x \leqslant x_0$

$$f^{(1A)}(x) - \sum_{s=0}^{r} \hat{f}_s^{(1A)}(x) < \sum_{h=1}^{m} x^{\varrho_2} (1/m)\,\varepsilon/\max\{1, x_0^{\varrho_2}\} \leqslant \varepsilon.$$

In consequence the series

$$\sum_{r=0}^{\infty} \hat{f}_r^{(1A)}(x),$$

([5]) $m > 0$ in view of (9) in cases (I) and (III), in view of (16) in cases (II) and (IV), and in view of (24) in case (V).

and thus also the series

$$\sum_{r=0}^{\infty} f_r^{(1A)}(x)$$

absolutely and uniformly converge in any interval $0 \leqslant x \leqslant x_0$, where x_0 is arbitrary.

(ii) The function $G_{\sigma,\tau}^{\mu,\nu}$ which appears in the formula defining $\Phi^{(1A)}$ is continuous in the interval $0 < x < \infty$. Point 0 is in general a branch point of the analytic function $G_{\sigma,\tau}^{\mu,\nu}$. We shall prove, however, that if β $(0 < \beta < 1)$ is sufficiently small, then the integral

$$\int_0^{\beta} |\Phi^{(1A)}(x)|\,dx \tag{58}$$

converges. According to a well-known theorem this integral converges if there exists a function $\Psi^{(1A)}$ defined in the interval $0 < x \leqslant \beta$ such that $|\Phi^{(1A)}(x)| \leqslant |\Psi^{(1A)}(x)|$ and if the integral

$$\int_0^{\beta} |\Psi^{(1A)}(x)|\,dx$$

is convergent. In each of the cases (I), (II), (III), (IV), (V) the formula

$$G_{\sigma,\tau}^{\mu,\nu}\left(\omega x \left| \begin{matrix} c_1, \ldots, c_\sigma \\ d_1, \ldots, d_\tau \end{matrix}\right.\right) = \sum_{h=1}^{\mu} D_h(d_h)(\omega x)^{d_h} \times$$
$$\times {}_\sigma F_{\tau-1}\left(\begin{matrix} 1+d_h-c_1, \ldots, 1+d_h-c_\sigma; \\ 1+d_h-d_1, \ldots * \ldots, 1+d_h-d_\tau; (-1)^{\sigma-\mu-\nu}\omega x \end{matrix}\right),$$

analogous to (55), may be applied for this purpose provided that in the case where some of the numbers (52) are natural, this formula must be understood in the sense of Remark 6. Hence

$$\Phi^{(1A)}(x) = (x/\eta)^{-\delta_{1A}-1}\exp(-x/t)\sum_{h=1}^{\mu} D_h(d_h)(\omega x)^{d_h} \times$$
$$\times {}_\sigma F_{\tau-1}\left(\begin{matrix} 1+d_h-c_1, \ldots, 1+d_h-c_\sigma; \\ 1+d_h-d_1, \ldots * \ldots, 1+d_h-d_\tau; (-1)^{\sigma-\mu-\nu}\omega x \end{matrix}\right).$$

If β $(0 < \beta < 1)$ is sufficiently small, then there exists such a constant $M > 0$ that

$$\left|{}_\sigma F_{\tau-1}\left(\begin{matrix} 1+d_h-c_1, \ldots, 1+d_h-c_\sigma; \\ 1+d_h-c_1, \ldots * \ldots, 1+d_h-d_\tau; (-1)^{\sigma-\mu-\nu}\omega x \end{matrix}\right)\right| \leqslant M$$
$$(0 \leqslant x \leqslant \beta,\ h = 1, \ldots, \mu),$$

whence

$$|\Phi^{(1A)}(x)| \leqslant M|(x/\eta)^{-\delta_{1A}-1}\exp(-x/t)|\sum_{h=1}^{\mu}|D_h(d_h)(\omega x)^{d_h}|$$

$$\leqslant M|\eta^{\delta_{1A}+1}|\max\{1, \exp(-\beta\operatorname{re}(1/t))\}\Big\{\sum_{h=1}^{\mu}|D_h(d_h)\,\omega^{d_h}|\Big\}x^{-1+\varepsilon^*}.$$

Since

$$\int_{\beta}^{\infty} x^{-1+\varepsilon^*}dx = \beta^{\varepsilon^*}/\varepsilon^*,$$

the integral (58) converges.

(iii) Notice first that in each of the cases (I), (II), (III), (IV), (V) for $r = 0, 1, \ldots$ and $h = 1, \ldots, m$ formulae (27) hold provided inequalities (28) are satisfied. In view of (i) and (ii) (compare Tests 1 and 2) the integration term by term in formula (50) may be performed (with the restriction that the integrals are summed first with respect to h and only then with respect to r) provided the integral

$$\int_{\beta}^{\infty} G_{1A}(x)\,dx$$

is convergent. Since in view of (i) and (ii) the function G_{1A} is continuous and non-negative in the interval $0 < x < \infty$, in the condition given above the number β may be replaced by any other positive number, i.e. it is sufficient to assume that condition (53) is fulfilled.

Assuming that all the conditions mentioned in the reasoning are fulfilled, we obtain from (57) and (27)

$$J = \sum_{r=0}^{\infty}\sum_{h=1}^{m}(1/r!)\,B_h(b_h)(t/\eta)^{1-b_h}\times$$
$$\times{}_{p+1}F_{q-1}^{*}\left(\begin{matrix}-r, 1-b_h+a_1, \ldots, 1-b_h+a_p;\\ 1-b_h+b_1, \ldots * \ldots, 1-b_h+b_q;\end{matrix}\ (-1)^{p-m-n+1}t/\eta\right)\times$$
$$\times G_{\sigma+1,\tau}^{\mu,\nu+1}\left(\omega t\,\middle|\,\begin{matrix}b_h-r, c_1, \ldots, c_\sigma\\ d_1, \ldots, d_\tau\end{matrix}\right)$$

i.e. formula (54). Thus the proof of Theorem 1A is ended.

Remark 7. The particular cases of Theorem 1A will be considered in § 5 (see [4]). Certain particular cases of this Theorem for $t = 1$ were also considered by Meijer in his papers [6]. Our condition (53) corresponds in [6] to the condition $\operatorname{re}\eta > \frac{1}{2}$ for $p = q-1$ and is superfluous for $p < q-1$. If in [6] we substituted η and ω by η/t and ωt respectively, we should receive the condition $\operatorname{re}(\eta/t) > \frac{1}{2}$ for $p = q-1$. Moreover, some conditions may be simplified by suitable substitutions. The author is at present unable to answer the question in what degree these observations may be transferred to the general case. We emphasize however, that the introduced parameter t has, at least for $p = q-1$, the convergence properties: the condition $\mathrm{re}(\eta/t) > \frac{1}{2}$ may be fulfilled provided we choose t properly, and in consequence the series on the right-hand side of formula (54), divergent for $t = 1$, will become convergent. For the above reasons the parameter t may be named the convergence parameter, and the factor $\exp(-x/t)$, the use of which is the essence of the method presented, the convergence factor. An analogous remark may be formulated for each of the Theorems 1B and 2A, 2B (see [4]).

THEOREM 1B. *Let $m, n, p, q, \mu, \nu, \sigma, \tau$ be integers, let t fulfil* (28) *and let one of the cases* (I), (II), (III), (IV), (V) *take place, where in* (II) *and* (IV) *we assume additionally* (30). *If for large x_0*

$$\int_{x_0}^{\infty} G_{1B}(x)\,dx \quad \textit{converges}, \tag{59}$$

then

$$G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}\left(\eta\omega\,\middle|\,\begin{matrix} b_1, \ldots, b_m, c_1, \ldots, c_\sigma, b_{m+1}, \ldots, b_q \\ a_1, \ldots, a_n, d_1, \ldots, d_\tau, a_{n+1}, \ldots, a_p\end{matrix}\right) \tag{60}$$
$$= (1/\eta\omega)\sum_{r=0}^{\infty}\sum_{h=1}^{\mu}(1/r!)D_h(d_h)(\omega/t)^{1+d_h}\times$$
$$\times {}_{\sigma+1}F_{\tau-1}\left(\begin{matrix}-r, 1+d_h-c_1, \ldots, 1+d_h-c_\sigma; \\ 1+d_h-d_1, \ldots * \ldots, 1+d_h-d_\tau; (-1)^{\sigma-\mu-\nu+1}\omega/t\end{matrix}\right)\times$$
$$\times G^{m,n+1}_{p+1,q}\left(1/\eta t\,\middle|\,\begin{matrix}-d_h-r, -a_1, \ldots, -a_p \\ -b_1, \ldots, -b_q\end{matrix}\right),$$

*where the asterisk * denotes that the number $1+d_h-d_h$ is to be omitted in the sequence $1+d_h-d_1, \ldots, 1+d_h-d_\tau$, and in the case if some of the numbers* (52) *are natural, formulae* (60) *and* (50) *must be understood in the sense of Remark* 6. *The connection between the branches of $G^{n+\mu,m+\nu}_{q+\sigma,p+\tau}$ and $G^{m,n+1}_{p+1,q}$ is determined by Remark* 2.

Proof. As in the previous proof, we first state that for any complex $t \neq 0$ in each of the cases (I), (II), (III), (IV), (V) we have

$$J = (1/\eta)\int_0^{\infty}\sum_{h=1}^{\mu}D_h(d_h)(\omega x)^{d_h}\exp(-tx)\sum_{r=0}^{\infty}(1/r!)(tx)^r\times \tag{61}$$
$$\times {}_{\sigma+1}F_{\tau-1}\left(\begin{matrix}-r, 1+d_h-c_1, \ldots, 1+d_h-c_\sigma; \\ 1+d_h-d_1, \ldots * \ldots, 1+d_h-d_\tau; (-1)^{\sigma-\mu-\nu+1}\omega/t\end{matrix}\right)\times$$
$$\times G^{m,n}_{p,q}\left(x/\eta\,\middle|\,\begin{matrix}-a_1, \ldots, -a_p \\ -b_1, \ldots, -b_q\end{matrix}\right)dx\,.$$

Let us write the integrand in the form

$$(\eta x)^{-\delta_{1B}-1}\exp(-tx)\,G^{m,n}_{p,q}\left(x/\eta\left|\begin{matrix}-a_1,\,\dots,\,-a_p\\-b_1,\,\dots,\,-b_q\end{matrix}\right.\right)\times$$
$$\times\sum_{h=1}^{\mu}\sum_{r=0}^{\infty}D_h(d_h)\,(1/r!)\,(\omega x)^{\delta_{1B}+d_h+1}(tx)^r\times$$
$$\times{}_{\sigma+1}F_{\tau-1}\left(\begin{matrix}-r,\,1+d_h-c_1,\,\dots,\,1+d_h-c_\sigma;\\1+d_h-d_1,\,\dots\,*\,\dots,\,1+d_h-d_\tau;\,(-1)^{\sigma-\mu-\nu+1}\omega/t\end{matrix}\right);$$

where

$$\delta_{1B}=-\max_{j=1,\dots,m}\operatorname{re}b_j-\varepsilon^*\qquad(\varepsilon^*>0\ \text{sufficiently small})$$

and let us introduce the notations

$$\Phi^{(1B)}(x)=(\eta x)^{-\delta_{1B}-1}\exp(-tx)\,G^{m,n}_{p,q}\left(x/\eta\left|\begin{matrix}-a_1,\,\dots,\,-a_p\\-b_1,\,\dots,\,-b_q\end{matrix}\right.\right),$$
$$f^{(1B)}_{r,h}(x)=(1/r!)\,D_h(d_h)\,\omega^{\delta_{1B}+d_h+1}(tx)^r\times$$
$$\times{}_{\sigma+1}F_{\tau-1}\left(\begin{matrix}-r,\,1+d_h-c_1,\,\dots,\,1+d_h-c_\sigma;\\1+d_h-d_1,\,\dots\,*\,\dots,\,1+d_h-d_\tau;\,(-1)^{\sigma-\mu-\nu+1}\omega/t\end{matrix}\right)$$
$$(r=0,1,\dots;\ h=1,\dots,\mu),$$
$$f^{(1B)}_r(x)=\sum_{h=1}^{\mu}x^{\delta_{1B}+d_h+1}f^{(1B)}_{r,h}(x)\,.$$

Applying as in the proof of the previous theorem Tests **1** and **2** from § 1, we state when it is possible to perform integration in (61) term by term with the restriction that the integrals will be summed first with respect to h and only then with respect to r. Moreover, we state when it is possible to evaluate these integrals from formula (29) with $r=0,1,\dots$ The required conditions are (28) and (59) in cases (I), (III), (V) and (28), (30), (59) in cases (II) and (IV), as may be easily verified.

Assuming that all the conditions mentioned in the reasoning are fulfilled, we obtain from (61) and (29)

$$J=\sum_{r=0}^{\infty}\sum_{h=1}^{\mu}(1/r!)\,D_h(d_h)\,(\omega/t)^{1+d_h}\times$$
$$\times{}_{\sigma+1}F_{\tau-1}\left(\begin{matrix}-r,\,1+d_h-c_1,\,\dots,\,1+d_h-c_\sigma;\\1+d_h-d_1,\,\dots\,*\,\dots,\,1+d_h-d_\tau;\,(-1)^{\sigma-\mu-\nu+1}\omega/t\end{matrix}\right)\times$$
$$\times G^{m,n+1}_{p+1,q}\left(1/\eta t\left|\begin{matrix}-d_h-r,\,-a_1,\,\dots,\,-a_p\\-b_1,\,\dots,\,-b_q\end{matrix}\right.\right),$$

i.e. formula (60), and thus the proof is ended.

Remark 8. Formula (60) may be rewritten in a slightly different way by taking advantage of the identity

$$(62)\qquad (1/\eta\omega)(\omega/t)^{1+d_h} G^{m,n+1}_{p+1,q}\left(1/\eta t\,\middle|\,\begin{matrix}-d_h-r,\,-a_1,\,\ldots,\,-a_p\\ -b_1,\,\ldots,\,-b_q\end{matrix}\right) = (\omega/t)^{d_h} G^{m,n+1}_{p+1,q}\left(1/\eta t\,\middle|\,\begin{matrix}1-d_h-r,\,1-a_1,\,\ldots,\,1-a_p\\ 1-b_1,\,\ldots,\,1-b_q\end{matrix}\right)$$

which results from (see [6], II, formula (41), p. 486)

$$(1/\eta t)\, G^{m,n+1}_{p+1,q}\left(1/\eta t\,\middle|\,\begin{matrix}-d_h-r,\,-a_1,\,\ldots,\,-a_p\\ -b_1,\,\ldots,\,-b_q\end{matrix}\right) = G^{m,n+1}_{p+1,q}\left(1/\eta t\,\middle|\,\begin{matrix}1-d_h-r,\,1-a_1,\,\ldots,\,1-a_p\\ 1-b_1,\,\ldots,\,1-b_q\end{matrix}\right).$$

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