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Integral transformations on product spaces

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Abstract. Let $\mathfrak{M}(Z)$ denote the (F)-space of (complex valued) measurable functions on a σ-finite measure space (Z, dz) and let (X, dx), (Y, dy) be two such spaces. For a kernel $K \in \mathfrak{M}(X \times Y)$ consider the transformation of the form $w \in \mathfrak{D}_K \subset \mathfrak{M}(XY) \mapsto Kw(y) = \int_X K(x, y) w(x, y) dx \in \mathfrak{M}(Y)$, where $\mathfrak{D}_K$ is the natural domain of K consisting of all w s.t. the integral $Kw(y)$ exists and is finite a.e. The objective of the research is to characterise various properties of the transformation K in terms of the properties of its kernel $K(x, y)$; corresponding study of the transformations of the form $u \in \mathfrak{D}_K \subset \mathfrak{M}(X) \mapsto Ku(y) = \int_X K(x; y) u(x) dx \in \mathfrak{M}(Y)$ was carried out in [1]. Here necessary and sufficient conditions are given for K to be continuous, closed or closable. Also the question of compatibility is investigated.

1. Introduction. In [1] a theory was developed of integral transformations of the form

$$(1.1)\qquad (Ku)(y) = \int_X K(x, y) u(x) d\mu(x),$$

where (X, μ), (Y, ν) are σ-finite measure spaces and $K(x, y)$ is a measurable complex valued function on $X \times Y$ – the kernel of K. The objective of this theory is twofold: 1) to relate various properties of K as a linear transformation between the spaces $\mathfrak{M}(X)$ and $\mathfrak{M}(Y)$ of measurable complex valued finite a.e. functions on X and Y respectively to some specific properties of its kernel $K(x, y)$ and 2) to study continuous extensions of K beyond its proper domain

$$(1.2)\qquad \mathfrak{D}_K = \left\{u \in \mathfrak{M}(X);\ \int_X |K(x, y) u(x)| d\mu(x) < \infty \text{ a.e. } y\right\}.$$

The theory, general as it is, leads to some interesting results for special integral transformations, in particular for Fourier transform; see [2], [3].

In this paper we propose to outline a similar theory for transformations of the form

$$(1.3)\qquad (K^{\#} w)(y) = \int K(x, y) w(x, y) d\mu(x)$$

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acting between the spaces $\mathfrak{M}(X\times Y)$ and $\mathfrak{M}(Y)$. It was originally hoped that a development of the second aspect of the theory would allow to include in a framework similar to that of [1] some basic results about singular integral transformations. This aim we were not able to accomplish, the difficulties encountered are illustrated by an example in Section 4. The first aspect of the theory, however, becomes considerably simpler in the case of the transformation (1.3) than in that of (1.1) and this fact alone seems to be of some interest and makes the study of transformations of the form (1.3) worthwhile. Even though some of the results that follow could be obtained directly without serious difficulties we preferred for the sake of uniform exposition to reduce them to the corresponding results in [1].

2. Some definitions and basic facts about transformations $K^{\#}$. Throughout this paper (X,μ), (Y,ν) will denote two fixed σ-finite measure spaces and for an arbitrary σ-finite measure space $(Z,\varkappa)$ we shall denote by $\mathfrak{M}(Z,\varkappa)=\mathfrak{M}(Z)$ the space of all complex valued, measurable finite almost everywhere functions on Z provided with the linear metric topology of convergence in measure on all subsets of finite measure. This topology may be given by the translation invariant metric

$$\varrho_Z(u,0)=\varrho_Z(|u|,0)=\varrho_Z(u)=\int\limits_Z \frac{|u(z)|}{1+|u(z)|}\varphi(z)\,d\varkappa(z), \tag{2.1}$$

where $\varphi\in\mathfrak{M}(Z)$, $\varphi>0$ a.e. and $\int\limits_Z\varphi(z)\,d\varkappa(z)<\infty$. In particular the function φ may be chosen so that the last integral is equal to 1.

We assume that some metrics of the form (2.1) have been chosen on $\mathfrak{M}(X)$, $\mathfrak{M}(Y)$, $\mathfrak{M}(X\times Y)$, which we shall denote by ϱ_X, ϱ_Y, $\varrho_{X\times Y}$.

For any $f\in\mathfrak{M}(Z)$ we shall denote by $|f|$ the function $z\to|f(z)|$.

If A, B are topological spaces, then we say that A is a topological subspace of B, $A\subset B$ if $A\subset B$ and the embedding $A\ni a\to a\in B$ is continuous.

By an (F)-space we shall mean a complete metric vector space, in particular $\mathfrak{M}(Z)$ is an (F)-space.

A topological subspace A of $\mathfrak{M}(Z)$ is an (FL)-subspace of $\mathfrak{M}(Z)$ if for every $u\in A$ and $v\in\mathfrak{M}(Z)$ the inequality $|v(z)|\leqslant|u(z)|$ a.e. implies $v\in A$. We shall denote by (F) and (FL) the families of (F)-subspaces and (FL)-subspaces of a given space $\mathfrak{M}(Z)$.

For a kernel $K(x,y)\in\mathfrak{M}(X\times Y)$ we consider the transformation

$$K^{\#}\colon \mathfrak{D}_{K^{\#}}\subset\mathfrak{M}(X\times Y)\to\mathfrak{M}(Y), \tag{2.2}$$

where the *proper domain* $\mathfrak{D}_{K^{\#}}$ of $K^{\#}$ is defined by

$$\mathfrak{D}_{K^{\#}}=\Big\{w\in\mathfrak{M}(X\times Y);\ \int|K(x,y)w(x,y)|\,d\mu(x)<\infty\ \text{a.e.}\Big\} \tag{2.3}$$

and for every $w \in \mathfrak{D}_{K^\#}$, $K^\# w$ is given by (1.3). In agreement with the notation introduced above $|K|^\#$ will denote the transformation of the form (1.3) with the kernel $|K(x, y)|$, and $\mathfrak{D}_{|K|^\#}$ the proper domain of this transformation.

It is immediate that $\mathfrak{D}_{K^\#}$ is vector subspace of $\mathfrak{M}(X \times Y)$, $K^\#$ is linear,

$$\mathfrak{D}_{K^\#} = \mathfrak{D}_{|K|^\#}, \tag{2.4}$$

and

$$w \in \mathfrak{D}_{K^\#}, \quad w' \in \mathfrak{M}(X \times Y), \quad |w'| \leqslant |w| \quad \text{a.e. imply } w' \in \mathfrak{D}_{K^\#}. \tag{2.5}$$

Moreover, we have

$$\mathfrak{D}_{K^\#} \text{ is dense in } \mathfrak{M}(X \times Y). \tag{2.6}$$

To prove (2.6) we take any function $f \in \mathfrak{M}(X)$, $f > 0$ a.e. such that $\int_X f\,d\mu(x) < \infty$ and then define $g \in \mathfrak{M}(X \times Y)$ by letting $g(x, y) = \dfrac{f(x)}{|K(x, y)|}$ if $K(x, y) \neq 0$ and $g(x, y) = f(x)$ otherwise. It follows that $g \in \mathfrak{D}_{K^\#}$ and for every $w \in \mathfrak{M}(X \times Y)$ the functions defined by $w_n(x, y) = w(x, y)$ if $|w(x, y)| \leqslant ng(x, y)$ and $w_n(x, y) = n\dfrac{|w(x, y)|}{|w(x, y)|}g(x, y)$ otherwise, belong to $\mathfrak{D}_{K^\#}$ and $w_n(x, y) \to w(x, y)$ a.e. and a fortiori in $\mathfrak{M}(X \times Y)$.

Remark. Using terminology of [1] one could restate property (2.5) by saying that every transformation $K^\#$ is non-singular.

Repeating the proof of Theorem 4.1 in [1] we obtain the following proposition.

PROPOSITION 2.1. *Let $K(x, y)$ be a kernel and define for every $w \in \mathfrak{D}_{K^\#}$*

$$\varrho_{K^\#}(w) = \varrho_{X \times Y}(w) + \varrho_Y(|K|^\# |w|). \tag{2.6}$$

Then

1) *$\varrho_{K^\#}$ is a complete metric on $\mathfrak{D}_{K^\#}$ and $\mathfrak{D}_{K^\#}$ with the metric $\varrho_{K^\#}$ is an (FL) subspace of $\mathfrak{M}(X \times Y)$.*

2) *The transformation $K^\#\colon \mathfrak{D}_{K^\#} \to \mathfrak{M}(Y)$ is continuous.*

We shall study now the following two problems:

1) Find necessary and sufficient conditions on the kernel $K(x, y)$ in order that the transformation $K^\#\colon \mathfrak{D}_{K^\#} \subset \mathfrak{M}(X \times Y) \to \mathfrak{M}(Y)$ is $\mathfrak{M}(X \times Y) \to (\mathfrak{M}Y)$ closable.

2) Find necessary and sufficient conditions on $K(x, y)$ in order that $K^\#$ be $\mathfrak{M}(X \times Y) \to \mathfrak{M}(Y)$ continuous.

The following three conditions on a kernel $K(x, y)$ were introduced in [1]. Let X' denote the non-atomic part of X and $\{a_n\}$ be all the atoms of X (which by σ-finiteness of X must form an at most countable set).

(A) $K(x, y) = 0$ a.e. on $X' \times Y$.

(B) Let $A_m = \{y \in Y; K(a_m, y) \neq 0\}$. Then $\nu(\limsup_{m\to\infty} A_m) = \nu\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m\right) = 0$.

(C) Let B_m denote the closed linear extension in $\mathfrak{M}(Y)$ of the set $\{K(a_n, y)\}_{n\geqslant m}$. Then $\bigcap_{m=1}^{\infty} B_m = \{0\}$.

Clearly conditions (B), (C) are meaningful only if $\{a_n\}$ is countable. If $\{a_n\}$ is finite, then A_m and B_m are empty for m sufficiently large.

The following results were obtained in [1], Theorem 5.1 and Theorem 5.2. Consider the transformation $K\colon \mathfrak{D}_K \subset \mathfrak{M}(X) \to \mathfrak{M}(Y)$ defined by (1.1) and (1.2) and assume that K is non-singular, i.e. there is $v \in \mathfrak{D}_K$, $v > 0$ a.e. Then K is $\mathfrak{M}(X) \to \mathfrak{M}(Y)$ closable if and only if the kernel $K(x, y)$ of K satisfies (A) and (C); K is continuous if and only if $K(x, y)$ satisfies (A) and (B).

The problem of finding such a direct description of closed transformations K is open and examples were given in [1] of transformations closed but not continuous and closable but not closed.

The corresponding results for transformations $K^{\#}$ depend on the following obvious remark.

Let $w \in \mathfrak{D}_{K^{\#}}$, $w \neq 0$ a.e. and define the mapping i_w of $\mathfrak{M}(X)$ into $\mathfrak{M}(X \times Y)$ by

$$(2.8) \qquad (i_w u)(x, y) = w(x, y) u(x), \quad u \in \mathfrak{M}(X).$$

Then i_w is a topological isomorphism of $\mathfrak{M}(X)$ onto a closed subspace of $\mathfrak{M}(X \times Y)$ and if we define $K_w(x, y) = K(x, y) w(x, y)$, then we have $u \in \mathfrak{D}_{K_w}$ if and only if $i_w u \in \mathfrak{D}_K$. Since the kernel $K_w(x, y)$ satisfies (A), (B) if and only if $K(x, y)$ satisfies (A), (B) the preceding remarks yield (a) $\Rightarrow$ (b) part of the following theorem.

THEOREM 2.1. *The following conditions are equivalent:*

(a) $K^{\#}$ *is* $\mathfrak{M}(X \times Y) \to \mathfrak{M}(Y)$ *continuous,*

(b) $K(x, y)$ *satisfies* (A) *and* (B),

(c) $\mathfrak{D}_{K^{\#}} = \mathfrak{M}(X \times Y)$.

To prove that (b) $\Rightarrow$ (a) we observe that if $w_n \in \mathfrak{D}_{K^{\#}}$ and $w_n \to 0$ in $\mathfrak{M}(X \times Y)$, then for every atom a_l of X $w_n(a_l, y) \to 0$ in $\mathfrak{M}(Y)$. If (A) is satisfied, then

$$(K^{\#} w_n)(y) = \sum K(a_l, y) w_n(a_l, y) \mu(a_l).$$

If $E \subset Y$ is any set of finite measure, then by (B) $\nu\left(E \cap \bigcap_{n=1}^{N} \bigcup_{m=n}^{\infty} A_m\right) \xrightarrow[N\to\infty]{} 0$ and it follows that for every $\varepsilon > 0$ there exists N such that $\nu\left(E \cap \bigcup_{m=N}^{\infty} A_m\right)$

$< \varepsilon$. Also for every fixed N, $\sum_{l=1}^{N} K(a_l, y) w_n(a_l, y) \xrightarrow[n\to\infty]{} 0$ in $\mathfrak{M}(Y)$, and it follows that $K^{\#}$ is continuous. (b) $\Rightarrow$ (c) is trivial and (c) $\Rightarrow$ (a) follows from Proposition 2.1 and closed graph theorem.

We take up now the question of closability. By an argument similar to that used in the necessity part of Theorem 2.1 we conclude using Theorem 5.1 of [1] that if $K^{\#}$ is closable, then $K_w(x, y)$ must satisfy (A) and (C) for every $w \in \mathfrak{D}_{K^{\#}}$, $w \neq 0$ a.e. We shall prove even more:

PROPOSITION 2.2. *In order that $K^{\#}$ be closable it is necessary that $K(x, y)$ satisfy* (B).

Proof. If $K^{\#}$ is closable, then $K_w(x, y)$ satisfies (A) and as already remarked $K(x, y)$ satisfies (A). If the set $(a_n\}$ of atoms of X is finite there is nothing to prove, thus we can assume that $\{a_n\}$ is infinite. Consider now the sequence $\{w_n\}$ defined as follows:

$$w_n(a_m, y) = 2^{n-m}\Big(\sum_{l=0}^{\infty} 2^{-l}\chi_{A_{n+l}}(y)\Big)^{-1}\cdot\big(\mu(a_m)K(a_m, y)\big)^{-}$$

for $m \geqslant n$, $K(a_m, y) \neq 0$ and $w_n(x, y) = 0$ otherwise. Then $w_n \in \mathfrak{M}(X \times Y)$, $n = 1, 2, \ldots$, and $w_n \to 0$ is $\mathfrak{M}(X \times Y)$. Moreover, recalling that $A_m = \{y\colon K(a_m, y) \neq 0\}$ we get

$$\begin{aligned} |K|^{\#}|w_n| &= \sum_{m=1}^{\infty} |K(a_m, y)|\,|w_n(a_m, y)|\,\mu(a_m) \\ &= \Big(\sum_{l=0}^{\infty} 2^{-l}\chi_{A_{n+l}}(y)\Big)^{-1} \sum_{m=n}^{\infty} 2^{n-m}\chi_{A_m} = \chi_{\tilde{A}_n}, \end{aligned}$$

where $\tilde{A}_n = \bigcup_{m=n}^{\infty} A_m$. This shows that $w_n \in \mathfrak{D}_{K^{\#}}$, $n = 1, 2, \ldots$, and the same computation gives $K^{\#} w_n = \chi_{\tilde{A}_n}$. Since $\chi_{\tilde{A}_n} \xrightarrow[n\to\infty]{} \chi_{\tilde{A}}$ in $\mathfrak{M}$, where $\tilde{A} = \bigcap_{n=1}^{\infty} \tilde{A}_m$ and by the hypothesis $K^{\#}$ is closable, it follows that $\chi_{\tilde{A}} = 0$ and $K(x, y)$ satisfies (B).

Proposition 2.2 and Theorem 2.1 give

THEOREM 2.2. *$K^{\#}$ is $\mathfrak{M}(X \times Y) \to \mathfrak{M}(Y)$ closable if and only if it is $\mathfrak{M}(X \times Y) \to \mathfrak{M}(Y)$ continuous.*

In particular the problem of characterisation of closed transformations K disappears in the case of transformations $K^{\#}$.

3. (F)-compatibility and (FL)-compatibility. Let U, V be linear Hausdorff topological spaces and $T\colon \mathfrak{D}_T \subset U \to V$ be a linear transformation. We recall, see [1], that T is A-semiregular if A is a linear topological subspace of U, $A \cap \mathfrak{D}_T$ is dense in A (in its topology) and $T|_{\mathfrak{D}_T \cap A}$

is $A \to V$ continuous. If V is complete and T is A-semiregular, then there is a unique continuous transformation $T_A\colon A \to V$, called *A-extension of T*, such that $T_A|_{\mathfrak{D}_T \cap A} = T|_{\mathfrak{D}_T \cap A}$.

If $\mathfrak{F}$ is a family of linear topological subspaces of U, then T satisfies the *$\mathfrak{F}$-compatibility condition* if for every A, $B \in \mathfrak{F}$ such that T is A-semiregular and B-semiregular we have $T_A|_{A \cap B} = T_B|_{A \cap B}$.

It is easy to show (Proposition 1.5 in [1]) that if T is closable, then T satisfies the $\mathfrak{F}$-compatibility condition, where $\mathfrak{F}$ is the family of all topological subspaces of U. If was also shown in [1], Theorem 7.1, that if $K\colon \mathfrak{D}_K \subset \mathfrak{M}(X) \to \mathfrak{M}(Y)$ is a non-singular integral transformation, then K satisfies the (F)-compatibility condition if and only if K is closable, note that if K is singular, i.e. $\mathfrak{D}_K = \{0\}$, then this result is trivial.

An analogous result remains true also in the case of transformations $K^{\#}$:

THEOREM 3.1. *A transformation $K^{\#}\colon \mathfrak{D}_{K^{\#}} \subset \mathfrak{M}(X \times Y) \to \mathfrak{M}(Y)$ satisfies the (F)-compatibility condition if and only if $K^{\#}$ is closable, i.e. if and only if $K^{\#}$ is continuous.*

Proof. It is easy to see using the remarks preceding Theorem (2.1) that if $K^{\#}$ satisfies the (F)-compatibility condition, then so does the transformation K_w for every $w \in \mathfrak{D}_{K^{\#}}$ such that $w \neq 0$ a.e. In particular for every such w, K_w must be closable and this, by the result of [1] stated above, implies that $K_w(x, y) = w(x, y)K(x, y)$ satisfies (A) and (C), in particular $K(x, y)$ must satisfy (A).

To verify that $K(x, y)$ satisfies (B) we use Proposition 1.6 of [1]. Assuming that $K^{\#}$ is not closable we shall construct a sequence $w_m \in \mathfrak{D}_{K^{\#}}$ with the following properties: (i) $w_m \xrightarrow[m\to\infty]{} 0$ in $\mathfrak{M}(X \times Y)$ and $K^{\#} w_m \xrightarrow[m\to\infty]{} v \neq 0$ in $\mathfrak{M}(Y)$, (ii) For every formal series $\sum a_n w_n$ and for any sequence of its partial sums $\sum_{n=1}^{n_k} a_n w_n = s_{n_k}$ the conditions $s_{n_k} \xrightarrow[k\to\infty]{} w$, $w \in \mathfrak{D}_{K^{\#}}$ imply $s_{n_k} \xrightarrow[k\to\infty]{} w$ in $\mathfrak{D}_{K^{\#}}$. By virtue of Proposition 1.6 of [1] this will imply that $K^{\#}$ fails to satisfy the (F)-compatibility condition.

Non-closability of $K^{\#}$ and condition (A), satisfied by $K(x, y)$, imply existence of a sequence $w'_n \in \mathfrak{D}_{K^{\#}}$ such that $w'_n \xrightarrow[n\to\infty]{} 0$ in $\mathfrak{M}(X \times Y)$ and $(K^{\#} w'_n)(y) = \sum_{m=1}^{\infty} K(a_m, y) w'_n(a_m, y) \xrightarrow[n\to\infty]{} v(y) \neq 0$ in $\mathfrak{M}(Y)$. If follows that for every integer N we have $\sum_{m=N}^{\infty} K(a_m, y) w'_n(a_m, y) \xrightarrow[n\to\infty]{} v(y)$ in $\mathfrak{M}(Y)$, also, since $w'_n \in \mathfrak{D}_{K^{\#}}$ there exist increasing sequences of integers N_k, n_k such that the functions defined by $w_k(a_l, y) = w'_{n_k}(a_l, y)$ for $N_k \leqslant l < N_{k+1}$ and $w_k(x, y) = 0$ otherwise satisfy $K^{\#} w_k \xrightarrow[k\to\infty]{} v$ in $\mathfrak{M}(Y)$ and, obviously $w_k \xrightarrow[k\to\infty]{} 0$ in $\mathfrak{M}(X \times Y)$. Thus $\{w_k\}$ satisfy (i); it is also easy to check that this sequence satisfies (ii) and the proof is complete.

As concerns the (FL)-compatibility we have a result similar to Theorem 9.1 in [1].

THEOREM 3.2. *For every* $f \in \mathfrak{M}(X \times Y)$ *define*

$$\tilde{\varrho}_{K^{\#}}(f) = \varrho_{X\times Y}(f) + \sup\{\varrho_Y(K^{\#}w);\ w \in \mathfrak{D}_{K^{\#}},\ |w| \leqslant |f| \ a.e.\}. \tag{3.1}$$

Then $\varrho_{K^{\#}}$ *is a complete metric on* $\mathfrak{M}(X \times Y)$ *with the following properties:*

(1) *The closure* $\tilde{\mathfrak{D}}_{K^{\#}}$ *of* $\mathfrak{D}_{K^{\#}}$ *in* $\mathfrak{M}(X \times Y)$ *with metric* $\tilde{\varrho}_{K^{\#}}$ *is an* (FL)-*subspace of* $\mathfrak{M}(X \times Y)$;

(2) $K\colon \mathfrak{D}_{K^{\#}} \to \mathfrak{M}(Y)$ *is* $\tilde{\varrho}_{K^{\#}}$ *continuous; denote by* $\tilde{K}^{\#}$ *the* $\tilde{\mathfrak{D}}_{K^{\#}}$ *extension of* $K^{\#}$,

(3) *For every* (FL)-*subspace* A *of* $\mathfrak{M}(X \times Y)$ *such that* $K^{\#}$ *is* A-*semiregular we have* $A \subset \tilde{\mathfrak{D}}_{K^{\#}}$ *and* $K^{\#}_A = \tilde{K}^{\#}|_A$.

The proof is quite similar to those of Proposition 9.1 and Theorem 9.1 in [1] and we omit it.

We also have the following result which again is obtained by a repetition of an argument of [1], Proposition 9.2.

PROPOSITION 3.1. *If* $K(x, y) \geqslant 0$, *then* $\tilde{\mathfrak{D}}_{K^{\#}} = \mathfrak{D}_{K^{\#}}$.

Consider now the function $v \in \mathfrak{M}(X \times Y)$ defined by $v(x, y) = K(x, y)^{-1} |K(x, y)|$ if $K(x, y) \neq 0$ and $v(x, y) = 1$ otherwise. Then, by immediate inspection $\tilde{\varrho}_{K^{\#}}(vf) = \tilde{\varrho}_{K^{\#}}(f)$ and it follows that $f \in \mathfrak{D}_{K^{\#}}$ if and only if $f \in \tilde{\mathfrak{D}}_{(Kv)^{\#}} = \tilde{\mathfrak{D}}_{|K|^{\#}} = \mathfrak{D}_{K^{\#}}$ by Proposition 3.1 and (2.4). Thus we get

PROPOSITION 3.2. $\tilde{\mathfrak{D}}_{K^{\#}} = \mathfrak{D}_{K^{\#}}$.

By Theorem 3.2 we also get the necessity part in the following

COROLLARY 3.1. *If* A *is an* (FL)-*subspace of* $\mathfrak{M}(X \times Y)$, *then* $K^{\#}$ *is* A-*semiregular if and only if* $A \subset \mathfrak{D}_{K^{\#}}$.

The sufficiency of the condition follows from the closed graph theorem.

4. An example. The transformation K with the kernel $K(x, y) = 1/(x-y)$, $x, y \in (0, 1) = X = Y$ is singular, i.e. $\mathfrak{D}_K = \{0\}$. In consideration of the Hilbert transform it is natural to study the transformation

$$u \in \mathfrak{M}(X) \to \int_0^1 \frac{u(x)-u(y)}{x-y}\,dx = K^{\#}w, \qquad w(x, y) = u(x)-u(y)$$

whose domain contains in particular all Hölder continuous integrable functions on (0.1). It was hoped that some results about the Hilbert transform could be obtained by studying continuous extensions of $K^{\#}$ to (F)-subspaces of $\mathfrak{M}(X \times Y)$. The following proposition together with Corollary 3.1 show that even L^2-extension of the Hilbert transform cannot be obtained in this manner within the framework of (FL)-subspaces of $\mathfrak{M}(X \times Y)$.

PROPOSITION 4.1. *Let $K(x,y) = 1/(x-y)$, $x, y \in (0,1)$; then for every p, $1 \leqslant p \leqslant \infty$ there is a function $u \in L^p(0,1)$ such that $\int |K(x,y)|\,|u(x)-u(y)|\,dx = \infty$ on a set of positive measure. In particular $u(x)-u(y) \notin \mathfrak{D}_{K^\#}$.*

Proof. For $0 < \varepsilon < 1$ we define

$$(T_\varepsilon f)(y) = \int\limits_{|x-y|>\varepsilon} \frac{|f(x)-f(y)|}{|x-y|}\,dx \quad \text{and} \quad (Tf)(y) = \int\limits_0^1 \frac{|f(x)-f(y)|}{|x-y|}\,dx.$$

It is immediate that for every p, $1 \leqslant p \leqslant \infty$, T_ε: $L^p(0,1) \to \mathfrak{M}(0,1)$ is a bounded mapping satisfying $\varrho_Y\big(T_\varepsilon(f+g)\big) \leqslant \varrho_Y(T_\varepsilon f) + \varrho_Y(T_\varepsilon g)$. If for every $f \in L^p(0,1)$, $Tf(y)$ were finite a.e. it would follow by the uniform boundedness principle that T: $L^p(0,1) \to \mathfrak{M}(0,1)$ is bounded. However, if $r_n(x) = \operatorname{sign}(\sin 2^n \pi x)$, $n = 1, 2, \ldots$, denote the sequence of Rademacher functions, then by an easy computation we get for $y \in \big(l2^{-n}, (l+1)2^{-n}\big)$, $0 \leqslant l < 2^n$,

$$\int\limits_0^1 \frac{|r_n(x)-r_n(y)|}{|x-y|}\,dx > 2 \sum_{l-2s-1\geqslant 0} \int\limits_{(l-2s-1)2^{-n}}^{(l-2s)2^{-n}} \frac{dx}{y-x} + 2 \sum_{l+2s+2\leqslant 2^n} \int\limits_{(l+2s+1)2^{-n}}^{(l+2s+2)2^{-n}} \frac{dx}{x-y}$$

$$\geqslant 2 \sum_{s \leqslant \frac{l-1}{2}} \frac{1}{2s+1} + 2 \sum_{s \leqslant \frac{2^n-l-2}{2}} \frac{1}{2s+1} - 2 \sum_{s=0}^{\infty} \frac{1}{(2s+1)^2} \xrightarrow[n\to\infty]{} \infty,$$

which shows that T: $L^p(0,1) \to \mathfrak{M}(0,1)$ cannot be bounded and the set of u's with the desired property is of 2nd category in $L^p(0,1)$.

Remark. By approximating the functions $r_n(x)$ by suitable piecewise linear functions we conclude that Proposition 4.1 remains valid if $L^p(0,1)$ is replaced by $C([0,1])$.

The above results suggest that a study of extensions of $K^\#$ in a more general class of (F)-subspaces of $\mathfrak{M}(X \times Y)$ would be of some interest.

References

[1] N. Aronszajn and P. Szeptycki, *On general integral transformations*, Math. Ann. 163 (1966), p. 127–154.

[2] P. Szeptycki, *On functions and measures whose Fourier transforms are functions*, ibidem 179 (1968), p. 31–41.

[3] — *On functions and measures whose Fourier transforms are functions*, Autoreferat, Zentralblat für Mathematik und ihre Grenzgebiete, 167 Band, Heft 2 (1969), p. 146.

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