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On a non-linear system of parabolic integro-differential inequalities in an unbounded region

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The following system of integro-differential inequalities is considered in a region D of time-space (t, X)

$$(1)\quad u_t^i \leqslant f^i\Big(t, X, U, u_x^i, u_{xx}^i, \int_{\sigma_t} U(t, Z)\, d\mu_z(t, X)\Big) \qquad (i = 1, 2, \ldots, m),$$

where $X = (x_1, \ldots, x_n)$, $Z = (z_1, \ldots, z_n)$, $U = (u^1, \ldots, u^m)$, u_x^i is the gradient with respect to X of the function $u^i(t, X)$ and u_{xx}^i is the matrix of its second derivatives.

The results of the paper are closely related to paper [4] of one of the authors. The difference consists in the fact that in the above-mentioned paper less restrictive assumptions are made with regard to the measure $\mu_z(t, X)$, whereas stronger conditions are assumed concerning the parabolicity of the solution $U(t, X)$, the class to which it belongs, and the right-hand sides of (1). From the theorem of the present paper follows the maximum principle and the uniqueness of the solution of the mixed problem with boundary values of Dirichlet's type for a system of equations of type (1). In the linear case the above principle and the uniqueness were proved by M. Krzyżański [2], [3] under similar assumptions to those made in paper [4].

In order to simplify the formulation of our theorem we first introduce some definitions.

Definitions and notations. We denote by D an open set contained in the zone $t_0 < t < t_0 + T$ such that for any t_1, $t_0 \leqslant t_1 < t_0 + T$, the intersection σ_{t_1} of the closure of D with the plane $t = t_1$ is non-void and unbounded. Σ will stand for that part of the boundary of D which is contained in the open zone $t_0 < t < t_0 + T$. For any set E of the time-space (t, X) we denote by E^h its intersection with the zone $t_0 \leqslant t \leqslant t_0 + h$, and by E_r the intersection of E with the cylinder $|X| < r$, where $|X| = \big[\sum_{i=1}^{n} x_i^2\big]^{1/2}$.

A function $u(t, X)$ defined in D is said to be *of class* $E_2(M, K)$ (M, K positive constants) if

$$|u(t, X)| \leqslant M \exp(K|X|^2) . \tag{2}$$

A function $u(t, X)$ is called *regular* in D if it is continuous in the closure of D and u_t, u_x, u_{xx} are continuous in D.

We recall that for two real symmetric matrices $R = (r_{ij})$, $\widetilde{R} = (\widetilde{r}_{ij})$ we write

$$R \leqslant \widetilde{R} \tag{3}$$

if the quadratic form in $(\lambda_1, \ldots, \lambda_n)$ $\sum\limits_{i,j=1}^{n} (r_{ij} - \widetilde{r}_{ij}) \lambda_i \lambda_j$ is non-positive.

A sequence of functions $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ being given, we say that the function $f^i(t, X, U, Q, R, S)$, where $X = (x_1, \ldots, x_n)$, $U = (u^1, \ldots, u^m)$, $Q = (q_1, \ldots, q_n)$, $R = (r_{ij})$ $(i, j = 1, \ldots, n)$, $S = (s_1, \ldots, s_m)$, is *elliptic* with respect to $U(t, X)$ at the point (t, X) if for any two real symmetric matrices $R, \widetilde{R}$ satisfying (3) the inequality

$$f^i\Big(t, X, U(t, X), u_x^i(t, X), R, \int\limits_{\sigma_t} U(t, Z)\, d\mu_z(t, X)\Big) \leqslant f^i\Big(t, X, U(t, X), u_x^i(t, X), \widetilde{R}, \int\limits_{\sigma_t} U(t, Z)\, d\mu_z(t, X)\Big)$$

holds true.

We write $A \leqslant B$ for two points $A = (a_1, \ldots, a_m)$, $B = (b_1, \ldots, b_m)$ such that $a_k \leqslant b_k$ $(k = 1, \ldots, m)$.

Using the above definitions and notations we formulate our theorem.

THEOREM. *Let the functions* $f^i(t, X, U, Q, R, S)$ $(i = 1, \ldots, m)$, *be defined for* $(t, X) \in D$ *and for arbitrary* U, Q, R, S. *Suppose that the function* f^i *is increasing with respect to the variables* $u^1, \ldots, u^{i-1}, u^{i+1}, \ldots, u^m$, $s_1, \ldots, s_m$ *and satisfies the inequality* (1)

$$[f^i(t, X, U, Q, R, S) - f^i(t, X, \widetilde{U}, \widetilde{Q}, \widetilde{R}, \widetilde{S})]\operatorname{sgn}(u^i - \widetilde{u}^i) \tag{4}$$
$$\leqslant L_0 \sum_{j,k} |r_{jk} - \widetilde{r}_{jk}| + (L_1|X| + L_2) \sum_j |q_j - \widetilde{q}_j| + (L_3|X|^2 + L_4) \sum_r |u^r - \widetilde{u}^r| + L_5 \sum_r |s_r - \widetilde{s}_r| ,$$

where L_s $(s = 0, 1, 2, 3, 4, 5)$ *are positive constants. Let* $\mu_z(t, X)$ *be a non-negative measure in the space* $(z_1, \ldots, z_n)$, *depending on* $(t, X) \in D$ *and such that the integral* $\int\limits_{R^n} \exp K|Z|^2 d\mu_z(t, X)$ *is finite and*

(1) $\operatorname{sgn} x$ denotes 1 for $x \geqslant 0$ and -1 for $x < 0$.

$$(5)\qquad \begin{aligned} &\lim_{r\to\infty} \sup_{(t,X)\in D_r} \int_{|Z|>r} \exp K|Z|^2 d\mu_z(t, X) = 0\,, \\ &\int_{R^n} \exp[2(K+1)|Z|^2]\,d\mu_z(t, X) \leqslant K_1 \quad \text{for} \quad (t, X)\in D\,. \end{aligned}$$

Assume $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ *and* $V(t, X) = \big(v^1(t, X), \ldots, v^m(t, X)\big)$ *to be regular functions of class* $E_2(M, K)$ *in* D. *Write* $\mathcal{E}^i = \{(t, X) \in D\colon u^i(t, X) > v^i(t, X)\}$. *For every index* j *we assume that at* $(t^*, X^*) \in \mathcal{E}^j$ *the function* f^j *is elliptic with regard to* $U(t, X)$ *and the following differential inequalities are satisfied*:

$$(6)\qquad \begin{cases} u_t^j(t^*, X^*) \\ \quad \leqslant f^j\Big(t^*, X^*,\ U(t^*, X^*)\ u_x^j(t^*, X^*),\ u_{xx}^j(t^*, X^*),\ \displaystyle\int_{\sigma_{t^*}} U(t^*, Z)\,d\mu_z(t^*, X^*)\Big)\,, \\ v_t^j(t^*, X^*) \\ \quad \geqslant f^j\Big(t^*, X^*, V(t^*, X^*), v_x^j(t^*, X^*), v_{xx}^j(t^*, X^*),\ \displaystyle\int_{\sigma_{t^*}} V(t^*, Z)\,d\mu_z(t^*, X^*)\Big)\,. \end{cases}$$

Suppose that the initial and boundary inequality

$$(7)\qquad U(t, X) \leqslant V(t, X) \quad \text{for} \quad (t, X) \in \sigma_{t_0} \cup \Sigma$$

holds true.

Under these assumptions we have

$$(8)\qquad U(t, X) \leqslant V(t, X) \quad \text{in } D\,.$$

The proof is modelled on that used in P. Besala's paper [1].

Proof. We introduce the growth damping factor

$$H(t, X) = \exp\left[\frac{(K+1)|X|^2}{1-\varkappa(t-t_0)} + \lambda t\right],$$

where

$$\lambda = 4[2n(K+1)(L_0+L_2) + m(L_4+K_1L_5) + 4]\,,$$
$$\varkappa = 4n^2(K+1)L_0 + 2n(L_1+L_2) + mL_3/(K+1)\,,$$

and the new functions

$$\tilde{u}^i(t, X) = u^i(t, X)[H(t, X)]^{-1}\,, \qquad \tilde{v}^i(t, X) = v^i(t, X)[H(t, X)]^{-1}\,.$$

We first prove (8) in $\bar{D}^h$, where $h < \min(T, 1/2\varkappa)$. Suppose that the contrary is true; then, we would have for some positive r_0

$$(9)\qquad p = \max_l \{ \max_{(t,X)\in \bar{D}^h_{r_0}} [\tilde{u}^l(t, X) - \tilde{v}^l(t, X)]\} > 0\,.$$

Write

$$C_r^h = \{(t, X) \in \bar{D}^h \colon |X| = r\}$$

and choose r_1 so great that $r_1 > r_0$,

(10) $$[H(t, X)]^{-1} 2M \exp(K|X|^2) < p \quad \text{for} \quad (t, X) \in C_{r_1}^h$$

and

(11) $$2M \int\limits_{|Z|>r_1} \exp(K|Z|^2)\, d\mu_z(t, X) < \frac{p}{mL_5} \quad \text{for} \quad (t, X) \in \bar{D}_{r_1}^h .$$

We obviously have

(12) $$\bar{D}_{r_1}^h = (\sigma_{t_0})_{r_1} \cup \Sigma_{r_1}^h \cup C_{r_1}^h \cup D_{r_1}^h .$$

By continuity there are a point $(t^*, X^*) \in \bar{D}_{r_1}^h$ and an index j such that

(13) $$\tilde{u}^j(t^*, X^*) - \tilde{v}^j(t^*, X^*) = \max_l \{ \max_{(t,X)\in \bar{D}_{r_1}^h} [\tilde{u}^l(t, X) - \tilde{v}^l(t, X)]\} \geqslant p > 0 .$$

In view of (7) and (13) the point (t^*, X^*) does not belong to $(\sigma_{t_0})_{r_1} \cup \Sigma_{r_1}^h$. It does not belong to $C_{r_1}^h$ either because of (2) satisfied by u^j and v^j, of (10) and (13). Hence, by (12), we must have $(t^*, X^*) \in D_{r_1}^h$. Since

$$\tilde{u}^j(t^*, X^*) - \tilde{v}^j(t^*, X^*) = \max_{(t,X)\in \bar{D}_{r_1}^h} [\tilde{u}^j(t, X) - \tilde{v}^j(t, X)] ,$$

it follows by a classical argument that

(14) $$\tilde{u}_t^j(t^*, X^*) - \tilde{v}_t^j(t^*, X^*) \geqslant 0 ,$$

(15) $$\tilde{u}_x^j(t^*, X^*) = \tilde{v}_x^j(t^*, X^*) ,$$

(16) $$\tilde{u}_{xx}^j(t^*, X^*) \leqslant \tilde{v}_{xx}^j(t^*, X^*) .$$

Now from (6) and (13) we obtain at the point (t^*, X^*)

(17) $$\begin{aligned}(\tilde{u}_t^j - \tilde{v}_t^j)H + (\tilde{u}^j - \tilde{v}^j)H_t &\leqslant \Big[f^j\Big(t^*, X^*, \tilde{U}H, Q^{\tilde{u}}, R^{\tilde{u}}, \int\limits_{\sigma_{t^*}} \tilde{U}(t^*, Z) H\, d\mu_z(t^*, X^*)\Big) - \\ &\quad - f^j\Big(t^*, X^*, \tilde{U}H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}}, \int\limits_{\sigma_{t^*}} \tilde{U}(t^*, Z) H\, d\mu_z(t^*, X^*)\Big)\Big] + \\ &\quad + \Big[f^j(t^*, X^*, \tilde{U}H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}}, \int\limits_{\sigma_{t^*}} \tilde{U}(t^*, Z) H\, d\mu_z(t^*, X^*)\Big) - \\ &\quad - f^j\Big(t^*, X^*, \tilde{V}H, Q^{\tilde{v}}, R^{\tilde{v}}, \int\limits_{\sigma_{t^*}} \tilde{V}(t^*, Z) H\, d\mu_z(t^*, X^*)\Big)\Big] ,\end{aligned}$$

where

$$Q^{\tilde{u}} = \tilde{u}^j_x H + \tilde{u}^j H_x\,, \qquad Q^{\tilde{v}} = \tilde{v}^j_x H + \tilde{v}^j H_x\,,$$

$$R^{\tilde{u}} = (\tilde{u}^j_{x_l x_k} H + \tilde{u}^j_{x_l} H_{x_k} + \tilde{u}^j_{x_k} H_{x_l} + \tilde{u}^j H_{x_l x_k}) \qquad (l, k = 1, 2, \ldots, n)\,,$$

$$R^{\tilde{v}} = (\tilde{v}^j_{x_l x_k} H + \tilde{v}^j_{x_l} H_{x_k} + \tilde{v}^j_{x_k} H_{x_l} + \tilde{v}^j H_{x_l x_k}) \qquad (l, k = 1, 2, \ldots, n)\,,$$

$$R^{\tilde{u},\tilde{v}} = (\tilde{v}^j_{x_l x_k} H + \tilde{u}^j_{x_l} H_{x_k} + \tilde{u}^j_{x_k} H_{x_l} + \tilde{u}^j H_{x_l x_k}) \qquad (l, k = 1, 2, \ldots, n)\,,$$

all values being taken at the point (t^*, X^*). By (16) and the ellipticity of f^j with regard to $U = \tilde{U}H$, the first difference in brackets on the right-hand side of (17) is non-positive. As to the second difference we rewrite it in the form

$$\begin{aligned}(18)\qquad &\Big[f^j\Big(t^*, X^*, \tilde{U}H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}}, \int\limits_{\sigma_{t^*}} \tilde{U}H\, d\mu_z(t^*, X^*)\Big) - \\ &\quad - f^j\Big(t^*, X^*, WH, Q^{\tilde{v}}, R^{\tilde{v}}, \int\limits_{\sigma_{t^*}} \tilde{V}H\, d\mu_z(t^*, X^*)\Big)\Big] + \\ &\quad + \Big[f^j\Big(t^*, X^*, WH, Q^{\tilde{v}}, R^{\tilde{v}}, \int\limits_{\sigma_{t^*}} \tilde{V}H\, d\mu_z(t^*, X^*)\Big) - \\ &\quad - f^j\Big(t^*, X^*, \tilde{V}H, Q^{\tilde{v}}, R^{\tilde{v}}, \int\limits_{\sigma_{t^*}} \tilde{V}H\, d\mu_z(t^*, X^*)\Big)\Big].\end{aligned}$$

where

$$W(t, X) = \big(w^1(t, X), \ldots, w^m(t, X)\big)\,, \qquad w^l(t, X) = \min[\tilde{u}^l(t, X), \tilde{v}^l(t, X)]$$

$(l = 1, 2, \ldots, m)$. By the monotonicity of f^j with respect to the variables $u^1, \ldots, u^{j-1}, u^{j+1}, \ldots, u^m$ the second difference in (18) is non-positive. Hence, using the obvious inequalities (see (13))

$$\tilde{u}^l(t^*, Z) \leqslant \tilde{v}^l(t^*, Z) + [\tilde{u}^j(t^*, X^*) - \tilde{v}^j(t^*, X^*)] \qquad (l = 1, 2, \ldots, m)$$

for $(t^*, Z) \in (\sigma_{t^*})_{r_1}$ and the monotonicity of f^j with respect to $s_1, \ldots, s_m$, we finally obtain at (t^*, X^*)

$$\begin{aligned}(\tilde{u}^j_t - \tilde{v}^j_t)H + (\tilde{u}^j - \tilde{v}^j)H_t \leqslant f^j\Big(&t^*, X^*, \tilde{U}H, Q^{\tilde{u}}, R^{\tilde{u},\tilde{v}}, \int\limits_{(\sigma_{t^*})_{r_1}} \tilde{V}H\, d\mu_z(t^*, X^*) + \\ &+ (\tilde{u}^j - \tilde{v}^j) \int\limits_{(\sigma_{t^*})_{r_1}} \tilde{H}\, d\mu_z(t^*, X^*) + \int\limits_{\sigma_{t^*} \cap |Z| > r_1} U\, d\mu_z(t^*, X^*)\Big) - \\ - f^j\Big(t^*, X^*, WH, Q^{\tilde{v}}, R^{\tilde{v}}, &\int\limits_{(\sigma_{t^*})_{r_1}} \tilde{V}H\, d\mu_z(t^*, X^*) + \int\limits_{\sigma_{t^*} \cap |Z| > r_1} V\, d\mu_z(t^*, X^*)\Big)\end{aligned}$$

where $\tilde{H} = (H, \ldots, H)$, Since

$$|\tilde{u}^l(t^*, X^*) - w^l(t^*, X^*)| \leqslant \tilde{u}^j(t^*, X^*) - \tilde{v}^j(t^*, X^*) \qquad (l = 1, \ldots, m)\,,$$

the last inequality implies, by (4) and (15)

$$(19)\quad [\tilde{u}^j_l(t^*, X^*)-\tilde{v}^j_l(t^*, X^*)]H \leqslant [\tilde{u}^j(t^*, X^*)-\tilde{v}^j(t^*, X^*)]\Big[F(H)+mL_5\int\limits_{(\sigma_{t^*})_{r_1}} H\,d\mu_z(t^*, X^*)\Big]+ + L_5\sum_l \int\limits_{\sigma_{t^*}\cap|Z|>r_1} (|u^l|+|v^l|)\,d\mu_z(t^*, X^*)$$

where

$$F(H) = L_0\sum_{l,k}|H_{x_lx_k}|+(L_1|X|+L_2)\sum_k|H_{x_k}|+m(L_3|X|^2+L_4)H-H_t\,,$$

taken at the point (t^*, X^*). Simple computation shows that (see [1] and also [5], § 65)

$$(20)\quad F(H) \leqslant \frac{H}{[1-\varkappa(t^*-t_0)]^2}\Big\{(K+1)|X^*|^2\Big[4(K+1)L_0n^2+2(L_1+L_2)n+\frac{mL_3}{K+1}-\varkappa\Big]+ + [2n(K+1)(L_0+L_2)+mL_4]-\lambda[1-\varkappa(t^*-t_0)]^2\Big\}.$$

Since $1/2\leqslant 1-\varkappa(t^*-t_0)\leqslant 1$, we obtain by the definition of H and by (5)

$$(21)\quad \int\limits_{(\sigma_{t^*})_{r_1}} H\,d\mu_z(t^*, X^*) \leqslant \exp(\lambda t^*)\int\limits_{R^n}\exp[2(K+1)|Z|^2]\,d\mu_z(t^*, X^*) \leqslant \frac{HK_1}{[1-\varkappa(t^*-t_0)]^2}\,.$$

Finally, by the inequality (2), satisfied by u^l and v^l and by (11) we have

$$(22)\quad \sum_{l=1}\int\limits_{\sigma_{t^*}\cap|Z|>r_1}(|u^l|+|v^l|)\,d\mu_z(t^*, X^*) \leqslant m\int\limits_{|Z|>r_1} 2M\exp(K|Z|^2)\,d\mu_z(t^*, X^*) < \frac{p}{L_5}\,.$$

Inequalities (19), (20), (21) and (22) together with the definition of λ and $\varkappa$ give

$$[\tilde{u}^j_l(t^*, X^*)-v^j_l(t^*, X^*)]H \leqslant -4H[\tilde{u}^j(t^*, X^*)-\tilde{v}^j(t^*, X^*)]+p\,.$$

But, since $H\geqslant 1$, we have by (13)

$$-4H[\tilde{u}^j(t^*, X^*)-\tilde{v}^j(t^*, X^*)]+p\leqslant -3p<0$$

and consequently $\tilde{u}^j_l(t^*, X^*)-\tilde{v}^j_l(t^*, X^*)<0$, which contradicts (14). This completes the proof of (8) in $\bar{D}^h$. We can now repeat our argument starting from the plane $t = t_0+h$, instead of the plane $t = t_0$, and thus after a finite number of steps we prove (8) in D.

From our theorem we obtain the following

UNIQUENESS CRITERION. *Under the assumptions of the theorem concerning the right-hand sides of* (1) *and the measure* $\mu_z(t, X)$, *the mixed problem with boundary values of Dirichlet's type for a system of equations*

$$(23)\qquad u_t^i = f^i\Big(t, X, U, u_x^i, u_{xx}^i, \int_{\sigma_t} U(t, Z)\, d\mu_z(t, X)\Big) \qquad (i = 1, \ldots, m),$$

consisting in finding in D *a regular and parabolic (i.e. such that all* f^i *are elliptic with rspect to* U (t, X)*) solution of class* E_2 *with prescribed values on* $\sigma_{t_0} \cup \Sigma$, *admits at most one solution.*

The next corollary to the theorem is the

MAXIMUM PRINCIPLE. *Let the right-hand sides of* (1) *and the measure* $\mu_z(t, X)$ *satisfy the assumptions of theorem. Let* $N = (n_1, \ldots, n_m) \geqslant 0$ *and suppose that*

$$f^i\Big(t, X, N, 0, 0, N \int_{\sigma_t} d\mu_z(t, X)\Big) \leqslant 0 \qquad (i = 1, \ldots, m).$$

Under these assumptions, if $U(t, X)$ *is a regular parabolic solution of system* (23), *of class* E_2 *in* D, *and satisfies the inequality*

$$U(t, X) \leqslant N \quad \text{for} \quad (t, X) \in \sigma_{t_0} \cup \Sigma,$$

then

$$U(t, X) \leqslant N \quad \text{in } D.$$

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