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On the mean values of an entire function and its derivatives represented by Dirichlet series

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1. Consider the Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$$

where $\lambda_{n+1} > \lambda_n$, $\lambda_1 \geqslant 0$, $s = \sigma + it$, and

$$\lim_{n\to\infty} \frac{\log n}{\lambda_n} = 0 , \tag{1.1}$$

$$\lim_{n\to\infty} |a_n|^{1/\lambda_n} = 0 . \tag{1.2}$$

Under conditions (1.1) and (1.2) $f(s)$ represents an entire function and if $M(\sigma) = \max_{-\infty<t<\infty} |f(\sigma+it)|$ and $\mu(\sigma) = \max_{n\geqslant 1} |a_n| e^{\sigma\lambda_n}$, then for functions of finite order,

$$\log M(\sigma) \sim \log \mu(\sigma) . \tag{1.3}$$

Throughout our discussions we shall assume that $f(s)$ is not an exponential polynomial and it satisfies (1.1) and (1.2).

The mean values of $f(s)$ are defined as

$$V(\sigma) \equiv V(\sigma, f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma+it)|^2 \, dt , \tag{1.4}$$

$$v_\delta(\sigma) \equiv v_\delta(\sigma, f) = \frac{1}{e^{\delta\sigma}} \int_0^{\sigma} V(x) e^{\delta x} dx \tag{1.5}$$

$$= \lim_{T\to\infty} \frac{1}{2Te^{\delta\sigma}} \int_0^{\sigma} \int_{-T}^{T} |f(x+it)|^2 e^{\delta x} dx\, dt, \qquad 0 < \delta < \infty .$$

If $f^{(m)}(s)$ denotes the mth derivative of $f(s)$, its mean values are similarly defined as

$$(1.6) \qquad V(\sigma, f^{(m)}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f^{(m)}(\sigma+it)|^2\, dt,$$

$$(1.7) \qquad v_\delta(\sigma, f^{(m)}) = \frac{1}{e^{\delta\sigma}} \int_0^\sigma V(x, f^{(m)}) e^{\delta x} dx$$

$$= \lim_{T\to\infty} \frac{1}{2Te^{\delta\sigma}} \int_0^\sigma \int_{-T}^{T} |f^{(m)}(x+it)|^2 e^{\delta x} dx\, dt.$$

If $f(s)$ is of order ϱ and lower order λ, it is known ([1]) that

$$(1.8) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log v_\delta(\sigma)}{\sigma} = \frac{\varrho}{\lambda},$$

and that $e^{\delta\sigma}V(\sigma)$ is a convex function of $e^{\delta\sigma}v_\delta(\sigma)$ for any finite positive δ.

In this paper we investigate into a few properties of $v_\delta(\sigma)$ and $v_\delta(\sigma, f^{(m)})$ and give an alternative proof of (1.8).

2. Theorem 1. *$v_\delta(\sigma)$ increases steadily with σ and $\log v_\delta(\sigma)$ is a convex function of σ for $\sigma > \sigma_0$.*

Proof. For all $\sigma < \infty$, it can be shown as in [3], p. 303, that

$$(2.1) \qquad V(\sigma) = \sum_{n=1}^{\infty} |a_n|^2 e^{2\sigma\lambda_n}.$$

Therefore, by (1.5),

$$v_\delta(\sigma) = \frac{1}{e^{\delta\sigma}} \int_0^\sigma \left(\sum_{n=1}^{\infty} |a_n|^2 e^{2x\lambda_n}\right) e^{\delta x} dx.$$

The series under the integral sign is a uniformly convergent series of continuous functions for $\sigma < \infty$ and hence,

$$(2.2) \quad v_\delta(\sigma) = \frac{1}{e^{\delta\sigma}} \sum_{n=1}^{\infty} \int_0^\sigma |a_n|^2 e^{2x\lambda_n} e^{\delta x} dx = \sum_{n=1}^{\infty} \frac{|a_n|^2 e^{2\sigma\lambda_n}}{2\lambda_n+\delta} - \frac{1}{e^{\delta\sigma}} \sum_{n=1}^{\infty} \frac{|a_n|^2}{2\lambda_n+\delta}.$$

That $v_\delta(\sigma)$ increases steadily with σ follows from (2.2). To prove the convexity of $\log v_\delta(\sigma)$ we see that

$$\frac{d\log v_\delta(\sigma)}{d\sigma} = \frac{\frac{1}{e^{\delta\sigma}} V(\sigma) e^{\delta\sigma} - \delta \frac{1}{e^{\delta\sigma}} \int_0^\sigma V(x) e^{\delta x} dx}{v_\delta(\sigma)} = \frac{V(\sigma) - \delta v_\delta(\sigma)}{v_\delta(\sigma)} = \left[\frac{V(\sigma)}{v_\delta(\sigma)} - \delta\right].$$

Now since $e^{\delta\sigma}V(\sigma)$ is a convex function of $e^{\delta\sigma}v_\delta(\sigma)$, the right hand side is a positive indefinitely increasing function of σ for $\sigma > \sigma_0$ [1] and so

$$\frac{d^2 \log v_\delta(\sigma)}{d\sigma^2} > 0 \quad \text{for} \quad \sigma > \sigma_0.$$

Hence the result.

THEOREM 2. *If $v_\delta(\sigma, f^{(1)})$ is the mean value of $f^{(1)}(s)$, the first derivative of $f(s)$, then*

$$v_\delta(\sigma, f^{(1)}) \geqslant v_\delta(\sigma)\left(\frac{\log v_\delta(\sigma)}{2\sigma}\right)^2 \tag{2.3}$$

for $\sigma > \sigma_0$.

Proof. We have

$$v_\delta(\sigma, f^{(1)}) = \lim_{T\to\infty} \frac{1}{2Te^{\delta\sigma}} \int_0^\sigma \int_{-T}^T |f^{(1)}(x+it)|^2 e^{\delta x} dx\, dt$$

$$= \lim_{T\to\infty} \frac{1}{2Te^{\delta\sigma}} \int_0^\sigma \int_{-T}^T \lim_{\varepsilon\to 0} \left| \frac{f(x+it) - f(\overline{x-x\varepsilon}+it)}{\varepsilon x} \right|^2 e^{\delta x} dx\, dt$$

$$\geqslant \lim_{T\to\infty} \frac{1}{2Te^{\delta\sigma}} \int_0^\sigma \int_{-T}^T \lim_{\varepsilon\to 0} \left\{ \frac{|f(x+it)| - |f(\overline{x-x\varepsilon}+it)|}{\varepsilon x} \right\}^2 e^{\delta x} dx\, dt.$$

Now, by Minkowski's inequality ([3], p. 384)

$$\left\{ \int_{-T}^T \big(|f(x+it)| - |f(\overline{x-x\varepsilon}+it)|\big)^2 dt \right\}^{1/2} \geqslant \left\{ \left(\int_{-T}^T |f(x+it)|^2 dt \right)^{1/2} - \left(\int_{-T}^T |f(\overline{x-x\varepsilon}+it)|^2 dt \right)^{1/2} \right\}.$$

Hence,

$$v_\delta(\sigma, f^{(1)}) \geqslant \lim_{T\to\infty} \lim_{\varepsilon\to 0} \frac{1}{e^{\delta\sigma} 2T\varepsilon^2} \times$$
$$\times \left[\int_0^\sigma \left\{ \left(\int_{-T}^T |f(x+it)|^2 dt \right)^{1/2} - \left(\int_{-T}^T |f(\overline{x-x\varepsilon}+it)|^2 dt \right)^{1/2} \right\}^2 \frac{e^{\delta x}}{x^2} dx \right]$$
$$\geqslant \lim_{T\to\infty} \lim_{\varepsilon\to 0} \frac{1}{2T\varepsilon^2 \sigma^2 e^{\delta\sigma}} \times$$
$$\times \left[\int_0^\sigma \left\{ \left(\int_{-T}^T |f(x+it)|^2 e^{\delta x} dt \right)^{1/2} - \left(\int_{-T}^T |f(\overline{x-x\varepsilon}+it)|^2 e^{\delta x} dt \right)^{1/2} \right\}^2 dx \right].$$

[1] σ_0 need not be the same at each occurrence.

Again using Minkowski's inequality,

$$v_\delta(\sigma, f^{(1)}) \geqslant \lim_{T\to\infty}\lim_{\varepsilon\to 0} \frac{1}{2T\varepsilon^2\sigma^2 e^{\delta\sigma}} \times$$

$$\times\left\{\left(\int_0^\sigma\int_{-T}^{T}|f(x+it)|^2 e^{\delta x}dx\,dt\right)^{1/2} - \left(\int_0^\sigma\int_{-T}^{T}|f(\overline{x-x\varepsilon}+it)|^2 e^{\delta x}dx\,dt\right)^{1/2}\right\}^2$$

$$\geqslant \lim_{\varepsilon\to 0}\left[\frac{\{v_\delta(\sigma)\}^{1/2} - \{v_\delta(\sigma-\sigma\varepsilon)\}^{1/2}}{\varepsilon\sigma}\right]^2.$$

Now let,

$$g(\sigma) = \frac{\log v_\delta(\sigma)}{\sigma}$$

then since $\log v_\delta(\sigma)$ is a steadily increasing convex function of σ for $\sigma > \sigma_0$, it follows that $g(\sigma)$ is a positive increasing function of σ and therefore

$$v_\delta(\sigma, f^{(1)}) \geqslant \lim_{\varepsilon\to 0}\left\{\frac{e^{\sigma g(\sigma)/2} - e^{(\sigma-\sigma\varepsilon)g(\sigma-\sigma\varepsilon)/2}}{\varepsilon\sigma}\right\}^2 \geqslant e^{\sigma g(\sigma)}\left\{\frac{g(\sigma)}{2}\right\}^2 = v_\delta(\sigma)\left\{\frac{\log v_\delta(\sigma)}{2\sigma}\right\}^2.$$

THEOREM 3. *If* $f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}$ *be an entire function of Ritt order* ϱ $(0 < \varrho < \infty)$, *lower order* λ, *type* $\varkappa$ *and lower type* ν, *then*

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log v_\delta(\sigma)}{\sigma} = \frac{\varrho}{\lambda}, \tag{2.4}$$

$$\lim_{\sigma\to\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log v_\delta(\sigma)}{e^{\varrho\sigma}} = \frac{2\varkappa}{2\nu}. \tag{2.5}$$

Proof. We have

$$v_\delta(\sigma) = \frac{1}{e^{\delta\sigma}}\int_0^\sigma V(x)e^{\delta x}dx \leqslant \frac{1}{e^{\delta\sigma}}\{M(\sigma)\}^2\int_0^\sigma e^{\delta x}dx = \frac{1}{\delta}\{M(\sigma)\}^2\{1-e^{-\delta\sigma}\}. \tag{2.6}$$

Now if $\mu(\sigma)$ be the maximum term of rank $N(\sigma)$ for $\mathrm{Re}(s) = \sigma$, in the series for $f(s)$, it follows from (2.2) that

$$\begin{aligned} v_\delta(\sigma) &= \sum_{n=1}^{\infty}\frac{|a_n|^2 e^{2\lambda_n\sigma}}{2\lambda_n+\delta} - \frac{1}{e^{\delta\sigma}}\sum_{n=1}^{\infty}\frac{|a_n|^2}{2\lambda_n+\delta} \\ &\geqslant \frac{|a_{N(\sigma)}|^2 e^{2\sigma\lambda_{N(\sigma)}}}{2\lambda_{N(\sigma)}+\delta} - \frac{1}{e^{\delta\sigma}}\cdot\frac{|a_{N(\sigma)}|}{2\lambda_{N(\sigma)}+\delta} \\ &= \frac{\{\mu(\sigma)\}^2}{2\lambda_{N(\sigma)}+\delta} - \frac{|a_{N(\sigma)}|}{(2\lambda_{N(\sigma)}+\delta)e^{\delta\sigma}}. \end{aligned} \tag{2.7}$$

Hence from (2.6) and (2.7), we have

$$\frac{\{\mu(\sigma)\}^2}{2\lambda_{N(\sigma)}+\delta} - o(1) \leqslant v_\delta(\sigma) \leqslant \frac{\{M(\sigma)\}^2}{\delta}.$$

Now making use of (1.3), we get

$$\log v_\delta(\sigma) \sim 2\log M(\sigma). \tag{2.8}$$

The results (2.4) and (2.5) easily follow from (2.8) since

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}$$

and

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{e^{\varrho\sigma}} = \begin{matrix}\varkappa\\ \nu\end{matrix}.$$

THEOREM 4. *If $N(\sigma)$ denotes the rank of the maximum term $\mu(\sigma)$ of $f(s)$ and $0<\lambda<\varrho<\infty$, then*

$$\limsup_{\sigma\to\infty} \frac{\log v_\delta(\sigma)}{\sigma\lambda_{N(\sigma)}} \leqslant 2\left(1-\frac{\lambda}{\varrho}\right)$$

and

$$\limsup_{\sigma\to\infty} \frac{\log v_\delta(\sigma)}{\lambda_{N(\sigma)}\log\lambda_{N(\sigma)}} \leqslant 2\left(\frac{1}{\lambda}-\frac{1}{\varrho}\right).$$

Proof. The results follow in view of (2.6) and the following results of Srivastav ([2], p. 84)

$$\limsup_{\sigma\to\infty} \frac{\log\mu(\sigma)}{\sigma\lambda_{N(\sigma)}} = \limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{\sigma\lambda_{N(\sigma)}} \leqslant \left(1-\frac{\lambda}{\varrho}\right)$$

and

$$\limsup_{\sigma\to\infty} \frac{\log\mu(\sigma)}{\lambda_{N(\sigma)}\log\lambda_{N(\sigma)}} = \limsup_{\sigma\to\infty} \frac{\log M(\sigma)}{\lambda_{N(\sigma)}\log\lambda_{N(\sigma)}} \leqslant \left(\frac{1}{\lambda}-\frac{1}{\varrho}\right).$$

COROLLARY. *If $f(s)$ is of linearly regular growth, then*

$$\lim_{\sigma\to\infty} \frac{\log v_\delta(\sigma)}{\sigma\lambda_{N(\sigma)}} = 0 = \lim_{\sigma\to\infty} \frac{\log v_\delta(\sigma)}{\lambda_{N(\sigma)}\log\lambda_{N(\sigma)}}.$$

3. In the first theorem we have shown that $\log v_\delta(\sigma)$ is an increasing convex function of σ for $\sigma>\sigma_0$. This implies that $\log v_\delta(\sigma)$ is differentiable almost everywhere with an increasing derivative. This enables us to write $\log v_\delta(\sigma)$ in the following form:

$$\log v_\delta(\sigma) = O(1) + \int_{\sigma_0}^{\sigma} \frac{v'_\delta(x)}{v_\delta(x)}\,dx. \tag{3.1}$$

This integral representation of $\log v_\delta(\sigma)$ helps us in proving the following theorem:

THEOREM 5. *If $v'_\delta(\sigma)$ denotes the derivative of $v_\delta(\sigma)$ with respect to σ, then*

$$(3.2)\qquad \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\{v'_\delta(\sigma)/v_\delta(\sigma)\}}{\sigma} = \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log v_\delta(\sigma)}{\sigma} = \frac{\varrho}{\lambda}.$$

Proof. From (3.1), we have

$$\log v_\delta(\sigma) \leqslant O(1) + (\sigma - \sigma_0)\frac{v'_\delta(\sigma)}{v_\delta(\sigma)}$$

and therefore

$$(3.3)\qquad \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log v_\delta(\sigma)}{\sigma} \leqslant \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\{v'_\delta(\sigma)/v_\delta(\sigma)\}}{\sigma}.$$

Further, for an arbitrary $\eta > 0$, $\sigma > \sigma_0$,

$$\log v_\delta(\sigma+\eta) \geqslant \int_\sigma^{\sigma+\eta} \frac{v'_\delta(x)}{v_\delta(x)}dx \geqslant \eta\frac{v'_\delta(\sigma)}{v_\delta(\sigma)}$$

and as $\sigma+\eta \sim \sigma$, we get

$$(3.4)\qquad \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log v_\delta(\sigma+\eta)}{\sigma+\eta} \geqslant \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\{v'_\delta(\sigma)/v_\delta(\sigma)\}}{\sigma}.$$

(3.3) and (3.4) give (3.2).

4. In this section we give a few applications of the above theorems.

(i) *If $0 < \lambda$, $\varrho < \infty$, then*

$$(4.1)\qquad v_\delta(\sigma) < v_\delta(\sigma, f^{(1)}) < v_\delta(\sigma, f^{(2)}) < \dots$$

for sufficiently large σ.

From Theorem 2, we have

$$\lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\{v_\delta(\sigma, f^{(1)})/v_\delta(\sigma)\}^{1/2}}{\sigma} \geqslant \lim_{\sigma\to\infty} \frac{\sup}{\inf} \frac{\log\log v_\delta(\sigma)}{\sigma} = \frac{\varrho}{\lambda}.$$

Therefore, for any $\varepsilon > 0$ and sufficiently large σ,

$$v_\delta(\sigma, f^{(1)}) > e^{2\sigma(\lambda-\varepsilon)} v_\delta(\sigma).$$

As $\lambda > 0$ and ε can be made arbitrarily small, it follows that

$$v_\delta(\sigma, f^{(1)}) > v_\delta(\sigma).$$

Writing the above inequality for $f^{(1)}(s), f^{(2)}(s), \dots$ and combining them, we get (4.1).

(ii) *For* $0 < \lambda, \ \varrho < \infty$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\{v_\delta(\sigma, f^{(m)})/v_\delta(\sigma)\}^{1/2m}}{\sigma} \geqslant \frac{\varrho}{\lambda}.$$

Writing the result of Theorem 2 for $f^{(m-1)}(s)$, we get

$$\frac{v_\delta(\sigma, f^{(m)})}{v_\delta(\sigma, f^{(m-1)})} \geqslant \frac{1}{2^2}\left(\frac{\log v_\delta(\sigma, f^{(m-1)})}{\sigma}\right)^2.$$

Taking $m = 1, 2, \ldots, m$ and multiplying the m inequalities thus obtained, we get

$$\frac{v_\delta(\sigma, f^{(m)})}{v_\delta(\sigma)} > \frac{1}{2^{2m}}\left(\frac{\log v_\delta(\sigma)}{\sigma}\right)^{2m},$$

since

$$v_\delta(\sigma) < v_\delta(\sigma, f^{(1)}) < v_\delta(\sigma, f^{(2)}) < \ldots$$

for $\lambda > 0$ and $\sigma > \sigma_0$. Hence

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\{v_\delta(\sigma, f^{(m)})/v_\delta(\sigma)\}^{1/2m}}{\sigma} \geqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log v_\delta(\sigma)}{\sigma} = \frac{\varrho}{\lambda}.$$

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References

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