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On Hermite expansions of x^k and $\ln|x|$

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Abstract. Hermite expansions of x^k and $\ln|x|$ are established and the asymptotic behaviour of their coefficients is estimated.

Introduction. The functions x^k and $\ln|x|$ do not expand into Hermite series in the classical sense. However, they do expand in Theory of Distributions. The aim of this note is to show what the expansions look like and to establish the asymptotic behaviour of the coefficients.

1. In [1] it is proved that every tempered distribution $f(x)$ expands uniquely into a Hermite series $\sum_{n=0}^{\infty} a_n h_n$, where

$$h_n(x) = (-1)^n \left(\sqrt{2\pi}\, n!\right)^{-\frac{1}{2}} e^{x^2/4} \left(e^{-x^2/2}\right)^{(n)}$$

and the coefficients a_n are inner products of h_n and distribution $f(x)$, which will be denoted by $\int_{-\infty}^{+\infty} f h_n$.

The functions x^k and $\ln|x|$, where k is a non-negative integer, are tempered distributions, and thus they expand into Hermite series.

2. For short, we shall write $V(n) = \sqrt{\frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}}$ for even n, and $U(n) = \sqrt{\frac{3}{2}\frac{5}{4}\cdots\frac{n}{n-1}}$ for odd n, assuming that $V(0) = U(1) = 1$.

THEOREM 1. *For even k, the function x^k expands into the Hermite series $\sum_{n=0}^{\infty} a_{kn} h_n$ with the coefficients*

$$a_{kn} = \begin{cases} W_k(n)\sqrt[4]{8\pi}\; V(n) & \text{for even } n, \\ 0 & \text{for odd } n, \end{cases}$$

where $W_k(n)$ is a polynomial of degree $k/2$ determined by the following recurrence equation:

$$W_k(n) = (n+1)W_{k-2}(n+2)+(2n+1)W_{k-2}(n)+nW_{k-2}(n-2)$$

for $k \geqslant 2$ and $W_0(n) = 1$.

For odd k, the coefficients are given by the formula

$$a_{kn} = \begin{cases} 0 & \text{for even } n, \\ W_k(n)\sqrt[4]{8\pi}\; U(n) & \text{for odd } n, \end{cases}$$

where $W_k(n)$ is a polynomial of degree $(k-1)/2$ determined by the following recurrence equation:

$$W_k(n) = (n+2)W_{k-2}(n+2)+(2n+1)W_{k-2}(n)+(n-1)W_{k-2}(n-2)$$

for $k \geqslant 3$ and $W_1(n) = 2$.

Proof. For $k = 0$, the coefficients are given by the formula $a_{0n} = \sqrt[4]{8\pi}\; V(n)$ for even n, and $a_{0n} = 0$ for odd n (see [1]).

Hence $W_0(n) = 1$ for every n. It is known that

$$xh_n = \sqrt{n+1}\, h_{n+1}+\sqrt{n}\, h_{n-1} \quad \text{for } n \geqslant 1 \tag{1}$$

(see [1]). We also have $xh_0 = h_1$. Multiplying both sides of the last equalities by x^{k-1} and integrating them, we obtain

$$\begin{aligned} a_{kn} &= \sqrt{n+1}\, a_{k-1,n+1}+\sqrt{n}a_{k-1,n-1} \quad \text{for } n \geqslant 1 \text{ and } k \geqslant 1, \\ a_{k0} &= a_{k-1,1}, \end{aligned} \tag{2}$$

where $a_{kn} = \int\limits_{-\infty}^{+\infty} x^k h_n$. For convenience we assume that $a_{kn} = 0$ for $n < 0$. Then formula (2) is true for $n \geqslant 0$. By a simple transformation of (2) we obtain

$$a_{kn} = \sqrt{(n+1)(n+2)}\, a_{k-2,n+2}+(2n+1)a_{k-2,n}+\sqrt{n(n-1)}\, a_{k-2,n-2} \tag{3}$$

for $n \geqslant 0$ and $k \geqslant 2$.

Let k be an even positive integer different from 0. It is evident that $a_{kn} = 0$ for odd n, because x^k is an even function (see [1]). We shall look for the coefficients a_{kn} in the form

$$a_{kn} = W_k(n)\sqrt[4]{8\pi}\sqrt{\frac{1}{2}\frac{3}{4}\cdots\frac{n-1}{n}} \quad \text{for } n \geqslant 2. \tag{4}$$

Putting this expression into formula (3) we obtain

$$W_k(n) = (n+1)W_{k-2}(n+2)+(2n+1)W_{k-2}(n)+nW_{k-2}(n-2) \tag{5}$$

for $n \geqslant 2$ and $k \geqslant 2$. It is very simple to show that $W_k(n)$ is a polynomial of degree $k/2$. From (5) we obtain

$$W_k(0) = W_{k-2}(2)+W_{k-2}(0). \tag{6}$$

We shall show that $a_{k0} = \sqrt[4]{8\pi}\, W_k(0)$. The proof is by induction. For $k = 0$, we have $W_0(n) = 1$. Assume that $a_{k-2,0} = \sqrt[4]{8\pi}\, W_{k-2}(0)$. In view of (3) we obtain for $n = 0$

$$a_{k0} = \sqrt{2}\, a_{k-2,2}+a_{k-2,0}.$$

From (4) it follows that $a_{k-2,2} = W_{k-2}(2)\sqrt[4]{8\pi}\,\sqrt{\tfrac{1}{2}}$. Hence

$$a_{k0} = \sqrt[4]{8\pi}\big(W_{k-2}(2)+W_{k-2}(0)\big).$$

The last equality and (6) imply the assertion. For odd k, the proof is like the first part of Theorem 1. It suffices to notice that $W_1(n) = 2$.

Indeed, from 2 we obtain

$$\begin{aligned} a_{1n} &= \sqrt{n+1}\, a_{0,n+1}+\sqrt{n}\, a_{0,n-1} = \sqrt[4]{8\pi}\big(\sqrt{n+1}\;\; V(n+1)+\sqrt{n}V(n-1)\big) \\ &= \sqrt[4]{8\pi}\big(U(n)+U(n)\big) = 2\sqrt[4]{8\pi}U(n) \end{aligned}$$

and the proof is complete.

We give below a few initial polynomials $W_k(n)$:

$$\begin{aligned} &W_0(n) = 1, && W_1(n) = 2, \\ &W_2(n) = 2(2n+1), && W_3(n) = 4(2n+1), \\ &W_4(n) = 4(4n^2+4n+3), && W_5(n) = 8(4n^2+4n+7), \\ &W_6(n) = 16(4n^3+6n^2+17n+15), && W_7(n) = 16(8n^3+12n^2+58n+27). \end{aligned}$$

The expansions of x, x^2, x^3, and x^4 are:

$$\begin{aligned} x &= 2\sqrt[4]{8\pi}\Big(h_1+\sqrt{\tfrac{3}{2}}h_3+\sqrt{\tfrac{3}{2}\tfrac{5}{4}}\,h_5+\sqrt{\tfrac{3}{2}\tfrac{5}{4}\tfrac{7}{6}}\,h_7+\ldots\Big), \\ x^2 &= 2\sqrt[4]{8\pi}\Big(h_0+5\sqrt{\tfrac{1}{2}}h_2+9\sqrt{\tfrac{1}{2}\tfrac{3}{4}}\,h_4+13\sqrt{\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}}\,h_6+\ldots\Big), \\ x^3 &= 4\sqrt[4]{8\pi}\Big(3h_1+7\sqrt{\tfrac{3}{2}}h_3+11\sqrt{\tfrac{3}{2}\tfrac{5}{4}}\,h_5+15\sqrt{\tfrac{3}{2}\tfrac{5}{4}\tfrac{7}{6}}\,h_7+\ldots\Big), \\ x^4 &= 4\sqrt[4]{8\pi}\Big(3h_0+27\sqrt{\tfrac{1}{2}}h_2+83\sqrt{\tfrac{1}{2}\tfrac{3}{4}}\,h_4+171\sqrt{\tfrac{1}{2}\tfrac{3}{4}\tfrac{5}{6}}\,h_6+\ldots\Big). \end{aligned}$$

3. We are going to prove

THEOREM 2. *The function* $\ln|x|$ *expands into the Hermite series* $\sum_{n=0}^{\infty} a_n h_n$, *where* $a_n = 0$ *for odd* n *and* $a_n = 2\sqrt[4]{8\pi}(u_n - C/4)V(n)$ *for even* n. C *is Euler's constant, and* u_n *is the sequence determined by the following recurrence equation:*

$$(n+1)u_{n+2} - u_n - nu_{n-2} = 1,$$

with

$$u_0 = 0 \quad \textit{and} \quad u_2 = 1.$$

Proof. Because $\ln|x|$ is an even function, we have $a_n = 0$ for odd n. The proof is based on the following well-known formulas (see [1]):

$$(7) \qquad \begin{aligned} -2h_n' &= \sqrt{n+1}\; h_{n+1} - \sqrt{n} h_{n-1} \quad \text{for } n \geqslant 1, \\ -2h_0' &= h_1. \end{aligned}$$

Multiplying both members of these equations by x and using (1), we obtain

$$\begin{aligned} -2xh_n' &= \sqrt{(n+1)(n+2)}\, h_{n+2} + h_n - \sqrt{n(n-1)}\, h_{n-2} \quad \text{for } n \geqslant 2, \\ -2xh_0' &= \sqrt{2}\, h_2 + h_0. \end{aligned}$$

Hence, multiplying by $\ln|x|$ and integrating the last expression, we obtain

$$(8) \qquad \sqrt{(n+1)(n+2)}\, a_{n+2} + a_n - \sqrt{n(n-1)}\, a_{n-2} = -2 \int_{-\infty}^{+\infty} x \ln x h_n',$$

where $a_n = \int_{-\infty}^{+\infty} \ln|x| h_n$. As in Theorem 1, we understand that $a_n = 0$ for $n < 0$. Then (8) is true for all even n. Transforming the right-hand side in (8), we obtain

$$(9) \qquad \sqrt{(n+1)(n+2)}\, a_{n+2} - a_n - \sqrt{n(n-1)}\, a_{n-2} = 2\sqrt[4]{8\pi}\, V(n)$$

for $n \geqslant 0$.

It remains to calculate the coefficient a_0:

$$a_0 = \int_{-\infty}^{+\infty} \ln|x| h_0 = \frac{2}{\sqrt[4]{2\pi}} \int_{-\infty}^{+\infty} \ln|x| e^{-x^2/4}.$$

The integral on the right side can easily be calculated by using the well-known Legendre formula:

$$\Gamma(a)\Gamma(a+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2a-1}} \Gamma(a).$$

In this way we obtain

$$\int_{-\infty}^{+\infty} \ln|x|\, e^{-x^2/4} = -\frac{C}{2}\sqrt{\pi}.$$

Hence $a_0 = -\frac{1}{2}C\sqrt[4]{8\pi}$, and from (9) we get $a_2 = 2\sqrt[4]{8\pi}(1-\frac{1}{4}C)$. We write a_n in the form

$$a_n = 2\sqrt[4]{8\pi}\left(u_n - \frac{C}{4}\right)V(n)$$

and look for suitable values of u_n. Putting this into (9), we obtain $(n+1)u_{n+2}-u_n-nu_{n-2} = 1$ for $n \geqslant 2$, and $u_0 = 0$, $u_2 = 1$, which completes the proof.

We shall now estimate the asymptotic behaviour of the coefficients a_{kn} in the expansion of x^k. We shall show that

$$\lim_{n\to\infty} \frac{a_{kn}}{\sqrt[4]{n^{2k-1}}} = 2^{k+1}.$$

The last limit is understood in the following sense. If k is even, then n takes only even values; if k is odd, n takes only odd values. From the Wallis formula

$$\pi = \lim_{n\to\infty} \frac{1}{n}\left(\frac{2\ 4\ \dots\ 2n}{1\ 3\ \dots\ 2n-1}\right)^2$$

and by the definition of $V(n)$ (see part 2) we obtain

$$\lim_{n\to\infty} \sqrt[4]{2n}\, V(2n) = \sqrt[4]{\frac{2}{\pi}}\,. \tag{10}$$

From (10) and the equality $U(2n-1) = \sqrt{2n-1}\ V(2n-2)$ we obtain

$$\lim_{n\to\infty} \frac{U(2n-1)}{\sqrt[4]{2n-1}} = \sqrt[4]{\frac{2}{\pi}} \tag{11}$$

by the definition of $U(n)$. Using the recurrence equation for the polynomials $W_k(n)$ which are given in Theorem 1, we obtain

$$\lim_{n\to\infty} \frac{W_{2k}(2n)}{(2n)^k} = 2^{2k} \quad \text{for } k \geqslant 0, \tag{12}$$

$$\lim_{n\to\infty} \frac{W_{2k-1}(2n-1)}{(2n-1)^{k-1}} = 2^{2k-1} \quad \text{for } k \geqslant 1. \tag{13}$$

The easy proofs are omitted. From (10), (12) and from the definition of $a_{2k,2n}$ we obtain

$$\lim_{n\to\infty}\frac{a_{2k,2n}}{\sqrt[4]{(2n)^{4k-1}}} = 2^{2k+1}.$$

Similarly from (11), (13) and the definition $a_{2k-1,2n-1}$ we obtain

$$\lim_{n\to\infty}\frac{a_{2k-1,2n-1}}{\sqrt[4]{2n-1)^{4k-3}}} = 2^{2k}.$$

This gives the required asymptotic behaviour of the coefficients. The two formulas can be fused into one expression, namely:

$$a_{kn} \sim 2^{k+1} n^{\frac{k}{2}-\frac{1}{4}}, \quad \text{as } n\to\infty,\ a_{kn} \neq 0.$$

References

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[3] J. Mikusiński, *On the expansions of the derivatives of the delta distribution*, Bull. Acad. Polon. Sci. (in press).

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