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A note on Liouville analytic spaces

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Abstract. In this paper, the spaces of bounded holomorphic, c-holomorphic and w-holomorphic functions on affine analytic sets are compared.

1. Introduction. Let X be an irreducible analytic subset of C^n. Keeping the notation of [4], we can state three properties connected with the Liouville Theorem:

(A) Each bounded holomorphic function on X is constant.

(B) Each bounded c-holomorphic function on X is constant.

(C) Each bounded w-holomorphic function on X is constant.

It is obvious that (B)$\Rightarrow$(A), (C)$\Rightarrow$(B) and (C)$\Rightarrow$(A). In the paper, we shall show that no other implication holds. In fact, we shall prove the following theorems:

THEOREM 1. *There exists an analytic subset X of C^4 for which* (A) *holds but* (B) *does not.*

THEOREM 2. *There exists an analytic subset X of C^4 for which* (B) *holds but* (C) *does not.*

Both proofs consist of two steps: we first construct an analytic space having the desired properties and then we embed it into C^4.

Our proofs are based on some results from the theory of Riemann surfaces ([1]), theory of Stein spaces ([2]) and complex analytic geometry ([3], [4]).

2. The implication (A)$\Rightarrow$(B). Let $D := \{z \in C\colon |z| < 1\}$ and $B(z_0, r) := \{z \in C\colon |z - z_0| < r\}$.

LEMMA 1. *Let $\{z_p\}_{p=1}^{\infty} \subset D$ be such that $\lim_{p\to\infty} |z_p| = 1$. Then there exists a sequence $\{m_p\}_{p=1}^{\infty}$ of positive integers such that each bounded holomorphic function on D with $f'(z_p) = \quad = f^{(m_p-1)}(z_p) = 0$, $p \geq 1$, is constant.*

Proof. There exist positive integers m_p such that

$$\sup_{|z| \leq 1/2} \left|\frac{z - z_p}{1 - \bar{z}_p z}\right|^{m_p - 1} \leq \frac{1}{4p}, \quad p \geq 1.$$

Let f be a bounded holomorphic function on D with

$$f'(z_p) = \quad = f^{(m_p-1)}(z_p) = 0, \quad p \geqslant 1$$

Applying the maximum principle, we get

$$|f(z)-f(z_p)| \leqslant 2C\left|\frac{z-z_p}{1-\bar{z}_p z}\right|^{m_p-1} \qquad z \in D,$$

where $C := \sup\limits_{z \in D} |f(z)|$.

Combining the above inequalities, we see that

$$|f(z_1)-f(z_2)| \leqslant C/p \quad \text{for } z_1, z_2 \in B(0, \tfrac{1}{2}),\ p \geqslant 1.$$

Now, letting $p \to \infty$ and applying the identity principle gives the required result.

Let $\{z_p\}_{p=1}^{\infty}$ and $\{m_p\}_{p=1}^{\infty}$ be sequences as in Lemma 1. There exists a sequence $\{r_p\}_{p=1}^{\infty} \subset \boldsymbol{R}_+$ such that the discs $U_p := B(z_p, r_p)$, $p \geqslant 1$, are contained in D and are pairwise disjoint.

The mapping $h_p\colon U_p \to B(0, r_p^{m_p+1}) \times B(0, r_p^{m_p})$, defined by

$$h_p(z) := \left((z-z_p)^{m_p+1}, (z-z_p)^{m_p}\right),$$

is a homeomorphism of the disc U_p onto the analytic subset

$$A_p := \{(u, v) \in B(0, r_p^{m_p+1}) \times B(0, r_p^{m_p})\colon u^{m_p} = v^{m_p+1}\}$$

of the bidisc $B(0, r_p^{m_p+1}) \times B(0, r_p^{m_p})$. Let us put $U_0 := D \backslash \{z_p\colon p \geqslant 1\}$ and $h_0 := \mathrm{id}_{U_0}$.

The collection $\{(U_p, h_p)\}_{p \in \boldsymbol{N}}$ gives a coordinate system in D. We shall denote by $\tilde{X}$ the analytic space defined by it ([4], Chap. 5, Def. 1E).

Proposition 1. *$\tilde{X}$ has property* (A) *but does not have property* (B).

Proof. We first observe that the function $I\colon \tilde{X} \ni z \to I(z) := z \in D$ is c-holomorphic, bounded and non-constant.

Now, let f be a bounded holomorphic function on $\tilde{X}$. Put $g := f \circ I^{-1}$. We finish the proof if we show that $g'(z_p) = \ldots = g^{(m_p-1)}(z_p) = 0$, $p \geqslant 1$. Since f is holomorphic, we have $f(z) = (f_p \circ h_p)(z)$, $z \in U_p$, where f_p is a function holomorphic on A_p.

Let

$$f_p(u, v) = \sum_{k,l \in \boldsymbol{N}} a_{k,l} u^k v^l, \qquad (u, v) \in A_p.$$

Consequently,

$$f(z+z_p) = \sum_{k,l \in \boldsymbol{N}} a_{k,l} z^{k(m_p+1)+lm_p}, \qquad |z| < r_p.$$

Differentiating the above equality $m_p - 1$ times, we get the desired conclusion.

Proof of Theorem 1. There exists a holomorphic homeomorphism F of $\tilde{X}$ onto an analytic subset X of C^4 ([2], Chap. IX, Th. 10B and Chap. VII, Th. 10C).

The function $I \circ F^{-1}$ is bounded, c-holomorphic and non-constant, hence, X does not have property (B).

Now, let f be bounded holomorphic function on X. By the proposition above the function $f \circ F$ is constant, thus f is constant too. The proof is complete.

Remarks. (a) The pair $(D, F \circ I^{-1})$ is a normalization of X. Thus X is a Liouville space with non-Liouville normalization.

(b) Since the analytic set X is simply connected, it is easy to check that each holomorphic function on X omitting two complex values is constant.

3. The implication (B)$\Rightarrow$(C). The following lemma will be used in the proof of Theorem 2.

LEMMA 2. *There exists a sequence* $\{A_p\}_{p=1}^{\infty}$ *of finite pairwise disjoint subset of the unit disc* D *with the following properties*:

(i) $\bigcup_{p=1}^{\infty} A_p$ *has no accumulation point in* D;

(ii) *any bounded holomorphic function on* D *constant on each* A_p *is constant on the whole of* D.

Proof. Let us define A_p: $\{k/(k+1)\colon\ k = p^p, \ldots, (p+1)^{(p+1)}-1\}$, $p \geqslant 1$. We claim that for this sequence the assertions of the lemma are fulfilled. As (i) is obvious we shall only prove (ii).

Let f be a bounded holomorphic function on D with $f|A_p = c_p$, $p \geqslant 1$. Applying the maximum principle, we get

$$|f(z)-c_p| \leqslant 2C \prod_{k=p^p}^{(p+1)^{(p+1)}-1} \left|\frac{z-k/(k+1)}{1-(k/(k+1))z}\right|,$$

where $C := \sup_{z \in D} |f(z)|$.

For every $z \in B(0, \frac{1}{2})$ we have

$$\left|\frac{z-k/(k+1)}{1-(k/(k+1))z}\right| \leqslant \left|\frac{3k+1}{3k+2}\right| \leqslant \left(\frac{k+1}{k+2}\right)^{1/3}$$

Combining the above inequalities we see that

$$|f(z_1)-f(z_2)| \leqslant 4C\left(\frac{p^p+1}{(p+1)^{p+1}+1}\right)^{1/3} \quad \text{for } z_1,\ z_2 \in B(0, \tfrac{1}{2}),\ p \geqslant 1.$$

Now, letting $p \to \infty$ and applying the identity principle we finish the proof.

Let $\{A_p\}_{p=1}^{\infty}$ be a sequence as in Lemma 2 and let $A_p = \{z_{p,1}, \ldots, z_{p,k_p}\}$. There exists a sequence $\{r_p\}_{p=1}^{\infty} \subset \boldsymbol{R}_+$ such that the discs $B_{p,q} := B(z_{p,q}, r_p)$, $p \geqslant 1$, $q = 1, 2, \ldots, k_p$, are contained in D and are pairwise disjoint.

The relation $\mathscr{R} := \mathrm{id}_D \cup \bigcup_{p=1}^{\infty} A_p^2$ is an equivalence relation in D. The quotient topological space $\tilde{X} := D/\mathscr{R}$ is a Hausdorff space and the quotient mapping h: $D \to \tilde{X}$ is a continuous surjection.

Let us define $U_p := \bigcup_{p=1}^{k_p} h(B_{p,q})$, $p \geqslant 1$. The mapping h_p: $U_p \to (B(0, r_p))^{k_p}$ defined by

$$h_p(h(z)) := (z - z_{p,1}, 0, \ldots, 0), \quad z \in B_{p,1},$$

$$h_p(h(z)) := (0, \ldots, 0, z - z_{p,k_p}), \quad z \in B_{p,k_p},$$

is a homeomorphism of U_p onto an analytic subset

$$A_p := \{z \in (B(0, r_p))^{k_p}: z_j = 0 \text{ except for at most one } j\}$$

of the polydisc $(B(0, r_p))^{k_p}$. Put $U_0 := \tilde{X} \setminus h(\bigcup_{p=1}^{\infty} A_p)$ and $h_0 := (h|h^{-1}(U_0))^{-1}$. The collection $\{(U_p, h_p)\}_{p \in N}$ gives a coordinate system in $\tilde{X}$ ([4], Chap. 5, Def. 1E).

PROPOSITION 2. *The analytic space* $\tilde{X}$ *has property* (B) *but does not have property* (C).

Proof. We first observe that h^{-1} is a non-constant, bounded w-holomorphic function on $\tilde{X}$.

Now, let f be a bounded c-holomorphic function on $\tilde{X}$. The function $f \circ h$ is bounded holomorphic on D and constant on each A_p. By the definition of $\tilde{X}$ the function $f \circ h$ is constant and hence so is f.

Proof of Theorem 2. There exists a holomorphic homeomorphism F of $\tilde{X}$ onto an analytic subset X of $\boldsymbol{C}^4$ ([2], Chap. IX, Th. 10B and Chap. VII, Th. 10C).

The function $h^{-1} \circ F^{-1}$ is bounded, w-holomorphic and non-constant, hence X does not have property (C).

Now, let f be a bounded c-holomorphic function on X. By Proposition 2 the function $f \circ F$ is constant and therefore f is constant too. The proof is complete.

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