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Difference methods for non-linear partial differential equations of the first order

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Abstract. The paper deals with the difference methods for the Cauchy problem

$$\text{(i)} \qquad \begin{aligned} &z_x(x, y) = f(x, y, z(x, y), z_y(x, y)), \\ &z(x^{(0)}, y) = \omega(y). \end{aligned}$$

The corresponding one-step difference method is of the form

$$\text{(ii)} \qquad \begin{aligned} &\Delta_0 w^{(i,j)} = \Phi(x^{(i)}, y^{(j)}, A w^{(i,j)}, [\![w^{(i,j)}]\!], \Delta w^{(i,j)}, h_i, k), \\ &w^{(0,j)} = \omega(y^{(j)}), \end{aligned}$$

where Δ_0 and Δ are difference operators.

We give sufficient conditions for the convergence of the sequence $\{u_m\}$ of solutions of (ii) to a solution $\bar{u}$ of (i). We also give an error estimate of the method, in terms of a power of the step h, indicating the order of the method.

1. Introduction. We consider the initial problem for the first order partial differential equation

$$\text{(1)} \qquad \begin{aligned} &z_x(x, y) = f(x, y, z(x, y), z_y(x, y)) \quad &&\text{for } (x, y) \in E, \\ &z(x^{(0)}, y) = \omega(y) &&\text{for } y \in E_0, \end{aligned}$$

where $y = (y_1, \ldots, y_n)$ and $z_y(x, y) = (z_{y_1}(x, y), \ldots, z_{y_n}(x, y))$. The sets E and E_0 are defined by

$$\text{(2)} \qquad E = \{(x, y)\colon x^{(0)} \leqslant x \leqslant x^{(0)} + a,\ |y_\tau - y_\tau^{(0)}| \leqslant b_\tau - M_\tau(x - x^{(0)}),\ \tau = 1, \ldots, n\}$$

where $a, b_\tau > 0$, $aM_\tau < b_\tau$, $\tau = 1, \ldots, n$, and

$$\text{(3)} \qquad E_0 = \{y\colon (x^{(0)}, y) \in E\}.$$

We are interested in establishing a method of approximating a solution of the Cauchy problem (1) by solutions of an associated difference equation and in estimating the difference between these solutions.

For h, $k_\tau > 0$, $\tau = 1, \ldots, n$, we define $x^{(i)} = x^{(0)} + ih$, $i = 0, 1, \ldots, n_0$, where $n_0 h = a$ and

(4) $$y_\tau^{(l)} = y_\tau^{(0)} + lk_\tau, \quad l = -n_\tau, -n_\tau+1, \ldots, -1, 0, 1, \ldots, n_\tau, \quad \tau = 1, \ldots, n, \text{ where } n_\tau k_\tau = b_\tau.$$

Let $j = (j_1, \ldots, j_n)$ and $y^{(j)} = (y_1^{(j_1)}, \ldots, y_n^{(j_n)})$. We define

(5) $$\begin{aligned} \tilde{\Gamma} &= \{(i, j): (x^{(i)}, y^{(j)}) \in E\}, \\ \Gamma^{(m)} &= \{(i, j) \in \tilde{\Gamma}: i = m, (i+1, j) \in \tilde{\Gamma}\}, \quad i = 0, 1, \ldots, n_0 - 1, \\ \Gamma &= \Gamma^{(0)} \cup \Gamma^{(1)} \cup \ldots \cup \Gamma^{(n_0-1)}, \end{aligned}$$

and

(6) $$E^* = \{(x^{(i)}, y^{(j)}): (i, j) \in \tilde{\Gamma}\}, \quad \Gamma_0 = \{j: (0, j) \in \tilde{\Gamma}\}.$$

Suppose that f is a function of the variables (x, y, p, q), $q = (q_1, \ldots, q_n)$, continuous on $E \times R^{1+n}$ and that the derivatives f_p, $f_q = (f_{q_1}, \ldots, f_{q_n})$ exist and are continuous on $E \times R^{1+n}$. Assume that for each i, $1 \leqslant i \leqslant n$, we have

(7) $$f_{q_i}(x, y, p, q) \leqslant 0 \quad \text{on } E \times R^{1+n} \quad \text{or} \quad f_{q_i}(x, y, p, q) \geqslant 0 \quad \text{on } E \times R^{1+n}.$$

Let

$$I_1 = \{i \in \{1, \ldots, n\}: f_{q_i}(x, y, p, q) \leqslant 0 \text{ on } E \times R^{1+n}\}$$

and $I_2 = \{1, \ldots, n\} \setminus I_1$. For a function u: $E^* \to R$ we write $u^{(i,j)} = u(x^{(i)}, y^{(j)})$, $(i, j) \in \tilde{\Gamma}$. We define the difference operators $\Delta_0, \Delta_1, \ldots, \Delta_n$ as follows:

$$\Delta_0 u^{(i,j)} = \frac{1}{h}[u^{(i+1,j)} - u^{(i,j)}],$$

$$\Delta_\tau u^{(i,j)} = \frac{1}{k_\tau}[u^{(i,j)} - u^{(i,j_1,\ldots,j_{\tau-1},j_\tau-1,j_{\tau+1},\ldots,j_n)}] \quad \text{for } \tau \in I_1,$$

$$\Delta_\tau u^{(i,j)} = \frac{1}{k_\tau}[u^{(i,j_1,\ldots,j_{\tau-1},j_\tau+1,j_{\tau+1},\ldots,j_n)} - u^{(i,j)}] \quad \text{for } \tau \in I_2.$$

Put $\Delta u^{(i,j)} = (\Delta_1 u^{(i,j)}, \ldots, \Delta_n u^{(i,j)})$. Suppose that there exists a solution $\bar{z}$ of (1).

Suppose that problem (1) is solved numerically by the difference method

(8) $$\begin{aligned} \Delta_0 u^{(i,j)} &= f(x^{(i)}, y^{(j)}, u^{(i,j)}, \Delta u^{(i,j)}), \quad (i, j) \in \Gamma, \\ u^{(0,j)} &= \omega(y^{(j)}) \quad \text{for } j \in \Gamma_0. \end{aligned}$$

We define the discretization error η by

$$\Delta_0 \bar{z}^{(i,j)} = f(x^{(i)}, y^{(j)}, \bar{z}^{(i,j)}, \Delta\bar{z}^{(i,j)}) + \eta^{(i,j)}, \qquad (i,j) \in \Gamma$$

and $\varepsilon(h, k_0) = \max\limits_{(i,j)\in\Gamma} |\eta^{(i,j)}|$, where $k_0 = \max\limits_{1\leq\tau\leq n} k_\tau$.

A simple sufficient condition for the convergence of the method (8) to a solution of (1) is given by the following theorem.

THEOREM (T). *Suppose that*

(i) $f \in C(E \times R^{1+n}, R)$, *the derivatives* f_p, f_q *exist and* $f_p, f_{q_i} \in C(E \times R^{1+n}, R)$, $i = 1, \ldots, n$.

(ii) *for each* i, $1 \leq i \leq n$, f_{q_i} *satisfies* (7) *and there exists a constant* $L > 0$ *such that* $|f_p(x, y, p, q)| \leq L$ *on* $E \times R^{1+n}$,

(iii) $k_\tau \leq hM_\tau$ *for* $\tau = 1, \ldots, n$ *and for* $(x, y, p, q) \in E \times R^{1+n}$ *we have*

$$1 + h\left[f_p(x, y, p, q) - \sum_{\tau=1}^{n} \frac{1}{k_\tau} |f_{q_\tau}(x, y, p, q)|\right] \geq 0,$$

(iv) $\bar{z}$ *and* $\bar{u}$ *are solutions of* (1) *and* (8), *respectively*.

Under these assumptions,

$$|\bar{z}^{(i,j)} - \bar{u}^{(i,j)}| \leq \frac{\varepsilon(h, k_0)}{L}(e^{Lhi} - 1), \qquad (i,j) \in \tilde{\Gamma}$$

and

$$\lim_{h,k_0 \to 0} |\bar{z}^{(i,j)} - \bar{z}^{(i,j)}| = 0, \qquad (i,j) \in \tilde{\Gamma}.$$

This theorem is an immediate consequence of [1], [2]. It was shown in [1] that arbitrary high accuracy can be achieved by method (8) by choosing the step h sufficiently small. The discretization error is roughly proportional to h, which means that the accuracy that can be achieved over a given domain E is proportional to h. It is shown in this paper that there are one-step methods for (1) which are much more effective than Euler's method (8) in the sense that the accuracy attainable with the step h (with respect to x) is proportional to h^α with $\alpha > 1$.

We prove the convergence of a general one-step difference method for (1). We also give an error estimate of the method, expressed in terms of a power of the step h, indicating the order of the method.

The basic tool in our investigations are theorems on difference inequalities.

We give some examples of the one-step difference methods of second order (based on Taylor's expansion or patterned on methods of Runge–Kutta type) and an example of the one-step difference method of third order (based on Taylor's expansion).

2. Difference inequalities. Suppose that $H \subset [0, h^{(0)}] \times [0, k^{(0)}]$ is a given set of parameters (ξ, η), $\eta = (\eta_1, \ldots, \eta_n)$, where $h^{(0)} > 0$, $k^{(0)} = (k_1^{(0)}, \ldots, k_n^{(0)})$, $k_\tau^{(0)} > 0$ for $\tau = 1, \ldots, n$, and $[0, k^{(0)}]$ is an interval in R_+^n, $R_+ = [0, +\infty)$. We assume that E and E_0 are given by (2) and (3). For $j = (j_1, \ldots, j_n)$ we denote $y^{(j)} = (y_1^{(j_1)}, \ldots, y_n^{(j_n)})$, where $y_\tau^{(l)}$, $\tau = 1, \ldots, n$, are defined by (4). Let

$$x^{(i+1)} = x^{(i)} + h_i, \qquad h_i \in (0, h^{(0)}],\ i = 0, 1, \ldots, \tilde{n}, \tag{9}$$
$$x^{(\tilde{n}+1)} = x^{(0)} + a,$$

where $(h_i, k) = (h_i, k_1, \ldots, k_n) \in H$ for $i = 0, 1, \ldots, \tilde{n}$. Let $h = (h_0, h_1, \ldots, h_{\tilde{n}})$. Suppose that $\tilde{\Gamma}$, Γ_0, Γ, E^* are sets defined in Section 1 with $x^{(i)}$ given by (9) and with $(h_i, k) \in H$. For an integer $r \geq 0$ we write $n^* = (2r+1)^n$ and $I(r) = \{-r, -r+1, \ldots, -1, 0, 1, \ldots, r\}$. Let $N(r) = \{s = (s_1, \ldots, s_n)\colon s_\tau \in I(r)$ for $\tau = 1, \ldots, n\}$ and $\Omega = E \times R \times R^{n^*} \times R^n$. Let p be a matrix

$$p = [p_s]_{\substack{s = (s_1, \ldots, s_n) \\ s \in N(r)}}.$$

Denote by $C(X, Y)$ the set of all continuous functions defined in X and taking values in Y; X, Y are arbitrary metric spaces.

Suppose that for each pair $(\xi, \eta) \in H$ there is given a function $\Phi(\cdot, \xi, \eta)\colon \Omega \to R$ of the variables (x, y, z, p, q), where $y = (y_1, \ldots, y_n)$, $q = (q_1, \ldots, q_n)$, $p = [p_s]_{s \in N(r)}$. We write $P = (x, y, z, p, q, \xi, \eta)$, where $(x, y, z, p, q) \in \Omega$, $(\xi, \eta) \in H$. If $j = (j_1, \ldots, j_n)$, $j' = (j'_1, \ldots, j'_n)$, then $j + j' = (j_1 + j'_1, \ldots, j_n + j'_n)$.

For a function $z\colon E \to R$ we define

$$Az(x, y) = \sum_{\substack{s = (s_1, \ldots, s_n) \\ s \in N(r)}} a_s z(x, y_1 + s_1 k_1, \ldots, y_n + s_n k_n),$$

$$B_\tau z(x, y) = \sum_{\substack{s = (s_1, \ldots, s_n) \\ s \in N(r)}} b_s^{(\tau)} z(x, y_1 + s_1 k_1, \ldots, y_n + s_n k_n), \qquad \tau = 1, \ldots, n,$$

where $a_s, b_s^{(\tau)} \in R$ for $s \in N(r)$, $\tau = 1, \ldots, n$. Suppose that $I_1, I_2 \subset \{1, \ldots, n\}$ are sets of integers such that $I_1 \cup I_2 = \{1, \ldots, n\}$ and $I_1 \cap I_2 = \emptyset$. (In particular, $I_1 = \emptyset$ or $I_2 = \emptyset$ are admitted.) We introduce the difference operators $\Delta_0, \Delta_1, \ldots, \Delta_n$. Let $z\colon E^* \to R$, $(h_i, k) \in H$, $rk_\tau \leq h_i M_\tau$, $i = 0, 1, \ldots, \tilde{n}$, $\tau = 1, \ldots, n$. We define

$$\Delta_0 z(x^{(i)}, y^{(j)}) = \frac{1}{h_i}[z(x^{(i)} + h_i, y^{(j)}) - Az(x^{(i)}, y^{(j)})],$$

$$\Delta_\tau z(x^{(i)}, y^{(j)}) = \frac{1}{k_\tau}[Az(x^{(i)}, y^{(j)}) - B_\tau(x^{(i)}, y^{(j)})] \quad \text{for } \tau \in I_1,$$

$$\Delta_\tau z(x^{(i)}, y^{(j)}) = \frac{1}{k_\tau}[B_\tau z(x^{(i)}, y^{(j)}) - Az(x^{(i)}, y^{(j)})] \quad \text{for } \tau \in I_2.$$

Let $\Delta z(x^{(i)}, y^{(j)}) = (\Delta_1 z(x^{(i)}, y^{(j)}), \ldots, \Delta_n z(x^{(i)}, y^{(j)}))$. Denote by $[\![z(x, y)]\!]$ the matrix defined by

$$[\![z(x, y)]\!] = [z(x, y_1 + s_1 k_1, \ldots, y_n + s_n k_n)]_{\substack{s=(s_1,\ldots,s_n) \\ s\in N(r)}}.$$

We consider the following difference method for the Cauchy problem (1):

$$(10)\qquad \begin{aligned} &\Delta_0 w^{(i,j)} = \Phi(x^{(i)}, y^{(j)}, Aw^{(i,j)}, [\![w^{(i,j)}]\!], \Delta w^{(i,j)}, h_i, k) \quad \text{for } (i, j)\in\Gamma, \\ &w^{(0,j)} = \omega(y^{(j)}) \quad \text{for } j\in\Gamma_0. \end{aligned}$$

ASSUMPTION H_0. Suppose that

(a_0) for each $(\xi, \eta)\in H$ we have $r\eta_\tau \leqslant M_\tau \xi$, $\tau = 1, \ldots, n$,

(b_0) if $(\xi, \eta)\in H$, then $\Phi(\cdot, \xi, \eta)\in C(\Omega, R)$,

(c_0) the derivatives $\Phi_z(\cdot, \xi, \eta)$, $\Phi_p(\cdot, \xi, \eta) = [\Phi_{p_s}(\cdot, \xi, \eta)]_{s\in N(r)}$, $\Phi_{q_\tau}(\cdot, \xi, \eta)$, $\tau = 1, \ldots, n$, $(\xi, \eta)\in H$, exist and are continuous on Ω,

(d_0) for $P\in\Omega\times H$ and for $s\in N(r)$ we have

$$a_s\left[1 + \xi\Phi_z(P) + \xi\sum_{\tau\in I_1}\frac{1}{\eta_\tau}\Phi_{q_\tau}(P) - \xi\sum_{\tau\in I_2}\frac{1}{\eta_\tau}\Phi_{q_\tau}(P)\right] + \\ + \xi\left[\Phi_{p_s}(P) - \sum_{\tau\in I_1}\frac{1}{\eta_\tau}b_s^{(\tau)}\Phi_{q_\tau}(P) + \sum_{\tau\in I_2}\frac{1}{\eta_\tau}b_s^{(\tau)}\Phi_{q_\tau}(P)\right] \geqslant 0.$$

Remark 1. Suppose that for each τ, $1\leqslant\tau\leqslant n$, the derivative $\Phi_{q_\tau}(\cdot, \xi, \eta)$ satisfies one of the following conditions:

$$\Phi_{q_\tau}(P)\leqslant 0 \quad \text{for } P\in\Omega\times H \quad \text{or} \quad \Phi_{q_\tau}(P)\geqslant 0 \quad \text{for } P\in\Omega\times H.$$

Suppose that $I_1 = \{\tau\in\{1, \ldots, n\}\colon \Phi_{q_\tau}(P)\leqslant 0 \text{ for } P\in\Omega\times H\}$, $I_2 = \{1, \ldots, n\}\setminus I_1$ and $a_s, b_s^{(\tau)}\geqslant 0$ for $s\in N(r)$, $\tau = 1, \ldots, n$. If

$$1 + \xi\left[\Phi_z(P) - \sum_{\tau=1}^{n}\frac{1}{\eta_\tau}|\Phi_{q_\tau}(P)|\right] \geqslant 0 \quad \text{and} \quad \Phi_{p_s}(P)\geqslant 0, \quad s\in N(r),$$

for $P\in\Omega\times H$, then condition (d_0) of Assumption H_0 is satisfied.

We are now able to prove the following theorem on difference inequalities.

THEOREM 1. *Suppose that*

(i_1) *Assumption* H_0 *is satisfied,*

(ii_1) $(h_i, k)\in H$ *for* $i = 0, 1, \ldots, \tilde{n}$,

(iii_1) u *and* v *are functions defined on* E^* *and*

$$(11)\qquad u^{(0,j)} \leqslant v^{(0,j)} \quad \textit{for } j\in\Gamma_0,$$

(iv$_1$) *the difference inequalities*

$$\begin{aligned}\Delta_0 u^{(i,j)} &\leqslant \Phi(x^{(i)}, y^{(j)}, Au^{(i,j)}, [\![u^{(i,j)}]\!], \Delta u^{(i,j)}, h_i, k),\\ \Delta_0 v^{(i,j)} &\geqslant \Phi(x^{(i)}, y^{(j)}, Av^{(i,j)}, [\![v^{(i,j)}]\!], \Delta v^{(i,j)}, h_i, k)\end{aligned} \tag{12}$$

are satisfied for $(i, j) \in \Gamma$.

Under these assumptions

$$u^{(i,j)} \leqslant v^{(i,j)} \quad \text{on } E^*. \tag{13}$$

Proof. Suppose that assertion (13) is false. Then the set

$$Z = \{i \in \{0, 1, \ldots, \tilde{n}+1\}: u^{(i,j)} > v^{(i,j)} \text{ for some } j = (j_1, \ldots, j_n)\}$$

is non-empty. Defining $l = \min Z$, it is clear from (11) that $l > 0$ and that there exists $c = (c_1, \ldots, c_n)$ such that

$$u^{(i,j)} \leqslant v^{(i,j)} \quad \text{for } 0 \leqslant i \leqslant l-1,\ (i, j) \in \tilde{\Gamma},$$

and

$$u^{(l,c)} > v^{(l,c)}, \quad (l, c) \in \tilde{\Gamma}. \tag{14}$$

Defining the function $\tilde{w}^{(i,j)} = u^{(i,j)} - v^{(i,j)}$, $(i, j) \in \tilde{\Gamma}$, we have $(l-1, c) \in \Gamma$ and

$$\tilde{w}^{(l,c)} = A\tilde{w}^{(l-1,c)} + h_{l-1}[\Delta_0 u^{(l-1,c)} - \Delta_0 v^{(l-1,c)}].$$

By the hypothesis of our theorem we obtain for a point $\tilde{P} = (x^{(l-1)}, y^{(c)}, \tilde{z}, \tilde{p}, \tilde{q}, h_{l-1}, k) \in \Omega \times H$ the inequalities

$$\begin{aligned}\tilde{w}^{(l,c)} \leqslant{}& A\tilde{w}^{(l-1,c)} +\\ &+ h_{l-1}[\Phi(x^{(l-1)}, y^{(c)}, Au^{(l-1,c)}, [\![u^{(l-1,c)}]\!], \Delta u^{(l-1,c)}, h_{l-1}, k) -\\ &\qquad - \Phi(x^{(l-1)}, y^{(c)}, Av^{(l-1,c)}, [\![v^{(l-1,c)}]\!], \Delta v^{(l-1,c)}, h_{l-1}, k)]\\ ={}& A\tilde{w}^{(l-1,c)} + h_{l-1}\{\Phi_z(\tilde{P})[Au^{(l-1,c)} - Av^{(l-1,c)}] +\\ &+ \sum_{s\in N(r)} \Phi_{p_s}(\tilde{P})[u^{(l-1,c+s)} - v^{(l-1,c+s)}] +\\ &+ \sum_{\tau=1}^{n} \Phi_{q_\tau}(\tilde{P})[\Delta_\tau u^{(l-1,c)} - \Delta_\tau v^{(l-1,c)}]\}\\ ={}& A\tilde{w}^{(l-1,c)} + h_{l-1}\{\Phi_z(\tilde{P})A\tilde{w}^{(l-1,c)} + \sum_{s\in N(r)} \Phi_{p_s}(\tilde{P})\tilde{w}^{(l-1,c+s)} +\\ &+ \sum_{\tau\in J_1} \frac{1}{k_\tau}\Phi_{q_\tau}(\tilde{P})[A\tilde{w}^{(l-1,c)} - B_\tau\tilde{w}^{(l-1,c)}] +\\ &+ \sum_{\tau\in J_2} \frac{1}{k_\tau}\Phi_{q_\tau}(\tilde{P})[B_\tau\tilde{w}^{(l-1,c)} - A\tilde{w}^{(l-1,c)}]\}\end{aligned}$$

$$= \sum_{s\in N(r)} \tilde{w}^{(l-1,c+s)} \Big\{ a_s \Big[1 + h_{l-1} \Phi_z(\tilde{P}) + h_{l-1} \sum_{\tau\in I_1} \frac{1}{k_\tau} \Phi_{q_\tau}(\tilde{P}) -$$

$$- h_{l-1} \sum_{\tau\in I_2} \frac{1}{k_\tau} \Phi_{q_\tau}(\tilde{P}) \Big] + h_{l-1} \Big[\Phi_{p_s}(\tilde{P}) - \sum_{\tau\in I_1} \frac{1}{k_\tau} b_s^{(\tau)} \Phi_{q_\tau}(\tilde{P}) +$$

$$+ \sum_{\tau\in I_2} \frac{1}{k_\tau} b_s^{(\tau)} \Phi_{q_\tau}(\tilde{P}) \Big] \Big\}.$$

These estimates together with condition (d_0) of Assumption H_0 lead to the inequality $\tilde{w}^{(l,c)} \leqslant 0$, which contradicts (14). Hence the set Z is empty and (13) follows.

Remark 2. It is easy to see that Theorem 1 is true in the unbounded zone

$$\tilde{E} \in \{(x^{(i)}, y^{(j)}):\ x^{(i+1)} = x^{(i)} + h_i,\ i = 0, 1, \ldots, \tilde{n},$$
$$y^{(j)} = (y_1^{(j_1)}, \ldots, y_n^{(j_n)}),\ y_\tau^{(l)} = lk_\tau,\ l = 0, \pm 1, \pm 2, \ldots, \tau = 1, \ldots, n\}.$$

We can omit the assumption $k_\tau r \leqslant h_i M_\tau$, $\tau = 1, \ldots, n$, $i = 0, 1, \ldots, \tilde{n}$, in this case.

3. The convergence of the difference method. We introduce

ASSUMPTION H_1. Suppose that

(a_1) for $s \in N(r)$, $\tau = 1, \ldots, n$, we have

$$\sum_{s\in N(r)} a_s = 1, \qquad \sum_{s\in N(r)} b_s^{(\tau)} = 1, \qquad \tau = 1, \ldots, n,$$

(b_1) there exist constants L^*, L_s, $s \in N(r)$, such that $|\Phi_z(P)| \leqslant L^*$ for $P \in \Omega \times H$ and $|\Phi_{p_s}(P)| \leqslant L_s$ for $P \in \Omega \times H$, $s \in N(r)$.

We define

$$L = L^* + \sum_{s\in N(r)} L_s$$

and

$$S^{(k)}(x, r) = \{y:\ (x, y_1 + s_1 k_1, \ldots, y_n + s_n k_n) \in E \ \text{for}\ s \in N(r)\}.$$

If u is a function of class C^1 on E, then

$$\Psi(x, y, u, \xi, k) = \begin{cases} \dfrac{u(x+\xi, y) - Au(x, y)}{\xi} & \text{for } \xi \neq 0, \\ f(x, y, u(x, y), u_y(x, y)) & \text{for } \xi = 0, \end{cases}$$

where $x \in [x^{(0)}, x^{(0)} + a]$, $y \in S^{(k)}(x, r)$.

THEOREM 2. *Suppose that*

(i_2) *Assumptions* H_0 *and* H_1 *are satisfied,* (ii$_2$) *$u: E \to R$ is a function of class C^1 on E and there exists a function γ: $[0, h^{(0)}]^{\bar{n}+1} \times [0, k^{(0)}] \to R_+$ such that*

$$(15)\quad \left|\Phi\big(x^{(i)}, y^{(j)}, Au(x^{(i)}, y^{(j)}), [\![u(x^{(i)}, y^{(j)})]\!], \Delta u(x^{(i)}, y^{(j)}), h_i, k\big) - \Psi(x^{(i)}, y^{(j)}, u, h_i, k)\right| \leqslant \gamma(h, k),$$

where $(i, j) \in \Gamma$, $(h_i, k) \in H$, $h = (h_0, h_1, \ldots, h_{\bar{n}})$,

(iii$_2$) *$u(x^{(0)}, y) = \omega(y)$ for $y \in E_0$ and $v: E^* \to R$ is a solution of* (10).

Under these assumptions

$$(16)\qquad |u^{(i,j)} - v^{(i,j)}| \leqslant \gamma(h, k)\frac{e^{i\tilde{h}L} - 1}{L} \quad \textit{for } (i, j) \in \tilde{\Gamma},$$

where $\tilde{h} = \max_{0 \leqslant i \leqslant \bar{n}} h_i$.

Proof. At first we prove that

$$(17)\qquad v^{(i,j)} \leqslant u^{(i,j)} + \gamma(h, k)\frac{e^{i\tilde{h}L} - 1}{L} \quad \text{for } (i, j) \in \tilde{\Gamma}.$$

Let

$$\tilde{w}^{(i,j)} = u^{(i,j)} + \gamma(h, k)\frac{e^{i\tilde{h}L} - 1}{L} \quad \text{for } (i, j) \in \tilde{\Gamma}.$$

We prove that $\tilde{w}$ satisfies the difference inequality

$$(18)\qquad \Delta_0 \tilde{w}^{(i,j)} \geqslant \Phi(x^{(i)}, y^{(j)}, A\tilde{w}^{(i,j)}, [\![\tilde{w}^{(i,j)}]\!], \Delta\tilde{w}^{(i,j)}, h_i, k), \qquad (i, j) \in \Gamma.$$

In virtue of the assumptions of our theorem we obtain, for $(i, j) \in \Gamma$ and for a point $\tilde{P} = (x^{(i)}, y^{(j)}, \tilde{z}, \tilde{p}, \tilde{q}, h_i, k) \in \Omega \times H$

$$\begin{aligned}
\Delta_0 \tilde{w}^{(i,j)} &= \frac{1}{h_i}[u^{(i+1,j)} - Au^{(i,j)}] + \gamma(h, k)\frac{e^{i\tilde{h}L}(e^{\tilde{h}L} - 1)}{\tilde{h}L} \\
&= \Psi(x^{(i)}, y^{(j)}, u, h_i, k) + \gamma(h, k)\frac{e^{i\tilde{h}L}(e^{\tilde{h}L} - 1)}{\tilde{h}L} \\
&= \Phi(x^{(i)}, y^{(j)}, A\tilde{w}^{(i,j)}, [\![\tilde{w}^{(i,j)}]\!], \Delta\tilde{w}^{(i,j)}, h_i, k) + \\
&\qquad + \gamma(h, k)\frac{e^{i\tilde{h}L}(e^{\tilde{h}L} - 1)}{\tilde{h}L} + [\Psi(x^{(i)}, y^{(j)}, u, h_i, k) - \\
&\qquad\qquad - \Phi(x^{(i)}, y^{(j)}, Au^{(i,j)}, [\![u^{(i,j)}]\!], \Delta u^{(i,j)}, h_i, k)] + \\
&\qquad + [\Phi(x^{(i)}, y^{(j)}, Au^{(i,j)}, [\![u^{(i,j)}]\!], \Delta u^{(i,j)}, h_i, k) - \\
&\qquad\qquad - \Phi(x^{(i)}, y^{(j)}, A\tilde{w}^{(i,j)}, [\![\tilde{w}^{(i,j)}]\!], \Delta\tilde{w}^{(i,j)}, h_i, k)] \\
&\geqslant \Phi(x^{(i)}, y^{(j)}, A\tilde{w}^{(i,j)}, [\![\tilde{w}^{(i,j)}]\!], \Delta\tilde{w}^{(i,j)}, h_i, k) +
\end{aligned}$$

$$+\gamma(h,k)\frac{e^{i\tilde{h}L}(e^{\tilde{h}L}-1)}{\tilde{h}L}-\gamma(h,k)-L^*|Au^{(i,j)}-A\tilde{w}^{(i,j)}|-$$

$$-\sum_{s\in N(r)}L_s|u^{(i,j+s)}-\tilde{w}^{(i,j+s)}|+\sum_{\tau=1}^{n}\Phi_{q_\tau}(\tilde{P})[\Delta_\tau u^{(i,j)}-\Delta_\tau \tilde{w}^{(i,j)}].$$

Next, we have

$$\begin{aligned}\Delta_0\tilde{w}^{(i,j)} &\geqslant \Phi(x^{(i)},y^{(j)},A\tilde{w}^{(i,j)},[\![\tilde{w}^{(i,j)}]\!],\Delta\tilde{w}^{(i,j)},h_i,k)+\\ &+\gamma(h,k)\frac{e^{i\tilde{h}L}(e^{\tilde{h}L}-1)}{\tilde{h}L}-\gamma(h,k)-L^*\Big|\sum_{s\in N(r)}a_s[u^{(i,j+s)}-\tilde{w}^{(i,j+s)}]\Big|-\\ &-\sum_{s\in N(r)}L_s\gamma(h,k)\frac{e^{i\tilde{h}L}-1}{L}+\\ &+\sum_{\tau\in I_1}\Phi_{q_\tau}(\tilde{P})\frac{1}{k_\tau}\Big[\sum_{s\in N(r)}(a_s-b_s^{(\tau)})(u^{(i,j+s)}-\tilde{w}^{(i,j+s)})\Big]+\\ &+\sum_{\tau\in I_2}\Phi_{q_\tau}(\tilde{P})\frac{1}{k_\tau}\Big[\sum_{s\in N(r)}(b_s^{(\tau)}-a_s)(u^{(i,j+s)}-\tilde{w}^{(i,j+s)})\Big]\\ &=\Phi(x^{(i)},y^{(j)},A\tilde{w}^{(i,j)},[\![\tilde{w}^{(i,j)}]\!],\Delta\tilde{w}^{(i,j)},h_i,k)+\\ &+\gamma(h,k)\left|\frac{e^{i\tilde{h}L}(e^{\tilde{h}L}-1)}{\tilde{h}L}-1\right|-\gamma(h,k)\frac{e^{i\tilde{h}L}-1}{L}\Big[L^*+\sum_{s\in N(r)}L_s\Big]+\\ &+\sum_{\tau\in I_1}\Phi_{q_\tau}(\tilde{P})\frac{1}{k_\tau}\left[-\gamma(h,k)\frac{e^{i\tilde{h}L}-1}{L}\right]\sum_{s\in N(r)}(a_s-b_s^{(\tau)})+\\ &+\sum_{\tau\in I_2}\Phi_{q_\tau}(\tilde{P})\frac{1}{k_\tau}\left[-\gamma(h,k)\frac{e^{i\tilde{h}L}-1}{L}\right]\sum_{s\in N(r)}(b_s^{(\tau)}-a_s).\end{aligned}$$

Using Assumption H_1 we obtain

$$\begin{aligned}\Delta_0\tilde{w}^{(i,j)} &\geqslant \Phi(x^{(i)},y^{(j)},A\tilde{w}^{(i,j)},[\![\tilde{w}^{(i,j)}]\!],\Delta\tilde{w}^{(i,j)},h_i,k)+\gamma(h,k)e^{i\tilde{h}L}\left[\frac{e^{\tilde{h}L}-1}{\tilde{h}L}-1\right]\\ &\geqslant \Phi(x^{(i)},y^{(j)},A\tilde{w}^{(i,j)},[\![\tilde{w}^{(i,j)}]\!],\Delta\tilde{w}^{(i,j)},h_i,k).\end{aligned}$$

This completes the proof of (18).

Since v satisfies (10) and $v^{(0,j)}=w^{(0,j)}$ for $j\in\Gamma_0$, we conclude by Theorem 1 that (17) holds true.

In a similar way we prove that

$$u^{(i,j)}-\gamma(h,k)\frac{e^{i\tilde{h}L}-1}{L}\leqslant v^{(i,j)},\qquad (i,j)\in\tilde{\Gamma}. \tag{19}$$

From (17) and (19) we obtain (16).

Remark 3. Suppose that $h_i = h^*$ for $i = 0, 1, \ldots, \tilde{n}$. It is clear that Theorem 2 holds with assumption (15) replaced by

$$|\Phi(x, y, Au(x, y), [\![u(x, y)]\!], \Delta u(x, y), h^*, k) - \Psi(x, y, u, h^*, k)| \leqslant \gamma^*(h^*, k)$$

for $x \in [x^{(0)}, x^{(0)}+a]$, $y \in S^{(k)}(x, r)$, $(h^*, k) \in H$ and for a function γ^*: $[0, h^{(0)}] \times [0, k^{(0)}] \to R_+$.

ASSUMPTION H_2. Suppose that

(a_2) $\bar{u}$ is a solution of (1), which is of class C^2 on E,

(b_2) the constants a_s, $b_s^{(\tau)}$ satisfy the conditions

$$\sum_{s \in N(r)} s_\tau a_s = 0 \qquad \text{for } \tau = 1, \ldots, n,$$

$$\sum_{s \in N(r)} s_\tau b_s^{(\tau')} = -1 \qquad \text{for } \tau = 1, \ldots, n, \ \tau' \in I_1,$$

and

$$\sum_{s \in N(r)} s_\tau b_s^{(\tau')} = 1 \qquad \text{for } \tau = 1, \ldots, n, \ \tau' \in I_2,$$

(c_2) $H = \{(\xi, \eta) \in [0, h^{(0)}] \times [0, k^{(0)}]: r\eta_\tau \leqslant M_\tau \xi, \ \tau = 1, \ldots, n\}$ and $\Phi \in C(\Omega \times H, R)$.

We adopt the following notation:

$$(20) \quad \begin{aligned}
\gamma_1(h, k) &= \max_{(i,j)\in\Gamma} |\Phi(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, [\![\bar{u}^{(i,j)}]\!], \Delta\bar{u}^{(i,j)}, h_i, k) - \\
&\qquad - \Phi(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, [\![\bar{u}^{(i,j)}]\!], \Delta\bar{u}^{(i,j)}, 0, 0)|, \\
\gamma_2(h, k) &= \max_{(i,j)\in\Gamma} |f(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, \Delta\bar{u}^{(i,j)}) - f(x^{(i)}, y^{(j)}, \bar{u}^{(i,j)}, \bar{u}_y^{(i,j)})|, \\
\gamma_3(h, k) &= \max_{\substack{(i,j)\in\Gamma \\ \tilde{\theta}\in[0,1]}} |f(x^{(i)}, y^{(j)}, \bar{u}^{(i,j)}, \bar{u}_y^{(i,j)}) - \\
&\qquad - f(x^{(i)}+\tilde{\theta}h_i, y^{(j)}, \bar{u}(x^{(i)}+\tilde{\theta}h_i, y^{(j)}), \bar{u}_y(x^{(i)}+\tilde{\theta}h_i, y^{(j)}))|, \\
\gamma_4(k) &= \max_{\substack{(i,j)\in\Gamma \\ \tilde{\theta}\in[0,1]}} \sum_{\tau,\tau'=1}^{n} \frac{M_\tau k_\tau}{2r} \Big| \sum_{s\in N(r)} a_s s_\tau s_{\tau'} \bar{u}_{y_\tau y_{\tau'}}(Q_{ij}) \Big|,
\end{aligned}$$

where $Q_{ij} = (x^{(i)}, y_1^{(j_1)} + \tilde{\theta}s_1 k_1, \ldots, y_n^{j_n} + \tilde{\theta}s_n k_n)$ and

$$\tilde{\gamma}(h, k) = \gamma_1(h, k) + \gamma_2(h, k) + \gamma_3(h, k) + \gamma_4(k).$$

THEOREM 3. *Suppose that*

(i_3) *Assumptions* H_1, H_2 *are satisfied and* v *is a solution of* (10), (ii$_3$) *f is continuous on $E \times R^{1+n}$ and conditions* (a$_0$)–(d$_0$) *of Assumption* H$_0$ *hold with* H *defined in* H$_2$,

(iii$_3$) *the function Φ satisfies the following consistency condition*:

$$\Phi(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, [\![\bar{u}^{(i,j)}]\!], \Delta\bar{u}^{(i,j)}, 0, 0) \tag{21}$$
$$= f(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, \Delta\bar{u}^{(i,j)}), \quad (i, j) \in \Gamma,$$

where $\bar{u}$ is a solution of (1),

(iv$_3$) *there exists $c_0 > 0$ such that $k_\tau/k_{\tau'} \leqslant c_0$, τ, $\tau' = 1, \ldots, n$.*

Under these assumptions the difference method (10) *is convergent to $\bar{u}$, i.e.*,

$$\lim_{h,k \to 0} (\bar{u}^{(i,j)} - v^{(i,j)}) = 0, \quad (i, j) \in \tilde{\Gamma}. \tag{22}$$

Proof. First we prove that

$$\Phi(x^{(i)}, y^{(j)}\ A\bar{u}^{(i,j)}, [\![\bar{u}^{(i,j)}]\!], \Delta\bar{u}^{(i,j)}, h_i, k) - \Psi(x^{(i)}, y^{(j)}, \bar{u}, h_i, k) \leqslant \tilde{\gamma}(h, k) \tag{23}$$

for $(i, j) \in \Gamma$, $(h_i, k) \in H$, $h = (h_0, h_1, \ldots, h_{\bar{n}})$.

By the assumptions of our theorem we obtain, for some $\bar{\theta}$, $\tilde{\theta} \in (0, 1)$,

$$\begin{aligned}
\Psi(x^{(i)}, y^{(j)}, \bar{u}, h_i, k) &= \frac{1}{h_i}[\bar{u}(x^{(i)} + h_i, y^{(j)}) - A\bar{u}(x^{(i)}, y^{(j)})] \\
&= \frac{1}{h_i}\{\bar{u}(x^{(i)}, y^{(j)}) + h_i \bar{u}_x(x^{(i)} + \bar{\theta}h_i, y^{(j)}) - \\
&\quad - \sum_{s \in N(r)} a_s \big[\bar{u}(x^{(i)}, y^{(j)}) + \sum_{\tau=1}^{n} s_\tau k_\tau \bar{u}_{y_\tau}(x^{(i)}, y^{(j)}) + \\
&\quad + \tfrac{1}{2} \sum_{\tau,\tau'=1}^{n} (s_\tau k_\tau)(s_{\tau'} k_{\tau'}) \bar{u}_{y_\tau y_{\tau'}}(x^{(i)}, y_1^{(j_1)} + \tilde{\theta}s_1 k_1, \ldots, y_n^{(j_n)} + \tilde{\theta}s_n k_n)\big]\} \\
&\geqslant f(x^{(i)} + \bar{\theta}h_i, y^{(j)}, \bar{u}(x^{(i)} + \bar{\theta}h_i, y^{(j)}), \bar{u}_y(x^{(i)} + \bar{\theta}h_i, y^{(j)})) - \gamma_4(k).
\end{aligned}$$

Thus

$$\begin{aligned}
&\Phi(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, [\![\bar{u}^{(i,j)}]\!], \Delta\bar{u}^{(i,j)}, h_i, k) - \Psi(x^{(i)}, y^{(j)}, \bar{u}, h_i, k) \\
&\quad \leqslant [\Phi(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, [\![\bar{u}^{(i,j)}]\!], \Delta\bar{u}^{(i,j)}, h_i, k) - \\
&\qquad - \Phi(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, [\![\bar{u}^{(i,j)}]\!], \Delta\bar{u}^{(i,j)}, 0, 0)] + \\
&\qquad + [f(x^{(i)}, y^{(j)}, A\bar{u}(x^{(i)}, y^{(j)}), \Delta\bar{u}(x^{(i)}, y^{(j)})) - \\
&\qquad - f(x^{(i)}, y^{(j)}, \bar{u}(x^{(i)}, y^{(j)}), \bar{u}_y(x^{(i)}, y^{(j)}))] + \\
&\qquad + [f(x^{(i)}, y^{(j)}, \bar{u}(x^{(i)}, y^{(j)}), \bar{u}_y(x^{(i)}, y^{(j)})) - \\
&\qquad - f(x^{(i)} + \bar{\theta}h_i, y^{(j)}, \bar{u}(x^{(i)} + \bar{\theta}h_i, y^{(j)}), \bar{u}_y(x^{(i)} + \bar{\theta}h_i, y^{(j)}))] + \gamma_4(k) \\
&\quad \leqslant \tilde{\gamma}(h, k).
\end{aligned}$$

In a similar way we obtain the inverse inequality with respect to (23). Thus condition (ii_2) of Theorem 2 is satisfied for the functions $u = \bar{u}$, $\gamma = \tilde{\gamma}$. Then we have

$$(24)\qquad |\bar{u}^{(i,j)} - v^{(i,j)}| \leqslant \tilde{\gamma}(h, k)\frac{e^{i\tilde{h}L}-1}{L} \quad \text{for } (i, j) \in \tilde{\Gamma},$$

where $\tilde{h} = \max_{0 \leqslant i \leqslant n} h_i$.

Now we prove that

$$(25)\qquad \lim_{h,k \to 0} \tilde{\gamma}(h, k) = 0.$$

It follows from (20) and from the continuity of f, Φ, $\bar{u}$, $\bar{u}_y$ that

$$\lim_{h,k \to 0} \gamma_1(h, k) = \lim_{h,k \to 0} \gamma_3(h, k) = \lim_{k \to 0} \gamma_4(k) = 0.$$

We prove that

$$(26)\qquad \lim_{h,k \to 0} \gamma_2(h, k) = 0.$$

Since $\bar{u}$ is of class C^2 on E, $|\bar{u}_{y_\tau y_{\tau'}}(x, y)|$, $|\bar{u}_{y_\tau}(x, y)| \leqslant C$, $(x, y) \in E$ for some $C \geqslant 0$. It follows from Assumptions H_1, H_2 that

$$(27)\qquad |A\bar{u}^{(i,j)} - \bar{u}^{(i,j)}| \leqslant C \sum_{s \in N(r)} \sum_{\tau=1}^{n} |s_\tau k_\tau a_s|$$

and

$$(28)\qquad |\Delta_\tau \bar{u}^{(i,j)} - \bar{u}^{(i,j)}_{y_\tau}| \leqslant \frac{C}{2k_\tau}\Big\{\sum_{s \in N(r)} \sum_{\tau',\tau''=1}^{n} [|a_s s_{\tau'} s_{\tau''} k_{\tau'} k_{\tau''}| + |b_s^{(\tau)} s_{\tau'} s_{\tau''} k_{\tau'} k_{\tau''}|]\Big\},$$

$$\tau = 1, \ldots, n.$$

Since $|k_\tau| \leqslant c_0 |k_{\tau'}|$, $\tau, \tau' = 1, \ldots, n$, we see that (27), (28) and definition (20) imply (26), which completes the proof of (25).

Condition (25) and estimate (24) lead to (22).

Remark 4. We assume in (21) that the consistency condition is satisfied along a given solution of (1). In Theorem 3 we can assume instead of (21) that

$$\Phi(x, y, z, p, q, 0, 0) = f(x, y, z, q), \quad (x, y, z, p, q) \in \Omega.$$

THEOREM 3′. *Suppose that*

(i_3') *assumptions* (i_3), (ii_3), (iv_3) *of Theorem* 3 *hold*, (ii'$_3$) *the function* Φ *satisfies the following consistency condition*:

$$(29)\quad \Phi(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, [\![\bar{u}^{(i,j)}]\!], \Delta\bar{u}^{(i,j)}, 0, 0) = f(x^{(i)}, y^{(j)}, \bar{u}(x^{(i)}, y^{(j)}), \Delta\bar{u}(x^{(i)}, y^{(j)})), \quad (i,j)\in\Gamma,$$

where $\bar{u}$ *is a solution of* (1).

Under these assumptions we have

$$\lim_{h,k\to 0} (\bar{u}^{(i,j)} - v^{(i,j)}) = 0, \quad (i,j)\in\tilde{\Gamma}.$$

This theorem can be deduced from Theorem 2. Estimate (15) is true for $\gamma(h,k) = \gamma_1(h,k)+\tilde{\gamma}_2(h,k)+\gamma_3(h,k)+\gamma_4(k)$, where $\gamma_1, \gamma_3, \gamma_4$ are defined by (20) and

$$\tilde{\gamma}_2(h,k) = \max_{(i,j)\in\Gamma} |f(x^{(i)}, y^{(j)}, \bar{u}^{(i,j)}, \Delta\bar{u}^{(i,j)}) - f(x^{(i)}, y^{(j)}, \bar{u}^{(i,j)}, \bar{u}_y^{(i,j)})|.$$

Remark 5. Clearly, condition (29) is satisfied if

$$\Phi(x, y, z, p, q, 0, 0) = f(x, y, p_\theta, q) \quad \text{on } \Omega, \text{ where } \theta = (0, \ldots, 0)\in R^n.$$

Theorem 2 gives sufficient conditions for the stability of the difference problem (10). Although Theorem 3 guarantees that a consistent and stable method is convergent, it is hardly useful for purposes of estimating the error of the method; in particular, it fails to indicate the order of the error. The next theorem states that if some additional conditions are satisfied, then the error of the difference method is of α-th order.

THEOREM 4. *Suppose that*

(i$_4$) *Assumptions* H_0 *and* H_1 *are satisfied*,

(ii$_4$) $\bar{u}$ *is a solution of* (1), *which is of class* C^1 *on* $\dot{E}$ *and* v *is a solution of* (10),

(iii$_4$) *there exist constants* $C > 0$, $\alpha > 0$ *such that*

$$(30)\quad |\Phi(x^{(i)}, y^{(j)}, A\bar{u}^{(i,j)}, [\![\bar{u}^{(i,j)}]\!], \Delta\bar{u}^{(i,j)}, h_i, k) - \Psi(x^{(i)}, y^{(j)}, \bar{u}, h_i, k)| \leqslant C\|h\|^\alpha, \quad (i,j)\in\Gamma,$$

where $(h_i, k)\in H$, $\|h\| = \max_{0\leqslant i\leqslant n} h_i$.

Under these assumptions the difference method (10) *is convergent and*

$$|\bar{u}^{(i,j)} - v^{(i,j)}| \leqslant C\|h\|^\alpha \frac{e^{i\|h\|L}-1}{L} \quad \textit{for } (i,j)\in\tilde{\Gamma}.$$

This theorem follows from Theorem 2 for $\gamma(h,k) = C\|h\|^\alpha$.

Remark 6. Suppose that $h_i = \tilde{h}$ for $i = 0, 1, \ldots, \tilde{n}$. It is clear that

Theorem 4 holds true if (30) is replaced by

$$|\Phi(x, y, A\bar{u}(x, y), [\![\bar{u}(x, y)]\!], \Delta\bar{u}(x, y), \tilde{h}, k) - \Psi(x, y, \bar{u}, \tilde{h}, k)| \leqslant C\tilde{h}^{\alpha}$$
$$\text{for } x \in [x^{(0)}, x^{(0)}+a],\ y \in S^{(k)}(x, r).$$

Remark 7. All theorems from this section can be reformulated for the case when E is the unbounded zone defined in Remark 2.

4. Examples. If $r \geqslant 0$ and $s = (s_1, \ldots, s_n) \in N(r)$, we write $\tau(s) = (s_1, \ldots, s_{\tau-1}, s_\tau+1, s_{\tau+1}, \ldots, s_n)$ and $-\tau(s) = (s_1, \ldots, s_{\tau-1}, s_\tau-1, s_{\tau+1}, \ldots, s_n)$. In particular, $\tau(\theta) = (0, \ldots, 0, 1, 0, \ldots, 0)$, $-\tau(\theta) = (0, \ldots, 0, -1, 0, \ldots, 0)$ (1, -1 standing on τ-th place), where $\theta = (0, \ldots, 0) \in R^n$.

I. Suppose that f satisfies assumptions (i)–(iii) of Theorem (T). Let $r = 1$, $a_\theta = 1$ and $a_s = 0$ for $s \neq \theta$, $s \in N(1)$. Let I_1 and I_2 be us in Section 1. Suppose that $\tau \in I_1$ and $b^{(\tau)}_{-\tau(\theta)} = 1$, $b^{(\tau)}_s = 0$ if $s \neq -\tau(\theta)$, $s \in N(1)$. Let $\tau \in I_2$; we define $b^{(\tau)}_{\tau(\theta)} = 1$ and $b^{(\tau)}_s = 0$ for $s \neq \tau(\theta)$, $s \in N(1)$. If $\Phi(x, y, z, p, q, \xi, \eta) = f(x, y, z, q)$, then the one-step method (10) coincides with Euler's method considered in Theorem (T).

II. Suppose that $r = 1$ and

$$a_{\tau(\theta)} = a_{-\tau(\theta)} = \frac{1}{2n}, \quad \tau = 1, \ldots, n,$$
$$a_s = 0 \quad \text{if } s \neq \tau(\theta) \text{ and } s \neq -\tau(\theta),\ \tau = 1, \ldots, n,\ s \in N(1). \tag{31}$$

Let

$$b^{(\tau)}_{\tau'(\theta)} = \frac{1}{2n} - \tfrac{1}{2}\delta_{\tau\tau'}, \quad \tau, \tau' = 1, \ldots, n,$$
$$b^{(\tau)}_{-\tau'(\theta)} = \frac{1}{2n} + \tfrac{1}{2}\delta_{\tau\tau'}, \quad \tau, \tau' = 1, \ldots, n, \tag{32}$$
$$b^{(\tau)}_s = 0 \quad \text{for all remaining } s \in N(1).$$

Consider the difference method

$$\Delta_0 w^{(i,j)} = f(x^{(i)}, y^{(j)}, Aw^{(i,j)}, \Delta w^{(i,j)}), \quad (i, j) \in \Gamma,$$
$$w^{(0,j)} = \omega(y^{(j)}), \quad j \in \Gamma_0, \tag{33}$$

where $\Delta = (\Delta_1, \ldots, \Delta_n)$, $x^{(i)} = x^{(0)} + ih$, $i = 0, 1, \ldots, n_0$, $n_0 h = a$, $y^{(j)}$ are defined by (4) and

$$Aw^{(i,j)} = \frac{1}{2n}\sum_{\tau=1}^{n} (w^{(i,\tau(j))} + w^{(i,-\tau(j))})$$
$$\Delta_0 w^{(i,j)} = \frac{1}{h}[w^{(i+1,j)} - Aw^{(i,j)}], \tag{34}$$
$$\Delta_\tau w^{(i,j)} = \frac{1}{2k_\tau}(w^{(i,\tau(j))} - w^{(i,-\tau(j))}), \quad \tau = 1, \ldots, n.$$

Assume that

$$\frac{k_\tau}{k_{\tau'}} \leqslant c_0, \quad \tau, \tau' = 1, \ldots, n, \tag{35}$$

for some $c_0 > 0$.

LEMMA 1. *Suppose that*

(α_1) *the function* f: $E \times R^{1+n} \to R$ *of the variables* (x, y, z, q) *is continuous, the derivatives* f_z, $f_q = (f_{q_1}, \ldots, f_{q_n})$ *exist, are continuous and bounded on* $E \times R^{1+n}$,

(β_1) $\bar{u}$ *is a solution of* (1), *which is of class* C^2 *on* E, *and* $k_i \leqslant hM_i$ *for* $i = 1, \ldots, n$,

(γ_1) $\bar{v}$ *is a solution of* (33), (34) *and for* $(x, y, z, q) \in E \times R^{1+n}$ *we have*

$$1 + hf_z(x, y, z, q) - n\frac{h}{k_i}|f_{q_i}(x, y, z, q)| \geqslant 0, \quad i = 1, \ldots, n.$$

Then

$$\lim_{h,k \to 0} (\bar{u}^{(i,j)} - \bar{v}^{(i,j)}) = 0, \quad (i, j) \in \tilde{\Gamma}.$$

This lemma follows from Theorem 3 for $\Phi(x, y, z, p, q, \xi, \eta) = f(x, y, z, q)$, $I_1 = \{1, \ldots, n\}$. We do not assume that f_{q_i}, $i = 1, \ldots, n$, satisfy (7).

Suppose that $r = 1$ and that the difference operators Δ_0, $\Delta = (\Delta_1, \ldots, \Delta_n)$ are defined by (31), (32), (34), (35). Consider the difference method

$$\begin{aligned} &\Delta_0 w^{(i,j)} = f(x^{(i)}, y^{(j)}, w^{(i,j)}, \Delta w^{(i,j)}), \quad (i, j) \in \Gamma, \\ &w^{(0,j)} = \omega(y^{(j)}), \quad j \in \Gamma_0. \end{aligned} \tag{36}$$

LEMMA 2. *Suppose that*

(α_2) *conditions* (α_1), (β_1) *of Lemma* 1 *hold*,

(β_2) $\bar{v}$ *is a solution of* (36) *and for* $(x, y, z, q) \in E \times R^{1+n}$ *we have*

$$f_z(x, y, z, q) \geqslant 0,$$

$$1 + 2nhf_z(x, y, z, q) - n\frac{h}{k_i}|f_{q_i}(x, y, z, q)| \geqslant 0, \quad i = 1, \ldots, n.$$

Then

$$\lim_{h,k \to 0} (\bar{u}^{(i,j)} - \bar{v}^{(i,j)}) = 0, \quad (i, j) \in \tilde{\Gamma}.$$

This property of (36) follows from Theorem 3 for

$$\Phi(x, y, z, p, q, \xi, \eta) = f(x, y, p_0, q), \quad I_1 = \{1, \ldots, n\}.$$

III. We consider the initial value problem

$$(37)\qquad \begin{aligned} &z_x(x, y) = g\big(x, y, z(x, y)\big) - Mz_y(x, y), \qquad (x, y)\in E,\\ &z(x^{(0)}, y) = \omega(y), \qquad y\in E_0, \end{aligned}$$

where x, y are scalars, $M > 0$ and

$$(38)\qquad \begin{aligned} &E = \{(x, y)\colon\ x\in[x^{(0)}, x^{(0)}+a],\ |y-y^{(0)}| \leqslant b - M(x-x^{(0)})\},\\ &E_0 = [y^{(0)}-b,\ y^{(0)}+b]. \end{aligned}$$

For h, $k > 0$ we define

$$\begin{aligned} &x^{(i)} = x^{(0)}+ih, \qquad i = 0, 1, \ldots, n_0,\\ &y^{(j)} = y^{(0)}+jk, \qquad j = -\tilde{n}, -\tilde{n}+1, \ldots, -1, 0, 1, \ldots, \tilde{n}, \end{aligned}$$

where $n_0 h = a$, $\tilde{n}k = b$.

ASSUMPTION H_3. Suppose that

(a_3) the function g of the variables (x, y, z) is of class C^1 on $E\times R$, the functions g, g_x, g_y, g_z are bounded on $E\times R$,

(b_3) the derivatives g_{xz}, g_{zz}, g_{yz} exist and are continuous on $E\times R$.

LEMMA 3. *If*

(α_3) *Assumption* H_3 *is satisfied and* $g_z(x, y, z) > 0$ *on* $E\times R$,

(β_3) $\bar{u}\colon E\to R$ *is a function of class* C^3 *and satisfies* (37),

(γ_3) $v\colon E^*\to R$ *is a solution of the difference problem*

$$(39)\qquad \begin{aligned} w^{(i+1,j)} = {} & \frac{w^{(i,j+1)}+w^{(i,j-1)}}{2} + \\ & + h\Phi\left(x^{(i)}, y^{(j)}, \frac{w^{(i,j+1)}+w^{(i,j-1)}}{2}, \frac{w^{(i,j+1)}-w^{(i,j-1)}}{2k}\right), \qquad (i, j)\in\tilde{\Gamma},\\ & w^{(0,j)} = \omega(y^{(j)}), \qquad y^{(j)}\in E_0, \end{aligned}$$

where $k = Mh$,

$$\tilde{\Gamma} = \{(i, j)\colon\ i = 0, 1, \ldots, n_0-1,\ (x^{(i)}, y^{(j)})\in E\}$$

and

$$(40)\qquad \begin{aligned} \Phi(x, y, z, q, h) = {} & g(x, y, z) - Mq + \tfrac{1}{2}h[g_x(x, y, z)+\\ & + g_z(x, y, z)g(x, y, z) - 2Mq\,g_z(x, y, z) - Mg_y(x, y, z)], \end{aligned}$$

then there exists $C > 0$ *such that*

$$|\bar{u}(x^{(i)}, y^{(j)}) - v(x^{(i)}, y^{(j)})| \leqslant Ch^2 \quad \textit{for } (x^{(i)}, y^{(j)})\in E.$$

Proof. This lemma follows from Theorem 3 for $n = 1$, $r = 1$, $\alpha = 2$,

$$\Delta_0 z(x, y) = \frac{1}{h}\left[z(x+h, y) - \frac{z(x, y+k)+z(x, y-k)}{2} \right], \tag{41}$$

$$\Delta_1 z(x, y) = \frac{1}{2k}[z(x, y+k) - z(x, y-k)]$$

(i.e., $a_{-1} = \frac{1}{2}$, $a_0 = 0$, $a_1 = \frac{1}{2}$, $b_{-1} = 1$, $b_0 = b_1 = 0$) and for Φ given by (40).

IV. Now we give an example of the difference method of the second order which is patterned on methods of Runge–Kutta type.

LEMMA 4. *Suppose that*

(α_4) *Assumption* H_3 *is satisfied and* $g_z(x, y, z) \geqslant 0$ *on* $E \times R$,

(β_4) $\bar{u}$: $E \to R$ *is of class* C^3 *and satisfies* (37),

(γ_4) v: $E^* \to R$ *is a solution of the difference problem* (39), *where* $k = Mh$ *and*

$$\Phi(x, y, z, q, h) \tag{42}$$

$$= (1-\beta) g(x, y, z) + \beta g\left(x + \frac{h}{2\beta}, y, z + \frac{h}{2\beta}(g(x, y, z) - Mq)) - \right.$$

$$\left. - M\tfrac{1}{2}h[g_y(x, y, z) + g_z(x, y, z) q] \right), \quad \beta \in R, \ \beta \neq 0.$$

Then there exists $C > 0$ *such that*

$$|\bar{u}(x^{(i)}, y^{(j)}) - v(x^{(i)}, y^{(j)})| \leqslant Ch^2 \quad \textit{for } (x^{(i)}, y^{(j)}) \in E.$$

This lemma follows from Theorem 3 for $n = 1$, $r = 1$, $\alpha = 2$, Δ_0, Δ_1 given by (41) and for Φ defined by (42).

V. Now we give an example of a method of the third order. We consider problem (1) for $n = 1$. Let E and E_0 be the sets defined in Section 1. Suppose that f is of class C^3 on $E \times R^2$ and $\bar{u}$ is a solution of (1), which is of class C^4 on E. Let $S = (x, y, \bar{u}(x, y), \bar{u}_y(x, y))$. Suppose that F is a function of the variables (x, y, z, q) and that F is of class C^1 on $E \times R^2$. We define

$$U_x F(S) = F_x(S) + F_z(S) F(S) + F_q(S)[F_y(S) + F_z(S)\bar{u}_y(x, y) + F_q(S)\bar{u}_{yy}(x, y)],$$

$$U_y F(S) = F_y(S) + F_z(S)\bar{u}_y(x, y) + F_q(S)\bar{u}_{yy}(x, y).$$

Suppose that the operators A, B satisfy

$$\sum_{i=-r}^{r} a_i = 1, \quad \sum_{i=-r}^{r} b_i = 1, \quad \sum_{i=-r}^{r} ia_i = 0, \quad \sum_{i=-r}^{r} ib_i = -1, \tag{43}$$

$$\sum_{i=-r}^{r} (a_i - b_i)^2 i^2 = 0, \quad \sum_{i=-r}^{r} (a_i - b_i)^3 i^3 = 0.$$

Let $P=(x, y, z, p, q, h, k)$, $p=(p_{-r}, \ldots, p_{-1}, p_0, p_1, \ldots, p_r)$, $Q=(x, y, p_0, q)$ and

$$V_x^{(h,k)}F(Q) = F_x(Q)+F_z(Q)F(Q)+F_q(Q)[F_y(Q)+F_z(Q)q+F_q(Q)G(z, p, h, k)],$$

$$V_y^{(h,k)}F(Q) = F_y(Q)+F_z(Q)q+F_q(Q)G(z, p, h, k),$$

where G is a given function.

Let Ψ be the exact relative increment function defined in Section 3. Then we have

$$\begin{aligned}(44)\quad \Psi(x, y, \bar{u}, h, k) &= \frac{1}{h}[\bar{u}(x+h, y)-A\bar{u}(x, y)]\\ &= \bar{u}_x(x, y)+\tfrac{1}{2}h\bar{u}_{xx}(x, y)+\frac{h}{3!}\bar{u}_{xxx}(x, y)+\frac{h}{4!}\bar{u}_{xxxx}(x+\theta_1 h, y)-\\ &\quad -\frac{k^2}{2h}\sum_{i=-r}^{r} i^2 a_i \bar{u}_{yy}(x, y)-\frac{k^3}{3!\,h}\bar{u}_{yyy}(x, y)\sum_{i=-r}^{r} i^3 a_i-\\ &\quad -\frac{k^4}{4!\,h}\sum_{i=-r}^{r} i^4 a_i \bar{u}_{yyyy}(x, y+\theta_2^{(i)}k),\end{aligned}$$

where θ_1, $\theta_2^{(i)}\in(0, 1)$ and

$$\begin{aligned}&\bar{u}_x(x, y) = f(S), \qquad \bar{u}_{xx}(x, y) = U_x f(S),\\ &\bar{u}_{xxx}(x, y) = U_x f_x(S)+f(S)U_x f_z(S)+f_z(S)U_x f(S)+\\ (45)\quad &\qquad + U_x f_q(S)U_y f(S)+f_q(S)[U_x f_y(S)+U_x f_z(S)\bar{u}_y(x, y)+\\ &\qquad +f_z(S)U_y f(S)+U_x f_q(S)\bar{u}_{yy}(x, y)+f_q(S)\bar{u}_{xyy}(x, y)],\\ &\bar{u}_{xyy}(x, y) = U_y f_y(S)+U_y f_z(S)\bar{u}_y(x, y)+\\ &\qquad +f_z(S)\bar{u}_{yy}(x, y)+U_y f_q(S)\bar{u}_{yy}(x, y)+f_q(S)\bar{u}_{yyy}(x, y).\end{aligned}$$

We define an increment function Φ by

$$\begin{aligned}(46)\quad \Phi(P) &= f(Q)+\tfrac{1}{2}hV_x^{(h,k)}f_x(Q)+\tfrac{1}{3}h\{V_x^{(h,k)}f_x(Q)+\\ &\quad +f(Q)V_x^{(h,k)}f_z(Q)+f_z(Q)V_x^{(h,k)}f(Q)+\\ &\quad +V_x^{(h,k)}f_q(Q)V_y^{(h,k)}f(Q)+f_q(Q)[V_x^{(h,k)}f_y(Q)+\\ &\quad +V_x^{(h,k)}f_z(Q)q+f_z(Q)V_y^{(h,k)}f(Q)+\\ &\qquad +V_x^{(h,k)}f_q(Q)G(z, p, h, k)+f_q(Q)\tilde{G}(Q)]\}-\\ &\quad -\frac{k^2}{2h}\sum_{i=-r}^{r} i^2 a_i G(z, p, h, k)-\frac{k^3}{3!\,h}G_0(z, p, h, k)\sum_{i=-r}^{r} i^3 a_i,\end{aligned}$$

where

$$(47)\qquad \tilde{G}(Q) = V_y^{(h,k)} f_y(Q) + V_y^{(h,k)} f_z(Q)\, q + f_z(Q)\, G(z, p, h, k) + \\ + V_y^{(h,k)} f_q(Q)\, G(z, p, h, k) + f_q(Q)\, G_0(z, p, h, k).$$

Suppose that $k = C_0 h$ for some $C_0 > 0$. In virtue of (28) we have

$$(48)\qquad |\Delta\bar{u}(x, y) - \bar{u}_y(x, y)| \leqslant D_1 h^3$$

and

$$(49)\qquad \left|f(x, y, \bar{u}(x, y), \Delta\bar{u}(x, y)) - f(x, y, \bar{u}(x, y), \bar{u}_y(x, y))\right| \leqslant D_1 h^3$$

for some $D_1 > 0$.

Now we define functions G and G_0. Let $r = 3$ and

$$(50)\qquad \begin{aligned} Az(x, y) &= \tfrac{1}{6} z(x, y-2k) + \tfrac{1}{2} z(x, y) + \tfrac{1}{3} z(x, y+k), \\ Bz(x, y) &= z(x, y-k). \end{aligned}$$

Then condition (28) holds. Let

$$(51)\qquad G(z, p, h, k) = \frac{1}{k^2}\left[\frac{3+\gamma}{6} p_{-2} - \frac{1+2\gamma}{3} p_{-1} + \frac{1+2\gamma}{2} p_0 - \frac{2\gamma}{3} p_1 + \frac{2+\gamma}{6} p_2 - z\right]$$

and

$$(52)\qquad G_0(z, p, h, k) = \frac{1}{k^3}\left[(\delta + \tfrac{1}{4}) p_{-3} - (6\delta + \tfrac{19}{12}) p_{-2} + (15\delta + \tfrac{7}{2}) p_{-1} - (20\delta + 2) p_0 + \\ + (15\delta + \tfrac{7}{12}) p_1 - (6\delta - \tfrac{1}{4}) p_2 + \delta p_3\right] - \frac{z}{k^3},$$

where $\gamma, \delta \in R$. Then there exists $D_2 > 0$ such that

$$\left|G(A\bar{u}(x, y), [\![\bar{u}(x, y)]\!], h, k) - \bar{u}_{yy}(x, y)\right| \leqslant D_2 h^3$$

and

$$\left|G_0(A\bar{u}(x, y), [\![\bar{u}(x, y)]\!], h, k) - \bar{u}_{yyy}(x, y)\right| \leqslant D_2 h^3.$$

It follows from these estimates that Φ defined by (46), (47), (50)–(52) satisfies

$$\left|\Phi(x, y, A\bar{u}(x, y), [\![\bar{u}(x, y)]\!], \Delta\bar{u}(x, y), h, k) - \Psi(x, y, \bar{u}, h, k)\right| \leqslant Dh^3,$$

where $D > 0$.

We do not go into the details of the method.

References

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