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The measurability of Lie groups *

by G. VRANCEANU (Bucharest)

A Lie group G_r transitive in n variables ($r > n$) is said to be *measurabel* if it admits one and only one integral invariant.

This notion is met in probability and the conditions for the groups to be measurable have been given by R. Deltheil in his book *Probabilités géométriques*; the conditions are given in the form of a system with total differentials.

In 1940, Siing Shen Chern, without mentioning Deltheil, considered the problem of establishing the conditions under which the determinant of n Pfaff forms, left invariant by the group, is an integral invariant. Later, Chern's problem was considered by Santalo in his book *Introduction to integral geometry* (1953), again without using Deltheil's results.

Recently, Deltheil's system has been expressed in a different form by Stoka, who has also considered the conditions of integrability of this system.

The aim of the present paper is to express both the systems of Deltheil and Stoka, and their conditions of integrability in different simpler forms, and also to give certain results, especially a theorem concerning the measurability of a group G, which possesses a subgroup which is simply transitive, perfect and invariant.

We also prove that a Lie group is Deltheil-measurable if it is Chern-measurable and conversely.

1. If a Lie group G_r with r parameters and n variables

$$y^i = f^i(x^1, \ldots, x^n, a^1, \ldots, a^r) \qquad (i = 1, \ldots, n) \tag{1}$$

is considered, a function $F(x^1, \ldots, x^n)$ is said to be an *integral invariant function* of the group G if we have

$$\int_{Ax} F(x^1, \ldots, x^n)dx^1 \ldots dx^n = \int_{Ay} F(y^1, \ldots, y^n)dy^1 \ldots dy^n \tag{2}$$

for every domain Ax on which the integral can be defined.

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According to formulas (1) it is necessary to have

$$F(x^1, \dots, x^n) = DF(y^1, \dots, y^n)\,, \tag{3}$$

where D is the functional determinant of transformation (1). Therefore, it is necessary for the second member of formula (3) to be independent of the parameters $a^1, \dots, a^r$.

R. Deltheil ([2], p. 28) has shown that this condition is equivalent to the fact that the function F satisfies the conditions

$$\frac{\partial}{\partial x^i}[\xi^i_h(x)F(x)] = 0\,, \tag{4}$$

where ξ^i_h $(h = 1, \dots, r)$ are the components of the r operators of the infinitesimal transformations of the group G_r.

Considering that these equations can be written in the form

$$\xi^i_h \frac{\partial F}{\partial x^i} + F\frac{\partial \xi^i_h}{\partial x^i} = 0 \tag{5}$$

it follows that, if the group is transitive, the system (4) is a system with total differentials, and consequently admits at most one solution, abstraction being made of the multiplication by an arbitrary constant.

M. Stoka ([4], p. 28) has shown that, in order for the system (4) of Deltheil to have, effectively, a solution in the case $r > n$, it is necessary for ξ^i_h to satisfy certain algebraic equations whose coefficients are the components of the tensor of structure of the group G_r.

In order to obtain Deltheil's equations we write the equations (1) of a group in the form (see [8], p. 68)

$$y^i = x^i + \xi^i_h(x)a^h + \tfrac{1}{2}\xi^i_{hk}(x)a^h a^k + \dots\,, \tag{6}$$

where the coefficients $\xi^i_{hk}(x)$ and also the coefficients of terms of higher order in a^h depend on ξ^i_h, provided ξ^i_h are known.

We, therefore, have

$$D = \left|\frac{\partial y^i}{\partial x^j}\right| = \left|\delta^i_j + \frac{\partial \xi^i_h}{\partial x^j}a^h + \dots\right|\,,$$

from which we obtain

$$D = 1 + \frac{\partial \xi^i_h}{\partial x^i}a^h + \dots\,,$$

the unwritten terms being of second order in a^h.

We also have

$$F(y^1, \dots, y^n) = F(x^1, \dots, x^n) + \frac{\partial F}{\partial x^i}\xi^i_h a^h\,.$$

By introducing these expressions in formula (3) and writing that the terms of first order in a^h are zero, we obtain Deltheil's conditions.

It also follows that (4) are the necessary conditions for the functions F to be an integral-invariant of the transformations (1) but it can be proved that (4) are also sufficient conditions if the transformations (1) form a group.

Indeed, here a well-known principle of Lie groups is verified, namely that it is sufficient that a property be true for the infinitesimal transformations of the group.

We now observe that equations (5) can be written in the form

$$\xi_h^i \frac{\partial f}{\partial x^i} + \frac{\partial \xi_h^i}{\partial x^i} = 0 \qquad (f = \log F) \tag{6'}$$

therefore, if these equations have two solutions, for instance F and F_1, then the function $\varphi = \log \frac{F_1}{F}$ satisfies the equation

$$\xi_h^i \frac{\partial \varphi}{\partial x^i} = 0 .$$

This shows that φ is an invariant-function of the group G_r. However, it is known that a group may have invariant functions only if it is intransitive. We therefore have Deltheil's theorem:

If an intransitive group admits an integral-invariant-function F, all the other integral-invariant functions are of the form

$$F_1 = Fe^{\varphi}$$

where φ is an invariant function.

We may also observe that, if the group G_r has an integral-invariant, we can suppose it to be equal to one, which means that the group has an invariant volume.

In this case, if we write

$$p_h = \frac{\partial \xi_h^i}{\partial x^i} \qquad (h = 1, \dots, r) , \tag{6''}$$

formulas (5) show that all p_h must be zero

We suppose that the determinant $\Delta = |\xi_u^i|$ $(u = 1, \dots, n)$ is different from zero. In this case we may write

$$\xi_a^i = \xi_u^i \sigma_a^u \qquad (a = n+1, \dots, r) , \tag{7}$$

because this means that we have

$$\sigma_a^u = \xi_a^i \bar{\xi}_i^u ,$$

where $\bar{\xi}_i^u$ are the cofactors of Δ.

We also observe that if ξ_a^i are the vectors of the stability group, then σ_a^u are equal to zero at the origin.

If we write system (6′) in the form

$$X_u f + p_u = 0 \quad \left(X_u f = \xi^i_u \frac{\partial f}{\partial x^i}\right), \qquad p_u \sigma^u_a = p_a\,, \tag{8}$$

it follows that the functions p_a are also zero at the origin.

Considering (7), the last equations (8) can be written as

$$\xi^i_u \frac{\partial \sigma^u_a}{\partial x^i} = 0\,. \tag{8'}$$

It follows that the problem consists in seeing whether the first equations (8) may have solutions and if there exist functions σ^u_a which are not constant and which satisfy equations (8′).

As a consequence, we have the theorem:

A necessary condition for the transformations (1) *to possess an integral-invariant function is that equations*

$$X_u f + p_u = 0 \quad \left(X_u f = \xi^i_u \frac{\partial f}{\partial x^i},\ p_u = \frac{\partial \xi^i_u}{\partial x^i}\right) \tag{8''}$$

should have a solution and that σ^u_a should be the solutions of equations (8′).

We now suppose that transformations (1) form a group. We therefore have the formulas

$$\xi^j_h \frac{\partial \xi^i_k}{\partial x^j} - \xi^j_k \frac{\partial \xi^i_h}{\partial x^j} = c^l_{hk} \xi^i_l\,, \tag{9}$$

which may also be written as

$$X_h(X_k f) - X_k(X_h f) = c^l_{hk} X_l f\,.$$

As a consequence of formulas (6′), these formulas may be written as:

$$\xi^j_h \frac{\partial p_k}{\partial x^j} - \xi^j_k \frac{\partial p_h}{\partial x^j} = c^l_{hk} p_l \tag{9'}$$

and therefore represent the conditions of integrability.

We therefore have the theorem:

In order that a group G_r ($r \geqslant n$) may admit an integral-invariant function it is necessary for ξ^i_u and ξ^i_a to satisfy equations (9′) *where the quantities p_h are defined by* (6″) *and where c^l_{hk} are the group constants of structure.*

For the case where the integral-invariant function is a constant, i.e. if the quantities p_h are zero, equations (8′) are the only equations not identically satisfied. There are $r-n$ such equations; therefore they admit an infinity of solutions. It follows that in general, if a group G_r, simply transitive, is given, there exists an infinity of groups G_r ($r > n$) which are measurable and include G_n as a subgroup.

We suppose that G_r is transitive and $r > n$. In this case, in formulas (9), we can make ξ^i_u and σ^u_a appear instead of ξ^i_a.

Indeed, formula (9) for $h, k \leqslant n$ can be written as

$$(9'') \qquad \xi^j_u \frac{\partial \xi^i_r}{\partial x^j} - \xi^j_r \frac{\partial \xi^i_u}{\partial x^j} = (c^t_{uv} + c^a_{uv}\sigma^t_\varrho)\,\xi^i_t\,.$$

In the same way, formulas (9) for $h \leqslant n$, $k > n$ can be written as

$$\xi^i_t\,\xi^j_u \frac{\partial \sigma^t_a}{\partial x^j} + \left(\xi^j_u \frac{\partial \xi^i_t}{\partial x^s} - \xi^j_t \frac{\partial \xi^i_u}{\partial x^j}\right)\sigma^t_a = (c^t_{ua} + c^\varrho_{ua}\sigma^t_\varrho)\,\xi^i_t\,,$$

so that, in view of (9″) and of the fact that the determinant $|\xi^i_t|$ is different from zero, we obtain the equations

$$(9''') \qquad \xi^j_u \frac{\partial \sigma^t_a}{\partial x^j} = \sigma^s_a(c^t_{su} + c^\varrho_{su}\sigma^t_\varrho) + c^t_{ua} + c^\varrho_{ua}\sigma^t_\varrho\,.$$

According to these equations, the operators $X_u(\sigma^t_a)$ are functions of σ.

If we consider the conditions of integrability

$$(9'''') \qquad X_u\big(X_\sigma(\sigma^t_a)\big) - X_v\big(X_u(\sigma^t_a)\big) = (c^s_{uv} + c^\varrho_{uv}\sigma^s_\varrho)\,X_s(\sigma^t_a)\,,$$

we obtain relations in finite terms in σ^j_a, which are evidently verified if the group G_r is a transitive group in n variables.

For h, k greater than n, equations (9) also give relations in finite terms in σ^t_a.

It follows that if some structure of a group G_r is given, we have a way of knowing whether that structure is compatible with a transitive group G_r in n variables.

If the group G_r is measurable, according to (9″), equations (8′) become

$$(10) \qquad -\sigma^s_a(m_s + c^\varrho_{ts}\sigma^t_\varrho) + m_a + c^\varrho_{ta}\sigma^t_\varrho = 0\,,$$

where we have written $m_l = c^t_{tl}$. We therefore have relations in finite terms in σ^a_t.

For the case where ξ^i_a are the vectors of the group of stability it follows that

$$(10') \qquad m_a = 0\,.$$

As a consequence, in the case where ξ^i_a are the vectors of the groups of stability, in order that a transitive group G_r be measurable it is necessary for the equations (10′) to be verified. In the second part of this work we shall see that this is also a sufficient condition.

Equations (10) have been given by Stoka in the form

$$(10'') \qquad c^l_{uv}\xi^t_l\bar\xi^u_t\,\xi^i_k\bar\xi^v_i - c^l_{uk}\xi^t_l\bar\xi^u_t = 0\,,$$

where k varies from 1 to r, but it can easily be seen that, if k varies from 1 to n, equations (10″) are identically zero, and that for $k = a$ they coincide with equations (10); this follows in view of the formulas

$$\xi_v^i \bar{\xi}_i^u = \delta_v^u , \qquad \xi_a^i \bar{\xi}_i^u = \sigma_a^u , \tag{10'''}$$

where δ_v^u is Kronecker's symbol.

2. Let us show how M. Stoka has given another form to Deltheil's equations.

We multiply equations (9), for $h = u = 1, \dots, n$, with $\bar{\xi}_s^u$ and take the sum relative to n.

We obtain

$$\frac{\partial \xi_k^i}{\partial x^j} = \xi_k^j \frac{\partial \xi_u^i}{\partial x^j} \bar{\xi}_s^u + c_{uk}^l \xi_l^i \bar{\xi}_s^u ,$$

so that, by making $i = s$ and summing, we obtain the following formulas for the quantities p_h:

$$p_h = \xi_h^j \frac{\partial \xi_u^i}{\partial x^j} \bar{\xi}_i^u + c_{uh}^l \xi_l^i \bar{\xi}_i^u .$$

On the other hand, if Δ is the determinant $|\xi_u^i|$ we have

$$\frac{\partial \log \Delta}{\partial \xi_u^i} = \bar{\xi}_i^u$$

so that, by multiplying with $\dfrac{\partial \xi_u^i}{\partial x^j}$ we obtain

$$\frac{\partial \log \Delta}{\partial x^j} = \frac{\partial \log \Delta}{\partial \xi_u^i} \cdot \frac{\partial \xi_u^i}{\partial x^j} = \frac{\partial \xi_u^i}{\partial x^j} \bar{\xi}_i^u ;$$

we therefore may write

$$p_h = \xi_h^j \frac{\partial \log \Delta}{\partial x^j} + c_{uh}^l \xi_l^i \bar{\xi}_i^u .$$

By introducing this relation in equations (4) we obtain Stoka's system

$$\xi_h^j \frac{\partial \log F \Delta}{\partial x^j} = c_{uh}^l \xi_l^i \bar{\xi}_i^u , \tag{11}$$

which, for the case where transformations (1) form a group, is equivalent to Deltheil's system.

In view of (10‴), we have the formulas

$$p_h = \xi_h^j \frac{\partial \log \Delta}{\partial x^j} + m_h + c_{uh}^\beta \sigma_\beta^u ,$$

where we have written

$$m_h = c^u_{uh} ,$$

and system (11) becomes

$$(11') \qquad \begin{aligned} \xi^j_u \frac{\partial \log F\Delta}{\partial x^j} &= m_u + c^a_{tu}\sigma^t_a , \\ \xi^j_u \sigma^u_l \frac{\partial \log F\Delta}{\partial x^j} &= m_a + c^\beta_{ja}\sigma^t_\beta . \end{aligned}$$

From the last of these equations, in view of the first ones, we obtain the equations (10) in finite terms.

The conditions of integrability of the first equations (11′) may be written as

$$X_u(X_v f) - X_v(X_u f) = c^l_{uv} X_l f ,$$

where we have put

$$X_l f = \xi^j_l \frac{\partial f}{\partial x^j} = m_l + c^a_{sl}\sigma^s_a .$$

Therefore, in view of the fact that $X_u(\sigma^t_a)$ are defined by (9‴), we have

$$\sigma^s_a(c^a_{tv}c^t_{su} + c^a_{tv}c^\varrho_{su}\sigma^t_\varrho) + c^a_{tv}c^t_{ua} + c^a_{tu}c^\varrho_{ua}\sigma^t_\varrho - \sigma^s_a(c^a_{tu}c^t_{sv} + c^a_{tu}c^\varrho_{sv}\sigma^t_\varrho) -$$
$$- c^a_{tu}c^t_{va} - c^a_{tu}c^\varrho_{va}\sigma^t_\varrho = c^l_{uv}m_l + c^l_{uv}c^a_{sl}\sigma^s_a .$$

But it can be easily verified that the quadratic terms in σ^t_a are identically zero while the linear terms are equal to zero according to Lie's quadratic relations. We therefore may write these relations in the form

$$(12) \qquad c^a_{tr}c^t_{ua} - c^a_{tu}c^t_{va} - c^l_{uv}m_l = 0 .$$

On the other hand, let us consider Lie's quadratic relations

$$(13) \qquad c^h_{sp}c^s_{uv} + c^h_{su}c^s_{rp} + c^h_{sr}c^s_{pu} + c^h_{ap}c^a_{uv} + c^h_{au}c^a_{vp} + c^h_{av}c^a_{pu} = 0 .$$

If we make $h = p = t \leqslant n$ and take the sum, we obtain the relations

$$-m_s c^s_{uv} + c^t_{su}c^s_{rt} + c^t_{sr}c^s_{tu} - m_a c^a_{uv} + c^t_{au}c^a_{rt} + c^t_{av}c^a_{tu} = 0 .$$

Because the second and third terms cancel, it follows that relations (12) are identically verified.

We, therefore, have the theorem:

A group G_r is measurable if σ^t_a verifies equations (10).

If we suppose that the group G is simply transitive, i.e. that $r = n$, equations (11′) become

$$(14) \qquad \xi^j_u \frac{\partial \log F\Delta}{\partial x^j} = c_u ,$$

where $c_u = c^s_{su}$ is the structure vector of the group.

The conditions for integrability of these equations can be written as

$$c_t c^t_{uv} = 0 \tag{14'}$$

and these conditions are verified if we consider Lie's quadratic relations.

Therefore, the following theorem results:

A group G_r, simply transitive, is always measurable.

It follows that, if the vector of structure is zero, we may take as the integral-invariant function F the quantity $1/\Delta$.

It also follows that, being given a measurable group G_r $(r > n)$, which has a simply transitive subgroup, we may always choose a system of coordinates relative to which the vectors ξ^i_h have a null divergence. Indeed, this means that we may conveniently proceed so that the integral-invariant function of the group G_r becomes equal to 1.

Also if the quantities ξ^i_u form a group G_n, simply transitive, we have $c^\varrho_{ts} = 0$ and, as a consequence, equations (10) may be written as

$$c^\varrho_{t\alpha}\sigma^t_p - \sigma^s_\alpha m_s + m_\alpha = 0 \tag{15}$$

and are, therefore, linear equations in σ^t_α.

If we suppose that the group G_n is an invariant subgroup in G_r we have $c^\varrho_{t\alpha} = 0$. In this case we have

$$c_s = m_s\,, \qquad m_s c^s_{uv} = 0$$

and, if the group G_n is perfect, conditions (15) become conditions (10').

We, therefore, have the theorem:

If a group G_r has a subgroup G_n, which is simply transitive, perfect and invariant, then the group G_r is measurable, provided relations (10') *are verified and conversely.*

In particular, this theorem may be applied to Cartan's symmetric spaces V_n, which have a group of motions G_{2n} and it results that these groups G_{2n} are measurable. Indeed, in this case the simply transitive group G_n is a simple and invariant subgroup in G_{2n}, i.e. an invariant and perfect group.

Relations (15) are also verified, because, in the case of G_{2n} the quantities $c^t_{s\alpha}$ are skew-symmetric in s and t.

We suppose that $r = n+1$. In this case, if we put $\sigma^s_{n+1} = \sigma_s$, equations (10) can be written in the form given by Stoka

$$c_s\sigma^s = c_{n+1}\,, \tag{16}$$

where $c_h = c^k_{kh}$ is the vector of structure of the group G_{n+1}. As a consequence, G_{n+1} is measurable if c_h is a null vector and, therefore, any group G_{n+1} which is perfect is measurable.

3. We will show now that conditions (10′) are sufficient for the group G_r to have a measure.

Indeed, it is known that, given a transitive group G_r in n variables $y^1, \ldots, y^n$, other variables $y^{n+1}, \ldots, y^r$, which since Elie Cartan are called secondary variables, can be associated in such a way that the group leaves invariant n Pfaff forms:

$$ds^u = \lambda_i^u dy^i , \qquad ds^a = \lambda_i^a dy^i \qquad (u, i = 1, \ldots, n;\ a, \beta > n) ,$$

where λ_i^u, λ_i^a, λ_β^a depend on the principal variables y^i and on the secondary variables y^β and the determinant $D = |\lambda_i^u|$ is different from zero; ds^u are therefore independent Pfaff forms in the space $Y_n(y^1, \ldots, y^n)$.

Concerning the r forms ds^u, ds^a they satisfy the relations of structure

$$(16') \qquad \begin{aligned} \Delta s^u &= c_{vt}^u ds^v \delta s^t + c_{v\beta}^u [ds^v ds^\beta] , \\ \Delta s^a &= c_{vt}^a ds^v \delta s^t + c_{v\beta}^a [ds^v ds^\beta] + c_{\beta\gamma}^a ds^\beta ds^\gamma , \end{aligned}$$

where we have written $[ds^v ds^\beta] = ds^v \delta s^\beta - ds^\beta \delta s^v$ and where Δ represents the operator $d\delta - \delta d$.

In this case, if the determinant D is independent of the secondary variables, we may take as volume in the space Y_n the quantity

$$(16'') \qquad V = \int_{\Delta y} D dy^1 \ldots dy^n ,$$

which can be written as

$$V = \int_{\Delta y} [ds^1 \ldots ds^n]$$

by using the notation of exterior product.

It can easily be shown that, if the formulas (15) are verified, D does not depend on the secondary variables, Indeed, it can be shown that we may always make a change of forms ds^u

$$\dot{ds}^u = c_v^u ds^v ,$$

in such a way that $\dot{ds}^u$ does not depend on the secondary variables and that c_v^u does not depend on the principal variables; in this case the determinant $\Delta = |c_v^u|$ satisfies the formulas (see [6], p. 159)

$$\frac{\partial \Delta}{\partial s^a} = -m_a \Delta .$$

Therefore, if (15) are verified, Δ is a constant and, considering we have $\bar{D} = \Delta D$, where $\bar{D}$ is the determinant $|\bar{\lambda}_i^u|$, it follows that $\bar{D}$ is independent of the secondary variables and we have Chern's theorem ([1], see also [3]):

In the space Y_n, a necessary and sufficient condition for the volume (16″) *to be invariant relative to the group G_r is that formulas* (10′) *be verified.*

These mean that a group G_r is measurable according to Deltheil or Chern at the same time.

We observe also that, according to formulas (16′), equations (14′) can be written as

$$c_s c^s_{uv} + c_a c^a_{uv} = 0\ , \qquad c_a c^a_{\beta\gamma} = 0\ , \qquad c_s c^s_{u\beta} + c_a c^a_{u\beta} = 0$$

and we also have

$$c_s = m_s + n_s\ , \qquad c_a = m_a + n_a \qquad (n_h = c^\beta_{\beta h})\ .$$

Considering that $c^a_{\beta\gamma}$ are the constants of structure of the stability subgroup of the group G_r, it follows that, for the case where G_{r-n} is perfect, the quantities c_a vanish.

On the other hand, for the group G_{r-n} we have also formulas (14′)

$$n_a c^a_{\beta\gamma} = 0\ .$$

Therefore, n_a also vanish and, as a consequence, (15) are verified.

We therefore have the theorem:

A group G_r having as subgroup of stability a perfect subgroup is measurable.

It follows thus that Cartan's symmetric spaces, with a simple group of stability, are measurable.

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