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Vector bundles on subcartesian spaces

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1. INTRODUCTION

The aim of this paper is to investigate various vector bundles (vector pseudo-bundles in the terminology of [4]) occurring in the process of differentiating of functions defined on subsets of a Euclidean space. If $X \subset \mathscr{R}^m$ and $f\colon X \to \mathscr{R}^1$, then f is differentiable at $x \in X$ if there is a neighborhood U of x in $\mathscr{R}^m$ and a differentiable function $\tilde{f}\colon U \to \mathscr{R}^1$ such that $f = \tilde{f}$ on $U \cap X$. The tangent space T_xX (see [9], [6], [2]) is the largest subspace of $\mathscr{R}^m$ such that the derivative $f'(x; u) := \tilde{f}'(x; u)$ for any $u \in T_xX$ is independent of the choice of the extension $\tilde{f}$. The bundle $TX = \bigcup \{\{x\} \times T_xX\colon x \to X\}$ is thus the domain of the derivative of every differentiable function on X. We can consider covariant vector fields on X (functions on TX linear on the fibers) and contravariant fields (sections of TX). To differentiate the former one can proceed as above to define the tangent bundle to TX — a bundle over TX — which thanks to the linear nature of the fibers of TX can be identified with a bundle over X — the second tangent bundle to X. This identification has been exploited by N. Aronszajn in the case of differentiable manifolds (personal communication), who also noticed the interest in considering coupled differentiable-linear structures. The process can be repeated to obtain higher order tangent bundles; the tangent bundle of any order is the natural domain of definition of the derivatives of functions on the tangent bundle of order by one less which are linear on fibers.

The bundles so obtained are of interest only at "singular" points of X; at points of smoothness they are easily calculated and do not seem to provide a new insight. At all points the bundles are local invariants of homeomorphisms locally extendable to diffeomorphisms between open

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sets in $\mathscr{R}^m$ and thus provide some insight into the nature of singular points.

In the above context it is natural to look for the notions of affine connections and covariant derivatives: these can be defined in a rather natural way, however (as one would expect), one cannot expect in general the existence of a smooth affine connection. Even if such exists, one with prescribed properties (e.g. Levi-Civitá connection) may not exist.

The bundles indicated above are connected with the operation of first order differentiation of *all* functions on some given bundle, which are linear on fibers. In the cases of special bundles and more restricted classes of functions, the derivatives of course are defined on the tangent bundle, but the latter may actually not be their maximal domain of definition. One of such cases is that of the bundle TX with the class of functions consisting of all derivatives of functions on X, the other one is that of the bundle $\otimes^k TX$, $k \geqslant 1$, with the class of functions consisting of all functions which are skew symmetric on fibers. The operation here is of course the exterior differentiation.

The bundles obtained in the first case are domains for derivatives of arbitrary order of functions defined on X; they provide a systematic set-up for the compatibility conditions to be satisfied by the derivatives of any (differentiable) extension of a function defined on X (see [2] for an example).

The bundles associated with the exterior derivative seem to present yet another framework for the exterior calculus and resulting theories (see [7], [5], [3]).

The examples occurring repeatedly in the paper are those of a polyhedral singularity and of the vertex of an algebraic cone (in most cases of 2nd degree). In these cases all the bundles in questions can be computed. It would be of interest to describe a wider class of sets with singularities where such computation would be feasible.

We found it expedient in this paper to use the setting of subcartesian spaces [1] even though all the considerations are local and concern subsets of a Euclidean space. In this setting the meaning of various concepts and of their invariance becomes apparent.

The content of the paper is as follows:

Section 2 sets up the notations and conventions as well as some basic notions concerning vector bundles over subcartesian spaces. Section 3 deals with bundles obtained from given ones by means of functors of linear algebra, Section 4 is devoted to the construction of the modified tangent bundle to a vector bundle, higher order tangent bundles and restricted tensor products. Section 5 contains further properties of the bundles introduced in Section 4 and Section 6 contains some results concerning bundles associated with higher order derivatives and with the exterior derivatives.

2. NOTATIONS AND PRELIMINARIES

2.1. For a function f we denote by D_f the domain of f, $C^k(\mathscr{R}^m, \mathscr{R}^n)$ is the class of functions $f: D_f \subset \mathscr{R}^m \to \mathscr{R}^n$, where D_f is open in $\mathscr{R}^m$ and f is k-times continuously differentiable when $k = 0, 1, \ldots$ or real analytic when $k = \omega$. We write $C^k = \bigcup\{C^k(\mathscr{R}^m, \mathscr{R}^n): m, n = 1, 2, \ldots\}$ and $C^k(\mathscr{R}^m) = C^k(\mathscr{R}^m, \mathscr{R}^1)$. For $f \in C^k$ $f'(x)u = f'(x; u) = \langle \nabla f(x), u\rangle$ denote the differential of f at x with increment u, $\langle\ ,\ \rangle$ denotes the scalar product; similar notations are used for partial and higher order differentials.

$\mathscr{R}^m \times \mathscr{R}^n$ denotes the space of pairs (x, y), $x \in \mathscr{R}^m$, $y \in \mathscr{R}^n$ with its product structure; in the cases when there is a need to identify it with $\mathscr{R}^{m+n}$ we use the notation $\mathscr{R}^m + \mathscr{R}^n$.

$C^kL(\mathscr{R}^m \times \mathscr{R}^n, \mathscr{R}^{m'} \times \mathscr{R}^{n'})$ denotes the class of functions of the form $(g, G): (x, u) \in \mathscr{R}^m \times \mathscr{R}^n \to \big(g(x), G(x, u)\big) \in \mathscr{R}^{m'} \times \mathscr{R}^{n'}$, where $g \in C^k(\mathscr{R}^m, \mathscr{R}^{m'}) D_G = D_g \times \mathscr{R}^n$, $G(x, \cdot)$ is linear for every $x \in D_g$ and $x \to G(x, \cdot)$ is of class C^k. We write $C^kL = \bigcup\{C^kL(\mathscr{R}^m \times \mathscr{R}^n, \mathscr{R}^{m'} \times \mathscr{R}^{n'}): m, n, m', n' = 1, 2, \ldots\}$ and $C^kL(\mathscr{R}^m \times \mathscr{R}^n) = C^kL(\mathscr{R}^m \times \mathscr{R}^n, \mathscr{R}^1)$ to denote the real valued functions linear in the second variable.

A homeomorphism in C^k is a homeomorphism $h \in C^k(\mathscr{R}^m, \mathscr{R}^m)$ for some $m = m(h)$ such that $h^{-1} \in C^k(\mathscr{R}^m, \mathscr{R}^m)$.

The notion of homeomorphism in C^kL is defined similarly.

The spaces $\mathscr{R}^n$, $n = 0, 1, \ldots$, are considered as subspaces of $\mathscr{R}^\infty$ — the space of all real sequences and are thus ordered by the canonical inclusions. The cartesian products $\mathscr{R}^m \times \mathscr{R}^n$ are considered as subspaces of $\mathscr{R}^\infty \times \mathscr{R}^\infty$ and are partially ordered by the canonical inclusions.

2.2. If S is a topological space, then φ is an $\{\mathscr{R}^n\}$- (or $\{\mathscr{R}^m \times \mathscr{R}^n\}$-) chart if for some m (or some m, n) φ is a homeomorphism $\varphi: D_\varphi \subset S \to \mathscr{R}^m$ (or respectively $\varphi: D_\varphi \subset S \to \mathscr{R}^m \times \mathscr{R}^n$) such that D_φ is open in S. It is not required that the image $\varphi(D_\varphi)$ of a chart φ be open in any of the spaces $\mathscr{R}^n$ ($\mathscr{R}^m \times \mathscr{R}^n$) containing it.

A C^k-atlas on S (or respectively a C^kL-atlas) is a collection of $\{\mathscr{R}^n\}$-charts $\Phi = \{\varphi\}$ (or respectively $\{\mathscr{R}^m \times \mathscr{R}^n\}$-charts) with $\bigcup\{D_\varphi: \varphi \in \Phi\} = S$ and such that the following condition of local extendability of connecting homeomorphisms holds: for any φ, $\psi \in \Phi$ such that $D_\varphi \cap D_\psi \neq \emptyset$ and for every $p \in D_\varphi \cap D_\psi$ there exists an open neighborhood U of p, $U \subset D_\varphi \cap \cap D_\psi$ and a C^k- (or respectively C^kL-homeomorphism) $h \in C^k$ such that $D_h \supset \varphi(U)$ and $h = \psi \circ \varphi^{-1}$ on $\varphi(U)$.

A C^k- (or C^kL-) space is a pair (S, Φ), where S is a topological space and Φ is a C^k- (C^kL-) atlas on S. We also say that Φ gives or defines a C^k- (C^kL-) structure on S.

The following concepts are defined in the usual way C^k- (C^kL)-compatibility of a chart in S with an atlas on S; C^k- (C^kL)-equivalence of two atlasses, a maximal atlas equivalent to a given one.

If (S, Φ), (S', Φ') are two C^k spaces, then $f\colon S \to S'$ is a C^k-mapping (morphism), $f \in C^k(S, S')$, provided for every $\varphi \in \Phi$, $\varphi' \in \Phi'$ the mapping $\varphi' \circ f \circ \varphi^{-1}$ is locally extendable about every point of its definition to a function in C^k. The notions of mono-, epi-, iso-, etc. morphisms are defined in the obvious way.

2.3. Let (S, Φ) be a C^k-space. A $C^l L$-vector bundle over (S, Φ) is a triple $(\Sigma, \pi, \tilde{\Phi})$, where $(\Sigma, \tilde{\Phi})$ is a $C^l L$-space, and $\pi\colon \Sigma \to S$ is continuous open mapping such that the following conditions are satisfied.

2.3.1. For every $p \in S$, $\pi^{-1}(p) = \Sigma_p$ — the fiber of Σ with the foot-point p (or at p) is a vector space; denote by O_p its zero vector. Also let $\Sigma_U = \bigcup\{\Sigma_p;\ p \in U\}$, $U \subset S$.

2.3.2. For every $\tilde{\varphi} \in \tilde{\Phi}$, $D_{\tilde{\varphi}} = \Sigma_U$ for some open $U \subset S$, $\tilde{\varphi}(D_{\tilde{\varphi}}) \subset \mathscr{R}^m \times \mathscr{R}^n$ and $\tilde{\varphi}(\xi) = \big(\varphi_1(O_p), \varphi_2(p, \xi)\big)$, where $\varphi_2(p, \cdot)\colon \Sigma_p \to \mathscr{R}^n$ is linear.

2.3.3. The atlas $\tilde{\Phi}\,|\,S$ is C^k-equivalent with Φ.

2.3.4. The term vector bundle is used here with connotations somewhat different from standard ones as no local triviality is assumed. A more precise term — vector pseudobundle — was introduced in [4]. We refer to this paper for a discussion of vector bundles with C^k-structures, as opposed to $C^k L$-structures on which we insist here.

2.3.5. If (Σ, π, Φ), (Σ', π', Φ') are C^l-vector bundles over C^k spaces S, S', then $f\colon \Sigma \to \Sigma'$ is a C^l bundle morphism, $f \in C^l L(\Sigma, \Sigma')$ if for every $p \in S$, $f\colon \Sigma'_p \to \Sigma'_f(O_p)$ is linear and if for every $\varphi \in \Phi$, $\varphi' \in \Phi'$, $\varphi' \circ f \circ \varphi^{-1}$ can be locally extended to a $C^l L$-function about every "foot-point" $\varphi(O_p)$ of its domain. We also write $C^l L(\Sigma') = C^l L(\Sigma, \mathscr{R})$.

The notions of mono-, epi-, iso-, etc. morphisms are defined in the obvious way.

2.4. The vector bundles of interest in this paper are those obtained from the tangent bundle to a C^k space by functors of linear algebra $\times$ (cartesian product), $\otimes$ (tensor product), $\otimes_s$ (symmetric tensor product) and $\otimes_a$ (skew symmetric tensor product). In the next section we shall also introduce the modified tangent bundle and restricted tensor products.

We recall the standard procedure of constructing a vector bundle over a C^k-space (S, Φ). Assume that we are given the following local data:

2.4.1. For every $\varphi \in \Phi$ with $\varphi(D_\varphi) \subset \mathscr{R}^m$ we are given a set $\Sigma^\varphi = \bigcup\big\{\{x\} \times \Sigma_x^\varphi\colon x \in \varphi(D_\varphi)\big\} \subset \mathscr{R}^m \times \mathscr{R}^n$, where Σ_x^φ is a vector subspace of $\mathscr{R}^n$ for every $x \in \varphi(D_\varphi)$.

2.4.2. For every pair φ, $\psi \in \Phi$ with $D_\varphi \cap D_\psi \neq \emptyset$ we are given a connecting homeomorphism $h_{\varphi\psi}\colon \Sigma^\varphi \to \Sigma^\psi$ such that

(i) For every $p \in D_\varphi \cap D_\psi$, $h_{\psi\varphi}(\varphi(p), 0) = \psi(p)$, and $h_{\psi\varphi}(\varphi(p), \cdot)\colon \Sigma^\varphi_{\varphi(p)} \to \Sigma^\psi_{\psi(p)}$ is a linear isomorphism.

(ii) If $\varphi, \psi, \chi \in \Phi$ and $D_\varphi \cap D_\psi \cap D_\chi \neq \emptyset$, then $h_{\psi\chi} \circ h_{\chi\varphi} = h_{\psi\varphi}$ on $\Sigma^\varphi_{\varphi(D_\varphi \cap D_\chi \cap D_\psi)}$.

(iii) About every point of $\varphi(D_\varphi \cap D_\psi)$ $h_{\psi\varphi}$ is locally extendable to a $C^l L$-homeomorphism. More precisely for every $p \in D_\varphi \cap D_\psi$ there is a neighborhood U of p, $U \subset D_\varphi \cap D_\psi$, there are integers $M \geqslant m$, m', $N \geqslant n$, n' ($\Sigma^\psi \subset \mathscr{R}^{m'} \times \mathscr{R}^{n'}$) and there is a homeomorphism (h, H) in $C^l(\mathscr{R}^M \times \mathscr{R}^N, \mathscr{R}^M \times \mathscr{R}^N)$ such that $\varphi(U) \subset D$ and $(h, H) = h_{\psi\varphi}$ on $\Sigma^\varphi_{\varphi(U)}$.

With the data 2.4.1, 2.4.2 we note that for $\xi^\varphi \in \Sigma^\varphi$, $\xi^\psi \in \Sigma^\psi$ the relation $\xi^\psi = h_{\psi\varphi}\xi^\varphi$ is an equivalence relation compatible with the vector space structure of the fibers in Σ^φ, Σ^ψ and we define for $p \in S$ the fiber Σ_p, over p, as the space of equivalence classes of vectors in $\Sigma^\varphi_{\varphi(p)}$ with the natural vector space structure. Then we let $\Sigma = \bigcup\{\{p\} \times \Sigma_p\colon p \in S\}$ and define the bundle $(\Sigma, \pi, \tilde{\Phi})$ by letting $\pi(\{p\} \times \Sigma_p) = p$ and $\tilde{\Phi} = \{\tilde{\varphi}\}_{\varphi\in\Phi}$, where $\tilde{\varphi}\big((p, \xi)\big) = (p, \xi^\varphi)$, where ξ^φ is the representative in Σ^φ of the vector ξ in Σ_p.

2.5. We recall the definition of the tangent bundle TS to (S, Φ). For $X \subset \mathscr{R}^m$, $x \in X$ we denote by $N_{X,x}$ the space of germs of $C^k(\mathscr{R}^m)$ functions vanishing on X at x. For $k \neq \omega$, $N_{X,x} = \{f|_U\colon f \in C^k(\mathscr{R}^m), D_f = \mathscr{R}^m, f|_X = 0, U - \text{a neighborhood of } x\}$. Let $T_x X = \bigcap\{\ker f'(x)\colon f \in N_{X,x}\}$. Then if h is a C^k-homeomorphism of some neighborhood of x in $\mathscr{R}^M$, $M \geqslant m$, then for $v \in T_x X$, $h'(x, v)$ depends only on $h|_X$ and belongs to $T_{h(x)} h(X)$.

For $\varphi \in \Phi$, $p \in D_\varphi$ we let $T^\varphi_p S = T_{\varphi(p)}\varphi(D_\varphi)$ and $h_{\psi\varphi}(x, v) = h'(x; v)$, where h is any extension about x of the connecting homeomorphism $\psi \circ \varphi^{-1}$. In view of the preceding remarks this provides the data needed to define the bundle TX as in 2.4. The corresponding atlas $\tilde{\Phi}$ is in this case denoted by $\Phi_* = \{\varphi_*\}_{\varphi\in\Phi}$. If S is a C^k-space, then TX is a $C^{k-1}L$ bundle over S.

If (S, Φ), (S_1, Φ_1) are C^k-spaces and $f\colon S \to S_1$, then the formula $(\varphi_1 \circ f \circ \varphi^{-1})' = \varphi_{1*} f_* \varphi_*^{-1}$, $\varphi_1 \in \Phi$, $\varphi \in \Phi_1$ defines a unique $C^{k-1}L$-morphism of the vector bundles TS, TS_1. f_* is referred to as the tangent mapping of f. In the case when $f = \varphi \in \Phi$ the notation coincides with the preceding.

2.6. We recall the notions of upper semicontinuity of a vector bundle over S. The following conditions are clearly invariant under C^k-homeomorphisms and are meaningful for abstract bundles. Let $\Sigma \subset \mathscr{R}^m \times \mathscr{R}^n$ be of the form $\Sigma = \bigcup\{\{x\} \times \Sigma_x\colon x \in X\}$, where $X \subset \mathscr{R}^m$ and $N_{\Sigma,x}$ denotes the space of germs at $x \in X$ of functions in $C^k L(\mathscr{R}^m \times \mathscr{R}^n)$ vanishing on Σ.

(i) For $x \in X$, $u \in \mathscr{R}^n$, the condition $f(x, u) = 0$ for every $f \in N_{\Sigma,x}$ implies $u \in \Sigma_x$.

(ii) For $\{x_l\} \subset X$, $u_l \in \Sigma_{x_l}$ the conditions $x_l \xrightarrow[l\to\infty]{} x \in X$, $u_l \xrightarrow[l\to\infty]{} u$ imply $u \in \Sigma_x$.

(iii) The function $y \in X \to \dim \Sigma_y$ is upper semicontinuous at $x \in X$, i.e., $\dim \Sigma_x \geqslant \dim \Sigma_y$ for every y in a neighborhood of x in X.

Clearly (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). For every S, TS satisfies (i) at every point of S. There are natural examples of vector bundles over S (5.5.2) satisfying (iii) but not (ii).

3. CARTESIAN AND TENSOR PRODUCTS OF VECTOR BUNDLES

3.1. If $\{\Sigma_i, \pi_i, \Phi_i\}$, $i = 1, 2$, are C^l-vector bundles over a C^k-space (S, Φ), then the bundles $\Sigma_1 \dotplus \Sigma_2$ (direct sum), $\Sigma_1 \otimes \Sigma_2$ can be defined in a canonical way provided the atlasses $\Phi_1|_S$, $\Phi_2|_S$ have a common refinement which we can assume to coincide with Φ. If the latter is the case then we can choose $\Phi_i = \{\varphi_i\}_{\varphi\in\Phi}$ so that $\varphi_i|_S = \varphi$, $i = 1, 2$. Let $\Sigma_1 \dotplus \Sigma_2 = \bigcup\{\{p\} \times \times (\Sigma_{1p} \dotplus \Sigma_{2p}) \colon p \in S\}$ (or respectively $\Sigma_1 \otimes \Sigma_2 = \bigcup\{\{p\} \times (\Sigma_{1p} \otimes \Sigma_{2p}) \colon p \in S\}$), $(\pi_1 \dotplus \pi_2)(\Sigma_{1p} \dotplus \Sigma_{2p}) = p$ (or $\pi_1 \otimes \pi_2 (\Sigma_{1p} \otimes \Sigma_{2p}) = p$) and for $\varphi\colon D_\varphi \subset S, \to \mathscr{R}^m$, $\varphi_i\colon D_{\varphi_i} \to \mathscr{R}^m \times \mathscr{R}^{n_i}$, $i = 1, 2$, and for $p \in D_\varphi$, $u_i \in \Sigma_{ip}$ let $(\varphi_1 \dotplus \dotplus \varphi_2)(p, u_1 + u_2) = \big(\varphi(p), \varphi_1(p, u_1) + \varphi_2(p, u_2)\big) \in \mathscr{R}^m \times (\mathscr{R}^{n_1} \times \mathscr{R}^{n_2})$ (or respectively $\varphi_1 \otimes \varphi_2 (p, u_1 \otimes u_2) = \varphi(p) \times \big(\varphi_1(p, u_1) \otimes \varphi_2(p, u_2)\big) \in \mathscr{R}^m \times (\mathscr{R}^{n_1} \otimes \otimes \mathscr{R}^{n_2})$) extended by linearity.

It is clear that different choices of the common refinements of Φ_1, Φ_2 give rise by the above construction to isomorphic bundles.

3.2. The condition in 3.1 is clearly satisfied if the bundles Σ_1, Σ_2 coincide. In particular if Σ is a C^l-vector bundle over S, then the following are well defined and are C^l-vector bundles over S: $\oplus^k \Sigma$ (kth direct sum), $\otimes^k \Sigma$ (kth tensor power), $\otimes^k_a \Sigma = \bigwedge^k \Sigma$ (skew symmetric tensor power) and $\otimes^k_s \Sigma = \odot^k \Sigma$ (symmetric tensor power).

3.3. If (Σ, π, Φ) is a vector bundle over S, then the bundles $\Sigma \times TS$, $\Sigma \otimes TS$ etc. are all well defined: we merely consider S with the atlas $\Phi | S$.

3.4. Proposition. *If the bundles Σ_1, Σ_2 over S satisfy at some point any of conditions 2.6, then the bundles $\Sigma_1 \dotplus \Sigma_2$, $\Sigma_1 \otimes \Sigma_2$ satisfy the corresponding condition. The same is true for the bundles $\otimes^r_s \Sigma$ and $\otimes^r_a \Sigma$, $r \geqslant 2$.*

Proof. It follows directly from the definition that all the bundles in question inherit (iii) and (ii) and that $\Sigma_1 \dotplus \Sigma_2$ inherits (i). If $\Sigma_l \subset \mathscr{R}^m \times \times \mathscr{R}^{n_l}$, $l = 1, 2$ and $f \in N_{\Sigma_{1,x}}$, then $(x, u \otimes v) \to f(x, u) g(x, v)$ is in $N_{\Sigma_1 \otimes \Sigma_{2,x}}$ for every $g \in C^k L(\mathscr{R}^m \times \mathscr{R}^{n_2})$. If $\sigma = \Sigma u_s \otimes v_s \in \mathscr{R}^{n_1} \otimes \mathscr{R}^{n_2}$ satisfies $a(x, \sigma) = 0$ for all $a \in N_{\Sigma_1 \otimes \Sigma_2}$, then assuming as we may that $\{u_s\}$ and $\{v_s\}$ are linearly independent and choosing g appropriately we conclude that

$f(x, u_s) = 0$ for every $f \in N_{\Sigma_{1,x}}$ and for every s. Thus if Σ satisfies (i) at x, then $u_s \in \Sigma_x$ for all s. Similarly $v_s \in \Sigma_{2x}$. The same reasoning applies to the bundles $\otimes_a^k \Sigma$ and $\otimes_s^k \Sigma$.

3.5. Morphisms of vector bundles induce in a canonical way morphism of the bundles obtained by means of functors of linear algebra described above.

4. TANGENTS TO VECTOR BUNDLES

4.1. Let $\Sigma = \bigcup \{\{x\} \times \Sigma_x;\ x \in X\} \subset \mathscr{R}^m \times \mathscr{R}^n$, where $X \subset \mathscr{R}^m$ and Σ_x is a subspace of $\mathscr{R}^n$ for every $x \in X$. As in 2.5 we define $N_{\Sigma,x}$ as the space of germs at x of functions in $C^k L(\mathscr{R}^m \times \mathscr{R}^n)$ which vanish on Σ and let for $(x, u) \in \Sigma$, $T_{(x,u)}\Sigma = \bigcap \{\ker f'(x, u)\colon f \in N_{\Sigma,x}\} \subset \mathscr{R}^m \times \mathscr{R}^n$, where

$$f'(x, u;\ v, w) = f'_x(x, u;\ v) + f'(x, w). \tag{4.1.1}$$

4.1.2. Remark. If $(v, w) \in T_{x,u}\Sigma$, then $v \in T_x X$. However, in general $w \notin \Sigma_x$.

To check the first part it suffices to notice that if $g \in N_{X,x}$ and b: $\mathscr{R}^n \to \mathscr{R}$ is linear, then $f(y, u) = g(y)b(u)$ is in $N_{\Sigma,x}$. For the second part one can take $\Sigma = TX$ and $x = (0, 0) \in X$, where $X \subset \mathscr{R}^2$ is the parabola $x_2 = x_1^2$.

4.1.3. Remark. The set $\bigcup \{\{x, u\} \times T_{x,u}\Sigma\colon (x, u) \in \Sigma\}$ is the representative of tangent bundle Y to Σ in the inclusion chart $\Sigma \subset \mathscr{R}^m \times \mathscr{R}^n$, where Σ is considered with its $C^k L$ structure given by this inclusion. This bundle is replaced in the next paragraph by a vector bundle over X carrying essentially the same information.

4.1.4. Remark. It is not clear, even in the case when $\Sigma = TX$ when do the spaces $T_{x,u}\Sigma$ and the tangent space to Σ at x, u in the C^k structure induced by the inclusion $\Sigma \subset \mathscr{R}^m + \mathscr{R}^n$ coincide. A sufficient condition is that the ideal generated by $N_{\Sigma,(x,u)}$ be dense in the ideal of germs at x, u of C^k functions in $\mathscr{R}^m + \mathscr{R}^n$ vanishing on Σ.

4.1.5. If f: $\Sigma \to \mathscr{R}$ is extendable to a $C^k L$-function in a neighborhood of x in $\mathscr{R}^m$, then the differential $f'(x, u;\ v, w) = f'_x(x, u;\ v) + f(x, w)$ for $(u, w) \in T_{x,u}\Sigma$ is independent of a particular extension of f.

4.2. For $x \in X$ let $T'_x\Sigma = [\{u \otimes v + w\colon (v, w) \in T_{x,u}\Sigma\}] \subset \mathscr{R}^n \otimes \mathscr{R}^m + \mathscr{R}^n$, where [ ] denotes the linear span. Also $T'\Sigma = \bigcup \{\{x\} \times T_x^1\Sigma\colon x \in X\}$.

Then $T'\Sigma$ has the following properties.

4.2.1. For f: $\Sigma \to \mathscr{R}$ as in 4.1.5 there is a well defined f_*: $T'\Sigma \to \mathscr{R}$ given by the linear extension to $T'_x\Sigma$ of $f_*(x, u \otimes v + w) = \tilde{f}'_x(x, u;\ v) + \tilde{f}(x, w)$, where $\tilde{f} \in C^k L(\mathscr{R}^m \times \mathscr{R}^n)$, $\tilde{f}|_\Sigma = f$. The value of $f(x, u \otimes v + w)$ does not depend on the choice of $\tilde{f}$.

4.2.2. If $(h, H): \in C^l L(\Sigma, \Sigma')$, $\Sigma' \subset \mathscr{R}^{m'} \otimes \mathscr{R}^{n'}$, then with $h(X) = X'$ we have that for $u \otimes v \dotplus w \in T'_x \Sigma$, $u' \otimes v' \dotplus w' \in T'_{x'} \Sigma'$, where $x' = h(x)$, $u' = H(x, u)$, $v' = Dh(x; v)$, $w' = H(x, w) + H'_x(x, u; v)$, the primed quantities being independent of the choice of a $C^l L$-extension of (h, H). The mapping $(x, u \otimes v \dotplus w) \to (x', u' \otimes v' \dotplus w')$ we denote by $(h, H)_*$; $(h, H)_* \in C^{l-1} L(T'\Sigma, T'\Sigma')$.

4.2.3. The procedure 2.4.2 can now be used to define the bundle $T'\Sigma$ for an arbitrary vector bundle (Σ, π, Φ) over a C^k-space S. The data 2.4.1, 2.4.2 are: for $\varphi \in \Phi$, $\varphi(D_\varphi) \subset \mathscr{R}^m \times \mathscr{R}^n$. $T'\Sigma^\varphi = T'\varphi(D_\varphi)$ and the connecting homeomorphisms are given by $h_{\psi\varphi} = (\psi \circ \varphi^{-1})_*$ with notations as in 4.2.2. The resulting bundle $T'\Sigma$ we refer to as the modified tangent bundle. Its charts will be denoted by $\varphi_*, \varphi \in \Phi$.

4.2.4. To every $f \in C^l L(\Sigma, \Sigma')$ there corresponds a unique $f_* \in C^{l-1} L(T'\Sigma, T'\Sigma')$ given by $(\varphi' \circ f \circ \varphi^{-1})_* = \varphi'_* \circ f_* \circ \varphi^{-1}$, $\varphi \in \Phi$, $\varphi' \in \Phi'$.

4.3. The form of the connecting homeomorphism in 4.2.2 implies that the "horizontal projection" π_h: $T'\Sigma \to \Sigma \otimes TX$ given by π_h: $\omega \dotplus w \in (\mathscr{R}^n \otimes \mathscr{R}^m) \dotplus \mathscr{R}^u \to \omega \in \mathscr{R}^n \otimes \mathscr{R}^m$ is a well defined bundle morphism. Define the restricted tensor product by $\Sigma \overset{\prime}{\otimes} TX = \pi_h(T'\Sigma)$.

More explicitly, $(\Sigma \overset{\prime}{\otimes} TX)_x$ is the subspace of $(\Sigma \otimes TX)_x$ spanned by the vectors $u \otimes v$ with the property

4.3.1. There exists $w \in \mathscr{R}^n$ such that $f'_x(x, u; v) + f(x, w) = 0$ for every $f \in N_{\Sigma, x}$.

4.4. There is no canonically defined vertical projection from the bundle $T'\Sigma$ (see 4.1.2). One can nevertheless consider mappings Γ: $\Sigma \overset{\prime}{\otimes} TS \to T'S$ which are linear on the fibers and satisfy the condition $\pi_h \circ \Gamma$ = identity on $\Sigma \overset{\prime}{\otimes} TS$. Any such mapping is called an affine connection in Σ. In the case when $\Sigma = TS$ we refer to Γ as an affine connection on S.

4.4.1. We avoid the term bundle morphism with regard to affine connections due to the fact that even in rather simple cases they need not be continuous.

4.4.2. Let Γ be an affine connection in $(\Sigma, \pi, \tilde{\Phi})$. For any $\tilde{\varphi} \in \tilde{\Phi}$ denote $\varphi = \tilde{\varphi}|_S$ and let $\Gamma^\varphi = \varphi_*^{-1} \circ \Gamma \circ (\tilde{\varphi} \otimes \varphi_*)^{-1}$ represent Γ in the chart $\tilde{\varphi}$. Then $\Gamma^\varphi(x, u \otimes v) = \big(x, u \otimes v \dotplus \gamma^\varphi(x, u \otimes v)\big)$, where $u \otimes v \to \gamma^\varphi(x, u \otimes v)$ is linear (or equivalently $(u, v) \to \gamma^\varphi(x, u, v)$ is bilinear).

The representations Γ^ψ, Γ^φ in two charts $\tilde{\psi}$, $\tilde{\varphi}$ at the same point $p \in S$ are connected by 4.2.2 with $x = \varphi(p)$, $x' = \psi(p)$ and $(h, H) = \tilde{\psi} \circ \tilde{\varphi}^{-1}$, in particular $\gamma^\psi(x', u' \otimes v') = H\big(x, \gamma^\varphi(x, u \otimes v)\big) + H'_x(x, u; v)$.

4.4.3. In the case when $\Sigma = TS$, $H(x, u) = h'(x; u)$ and the above formula becomes $\gamma^\psi(x', u' \otimes v') = h'\big(x; \gamma^\varphi(x, u \otimes v)\big) + h''(x; u, v)$, where $x' = h(x)$, $u' = h'(x; u)$, $v' = h'(x; v)$.

This is formally the same as the transformation law for connections in the classical case.

4.4.4. If for some atlas $\tilde{\Phi}$ on Σ we are given in the image of every chart φ a mapping Γ^φ (or γ^φ): $\tilde{\varphi}(D_{\tilde{\varphi}}) \acute{\otimes} T_\varphi(D_{\tilde{\varphi}}) \to T'\tilde{\varphi}(D_{\tilde{\varphi}})$ as in 4.4.2, then a connection on Σ can be reconstructed from these local data by means of a partition of unity (assuming that S is paracompact).

4.5. Let $f \in C^l L(\Sigma, \Sigma')$ and f_* be as in 4.2.4. If Γ is an affine connection in Σ, then the corresponding covariant derivative of f is given by $\nabla_\Gamma f = f_* \circ \Gamma$: $\Sigma \acute{\otimes} TS \to T'\Sigma'$. In particular if $f \in C^l L(\Sigma)$, then $\nabla_\Gamma f$: $\Sigma \acute{\otimes} TS \to \mathcal{R}$.

4.6. If $(\Sigma, \pi, \tilde{\Phi})$ is a vector bundle over (S, Φ), $\Phi = \tilde{\Phi}|_S$, then u: $S \to \Sigma$ is a section of Σ provided $\pi \circ u =$ identity on Σ. A section u is of class C^l, $u \in C^l(S, \Sigma)$, provided for every $\varphi \in \Phi$, $\varphi = \tilde{\varphi}|_S$, $\varphi \circ u \circ \varphi^{-1}$ is locally extendable about every point of its domain to a C^l function.

4.6.1. PROPOSITION. *If u: $S \to \Sigma$ is a C^l-section of a C^l-vector bundle Σ, if $p \in S$ and if $u(p) = (p, u_p) \in \Sigma$, then for every $v \in T_p S$, $u_p \otimes v \in (\Sigma \acute{\otimes} TS)_p$.*

It suffices to verify the claim in the case when $\Sigma \subset \mathcal{R}^m \times \mathcal{R}^n$ is as in 4.1 and $u(y) = (y, u_y)$, where u: $X \to \mathcal{R}^n$ and $u_y \in \Sigma_y$, $y \in X$. If $a \in N_{\Sigma,x}$, then $a \circ \tilde{u}$ is in $N_{X,x}$ for every C^l-extension $\tilde{u}$ about x of the function u; hence for every $v \in T_x X$, $a'_x(x, u(x); v) + a(x, u'(x; v)) = 0$ establishing the claim.

The converse to 4.6.1 is false.

4.6.2. For an affine connection Γ in a C^l-vector bundle Σ over S and for a section u: $S \to \Sigma$ one can define the covariant derivative $\nabla_\Gamma u(p;\ v)$ of u at $p \in S$ with increment $v \in T_p S$ by the condition $(f \circ u)_*(p;\ v) = \nabla_\Gamma f(p;\ u(p) \otimes v) + f(p, \nabla_\Gamma u(p;\ v))$ for every $f \in C^l L(\Sigma)$.

The first term on the right-hand side is meaningful because of 4.6.1, also in local coordinates with $\Gamma(x, u \otimes v) = (x, \gamma(x, u \otimes v))$ one easily gets

$$\nabla_\Gamma u(x;\ v) = u'(x;\ v) - \gamma(x, u(x) \otimes v).$$

It can be checked directly that $\nabla_\Gamma v(x;\ u) \in \Sigma_x$ provided Σ satisfies at x the upper semicontinuity condition 2.6(i), or if γ is continuous at x and Σ satisfies at x, 2.6(ii).

5. FURTHER PROPERTIES OF THE MODIFIED TANGENT BUNDLES AND EXAMPLES

5.1. We assume from now on that spaces and bundles under consideration are of class at least C^∞.

The functors T' and $\acute{\otimes}$ can be iterated. In the case when $\Sigma = TS$

we write $T^kS = T'T^{k-1}S$, $T'S = TS$ and $\overset{\prime}{\otimes}{}^kTS = (\overset{\prime}{\otimes}{}^{k-1}\, TS)\, \overset{\prime}{\otimes} TS$, $k \geqslant 2$.

5.2. Recall that $p \in S$ is a point of homogeneity provided $\dim_q S = \dim_p S$ for every q in some neighborhood of p in S. Also recall that $\dim_q S = \dim T_p S$.

5.2.1. PROPOSITION. *If $p \in S$ is a point of homogeneity of S, then* $(\overset{\prime}{\otimes}{}^kTS)_p = (\otimes^kTS)_p$.

Proof. It suffices to check the statement for the image of any chart about p. But in the tangential chart the statement is obvious.

5.2.2. 5.2.1 applies in the case when S is a manifold or the closure of an open subset in a manifold.

5.2.3. In a neighborhood of a point of homogeneity $p \in S$ one can define a C^∞-affine connection as follows. By choosing a tangential chart φ at p one can assume that $\varphi(D_\varphi) \subset \mathscr{R}^m$, where $m = \dim_q S$ for every $q \in D_\varphi$. Then $T_x\varphi(D_\varphi) = \mathscr{R}^m$, $\big(\otimes^2 T\varphi(D_\varphi)\big) = \mathscr{R}^m \otimes \mathscr{R}^m$ for every $x \in \varphi(D_p)$. Choose now any function γ (in $C^\infty L\big(\mathscr{R}^m \times (\otimes^2\mathscr{R}^m), \mathscr{R}^m\big)$) and let $\gamma^\varphi = \gamma|_{X \times \mathscr{R}^m \otimes \mathscr{R}^m}$. In any other chart ψ about p (tangential or not) γ^ψ is then defined as in 4.4.3. In the case when S is a C^∞-manifold and only the tangential charts are considered this coincides with the standard notion of an affine connection.

5.3. The next case of interest is when S is locally a union of finite number of sets as in 5.2. More precisely, assume that $X \subset \mathscr{R}^m$ is of the form $X = \bigcup_{i=1}^{N} X_i$, that $x \in \bigcap_{i=1}^{N} X_i$ and that x is a point of homogeneity of X_i for $i = 1, \ldots, N$. For $k \geqslant 1$ let $W = \sum_{i=1}^{N} (\otimes^k TX_i)_x \subset \otimes^k \mathscr{R}^m$.

5.3.1. PROPOSITION. *With the hypotheses as above we have* $W \subset \overset{\prime}{\otimes}{}^k TX_x$.

Proof. If the result is true for some k, then $N_{\overset{\prime}{\otimes}{}^kTX,x} \subset N_{\otimes^kTX_{ix}}$ for $i = 1, \ldots, N$ which by 5.2.1 and the definition of $\overset{\prime}{\otimes}$ implies that $(\otimes^{k+1}TX)_x \subset \overset{\prime}{\otimes}{}^{k+1}TX_x$. ∎

5.3.2. The following example shows that the inclusion in 5.3.1 is in general strict. Let $X = \big\{(x_1, x_2) \in \mathscr{R}^2\colon x_2\big(x_2 - \exp(-x_1^{-2})\big) = 0\big\}$. Write $a \in C^\infty L(\mathscr{R}^2 \times \mathscr{R}^2, \mathscr{R})$ in the form $a(x, v) = a_1(x)v^1 + a_2(x)v^2$. Then $a \in N_{TX}$ if and only if $a_1(x_1, 0) = 0$ and $a_1 + a_2 2x_1^{-3}\exp(-x_1^{-2}) = 0$ for $x_2 = \exp(-x_1^{-2})$. It follows that $a_1(x_1, x_2) = b_1(x_1, x_2)x_2$ and that $x_1^3 b_1 + 2a_2 = 2b_2$, where $b_2\big(x_1, \exp(-x_1^{-2})\big) = 0$ and that $a(x, v) = b_1(x_2v^1 - \tfrac{1}{2}x_1^3v^2) + b_2v^2$. Condition 4.3.1 is equivalent to $v^1u^2 = 0$ and $v^2u^2 = 0$. (Note that $\partial b_2/\partial x_1 = 0$ at 0.) It follows that $(\overset{\prime}{\otimes}{}^2TX)_0 = [e_1 \otimes e_1, e_2 \otimes e_1]$. If $X_1 = \{(x_1, 0)\colon x_1 \in \mathscr{R}^1\}$, $X_2 = \{x_2 = \exp(-x_1^2)\colon x_1 \in \mathscr{R}^1\}$, then $(TX_1 \otimes TX_1)_0 = (TX_2 \otimes TX_2) = [e_1 \otimes e_1]$.

5.3.3. Remark. 5.3.2 shows that $TX\acute{\otimes}TX$ need not be symmetric, i.e., $u\otimes v\in(\acute{\otimes}^2TX)_x$ does not in general imply that $v\otimes u\in(\acute{\otimes}^2TX)_x$.

5.3.4. Neither of the peculiarities in the example above may occur if the singularity is a point of finite order tangency of two manifolds:

Let $g\in C^\infty(\mathscr{R}^{m-1})$, $m\geqslant 2$, $g^{(l)}(0)=0$, $l=0,1,\ldots,k-1$, $k\geqslant 2$ and let $X=\{(x',x_m)\in\mathscr{R}^m\colon x_m(x_m-g(x'))=0\}$. Then $(\acute{\otimes}^2TX)_0=\mathscr{R}^{m-1}\otimes\otimes\mathscr{R}^{m-1}$.

We consider next a case when the inclusion in 5.3.1. is an equality.

5.3.5. Proposition. *With the notations as in* 5.3.1 *assume that, for* $i=1,\ldots,N$, X_i *is a closed convex set. Then for* $x\in\bigcap_{i=1}^N X_i$ *and any* $k\geqslant 2$, $(\acute{\otimes}^kTX)_x=\sum_{i=1}^N(\otimes^kTX_i)_x$.

Proof. We can assume that $\dim_x X=m$ and that $x=0$. For $i=1,\ldots,N$ let $P_i\colon\mathscr{R}^m\to[X_i]$ denote the orthogonal projection onto the span $[X_i]$ of X_i. If $a\in C^\infty(\mathscr{R}^m\times\mathscr{R}^m,\mathscr{R})$ we write $a(x,v)=\langle\tilde a(x),v\rangle$, $a\in C^\infty(\mathscr{R}^m,\mathscr{R}^m)$. If $a\in N_{TX,0}$, then $\tilde a(0)=0$ and $\tilde a(x)=A(x)x$, where $A\in C^\infty(U,L(\mathscr{R}^m))$, where U is a neighborhood of 0 which we can assume to be convex. Let $U_i=U\cap P_i^{-1}(X_i)$, then $0\in U_i$, U_i is convex and U_i has non-void interior. If $x\in U_i$, $v\in\mathscr{R}^m$, then $P_ix\in X_i$, $P_iv\in T_{p_ix}X_i$ and $\langle A(P_ix)P_ix,P_iv\rangle=0$. Replacing x by tx, letting $t\to 0$ we conclude that $\langle A(0)P_ix,P_iv\rangle=0$ for $x\in U$, $v\in\mathscr{R}^m$. By linearity the equation remains valid for all $x\in\mathscr{R}^m$ and

$$P_iA(0)P_i=0.$$

Conversely if $A(0)$ satisfies the above condition, then $(x,v)\to\langle A(0)x,v\rangle$ is in $N_{TX,0}$. Definition 4.3 of $(TX\acute{\otimes}TX)_0$ becomes $v\otimes u\in(TX\acute{\otimes}TX)_0$ $\Leftrightarrow\langle Au,v\rangle=0$ for all A satisfying $P_iAP_i=0$. With the usual identification of $L(\mathscr{R}^m)$ with $\mathscr{R}^m\otimes\mathscr{R}^m$ we let $V=\{A\in\mathscr{R}^m\otimes\mathscr{R}^m\colon P_iAP_i=0,\ i=1,\ldots,N\}$. Then $(TX\acute{\otimes}TX)_0=V^\perp$, on the other hand it is easily checked that $\big(\sum_{i=1}^N TX_i\otimes TX_i\big)^\perp=V$ which proves the claim for $k=2$. Assuming the claim established for $k-1$ we write $a\in N_{\acute{\otimes}^{k-1}TX,0}$ in the form $a(x,v)=\langle\tilde a(x),v\rangle$, $v\in\otimes^{k-1}\mathscr{R}^m\tilde a\in C^\infty(\mathscr{R}^m,\otimes^{k-1}\mathscr{R}^m)$ and easily conclude that $\tilde a$ can be represented in the form $\tilde a(x)=\tilde a(0)+A(x)\cdot x$, where $A\in C^\infty(\mathscr{R}^m,\otimes^k\mathscr{R}^m)$, $\cdot$ denotes the contraction with respect to the last entry and $\tilde a(0)$, $A(0)$ satisfy $\langle\tilde a(0),P_iv_i\times\ldots\times P_iv_{k-1}\rangle=0$, $\langle A(0)P_ix,P_iv_i,\ldots,P_iv_{k-1}\rangle=0$ for all $x,v_i,\ldots,v_{k-1}\in\mathscr{R}^m$ and $i=1,\ldots,N$. 5.3.1 now becomes $v\otimes u\in(\acute{\otimes}^kTX)_0$ if and only if $\langle A(0)\cdot u,v\rangle+\langle\tilde a(0),w\rangle=0$ for some $w\in\otimes^{k-1}\mathscr{R}^m$ and every $\tilde a(0)$, $A(0)$ satisfying the conditions above. Choosing in particular $\tilde a(0)=0$ and using the same argument as for $k = 2$ we conclude that $(\otimes^k TX)_0 \subset \sum_{i=1}^{N} \otimes^k TX_i$ which is the opposite inclusion to the one 5.3.1. ■

5.3.6. PROPOSITION. *With the notations and hypotheses of Proposition 5.3.5 assume that for every $1 \leqslant i, j \leqslant N$, $X_j \cap X_l \subset \{X_i, i = 1, \dots, N\}$ and for $1 \leqslant j \leqslant N$ let $I_j = \{i: X_j \subset X_i\}$. Then $\big((\otimes^k TX) \otimes' TX\big)_x = \sum_{j=1}^{N} \big(\Sigma_{i_1,\dots,i_k \in I_j} TX_{i_1} \otimes \dots \otimes TX_{i_k}\big) \times TX_j$.*

The proof is similar to that of Proposition 5.3.5 and is omitted.

5.3.7. COROLLARY. *If X is as in 5.3.5, then the mapping $(\otimes'^2 TX)_y \to 0 \in \mathscr{R}^m$, $y \in X$, defines, in the inclusion chart $X \subset \mathscr{R}^m$, a C^∞-affine connection.*

Recall that a C^∞-space S is of polyhedral type if for every $p \in S$ there is a chart φ at p in the maximal atlas defining the C^∞ structure on S, such that $\varphi(D_\varphi) = UX_i \subset \mathscr{R}^m$, where X_i are closed simplices.

It follows that on a space of polyhedral type there always exists a C^∞-affine connection.

5.4. We consider here some additional properties of the bundle $\otimes'^k TS$.

5.4.1. PROPOSITION. *Suppose that $X \subset \mathscr{R}^m$, $x \in X$. If $u_1 \otimes \dots \otimes u_k \in (\otimes'^k TX)_x$, then $u_{i_1} \otimes \dots \otimes u_{i_l} \in (\otimes'^l TX)_x$ for $1 \leqslant i_1 < \dots < i_l \leqslant k$.*

Proof. For $k = 2$ this is Remark 4.1.2. For $k > 2$ we can proceed by induction. If $i_l < k$, then there is nothing to prove, thus we can assume that $i_l = k$. Write $\{1, \dots, k\} = \{i_1, \dots, i_l\} \cup \{j_{l+1}, \dots, j_k\}$. If $a \in N'_{\otimes^{l-1} TX, x}$ then, using the induction hypothesis the function $a_1(x, v_1 \otimes \dots \otimes v_{k-1}) = a(x, v_{i_1}, \dots, v_{i_{l-1}}) \langle b, v_{j_{l+1}} \otimes \dots \otimes v_{j_k} \rangle$ is in $N'_{\otimes^{k-1} TX, x}$ for every $b \in \otimes^{k-l} \mathscr{R}^m$. Applying 5.3.1 to this function: there is $w \in \otimes^{k-1} \mathscr{R}^m$ such that $D_x a(x, v_{i_1}, \dots, v_{i_{l-1}}; v_k) \langle b, v_{j_{l+1}} \otimes \dots \otimes v_{j_k} \rangle + a_1(x, w) = 0$. Choose b so that $\langle b, v_{j_{l+1}} \otimes \dots \otimes v_{j_k} \rangle = 1$ and let $w = \sum_a w_1^a \otimes \dots \otimes w_{k-1}^a$. Then $a'_x(x, v_{i_1} \otimes \dots \otimes v_{i_{l-1}}; v_k) + a(x, \tilde{w}) = 0$, where $\tilde{w} = \sum_a \langle b, w_{j_{l+1}}^a \otimes \dots \otimes w_{j_k}^a \rangle \, w_{i_1}^a \otimes \dots \otimes w_{i_{l-1}}^a$. ■

Remark. The above proof remains valid if $i_1, \dots, i_l$ are distinct, not necessarily increasing provided $i_s = k \Rightarrow s = l$.

5.4.2. If (S_1, Φ_1), (S_2, Φ_2) are C^∞-spaces, then on the space $S_1 \times S_2$ we can consider the atlas $\Phi_1 \times \Phi_2 = \{\varphi_1 + \varphi_2 : \varphi_1 \in \Phi_1, \varphi_2 \in \Phi_2\}$, where $\varphi_1 + \varphi_2 : (p_1, p_2) \in D_{\varphi_1} \times D_{\varphi_2} \to \varphi_1(p_1) + \varphi_2(p_2) \subset \mathscr{R}^{n_{\varphi_1}} + \mathscr{R}^{n_{\varphi_2}}$. We identify here in the standard way $\mathscr{R}^{n_{\varphi_1}} + \mathscr{R}^{n_{\varphi_2}}$ with $\mathscr{R}^{n_{\varphi_1} + n_{\varphi_2}}$. It is clear then that $(S_1 \times S_2, \Phi_1 \times \Phi_2)$ is a C^∞-space.

The bundle $T(S_1 \times S_2)$ can be identified in a canonical way with the bundle $TS_1 + TS_2 = \bigcup \big\{\{(p_1, p_2)\} \times (T_{p_1} S_1 + T_{p_2} S_2) : p_1 \in S_1, p_2 \in S_2\big\}$. In the case when $S_1 = S_2 = S$ this bundle is not to be confused with $TS+TS$ introduced in 3.1 — the latter is the restriction of the former to the diagonal of $S\times S$.

5.4.3. PROPOSITION. (a) *If* $X_i \subset \mathscr{R}^{m_i}$, $x_i \in X_i$, $i=1,2$, *and if* $(v_1+ v_2)\otimes(u_1+u_2) \in \acute{\otimes}^2 T(X_1\times X_2)_{(x_1,x_2)}$, *then* $v_i\otimes u_i \in (\acute{\otimes}^2 TX_i)_{x_i}$, $i=1,2$.

(b) *If* x_2 *is a point of homogeneity of* X_2, *then the converse of* (a) *is true, in particular* $v\otimes u = (v+0)\otimes(0+u) \in \acute{\otimes} T(X_1\times X_2)_{(x_1,x_2)}$ *if* $v\in T_{x_1}X_1$ *and* $u\in T_{x_2}X_2$.

(c) *If* x_2 *is a point of homogeneity of* X_2, *then* $(u_1+v_1)\otimes \dots \otimes(u_k+ v_k) \in \acute{\otimes}^s T(X_1\times X_2)_{(x_1,x_2)}$ *if and only if* $u_1\otimes\dots\otimes u_k \in (\acute{\otimes} TX_1)_{x_1}$ *and* $v_1,\dots,v_k\in T_{x_2}X_2$. *Also if for some* $0\leqslant s\leqslant k$, $u_1\otimes\dots\otimes u_s \in (\acute{\otimes}^s TX_1)_{x_1}$ *and* $u_{s+1},\dots,v_k\in T_{x_2}X_2$, *then* $u_1\otimes\dots\otimes u_s\otimes v_{s+1}\otimes\dots\otimes v_k \in \acute{\otimes}^k T(X_1\times X_2)_{(x_1,x_2)}$.

We omit the details of the proof which uses the following observations. 1° If $a\in N_{TX_1,x_1}$, then the function $((y_1,y_2),u_1+u_2)\to a(y_1,u_1)$ is in $N_{T(X_1\times X_2),(x_1,x_2)}$, similarly for $a\in N_{TX_2,x_2}$. 2° If $a\in N_{T(X_1\times X_2),(x_1,x_2)}$, then writing $a(y_1,y_2,u_1+u_2) = a_1(y_1,y_2,u_1)+a_2(y_1,y_2,u_2)$ we have:

$$y_2\mapsto a_1(x_1,y_2,u_1) \quad \text{is in } N_{X_2,x_2};$$
$$y_1,u_1\mapsto a_1(y_1,x_2,u_1) \quad \text{is in } N_{TX_1,x_1}$$

and similarly for a_2.

5.5. We next describe the bundle $\acute{\otimes} TX$ at certain conical singularities.

5.5.1. EXAMPLE. Suppose that A is an $m\times m$ symmetric and non-singular matrix, $m>2$, and $X = \{x\in\mathscr{R}^m\colon \langle Ax,x\rangle = 0\}$. Then $(TX\acute{\otimes}TX)_0 = [\{u\otimes v\colon \langle Au,v\rangle = 0\}]$.

To check the claim: suppose that $a\in N_{TX,0}$. Since $T_0X = \mathscr{R}^m$ this implies that $a(0,u)=0$ for every $u\in\mathscr{R}^m$ and that $a(x,u) = \langle B(x)x,u\rangle$, where $B\in C^\infty(\mathscr{R}^m, L(\mathscr{R}^m))$. Also for $x\in X$, $x\neq 0$, $T_xX = \{u\colon \langle Ax,u\rangle = 0\}$ and $\langle B(x)x,u\rangle = 0$ whenever $\langle Ax,u\rangle = 0$. Replacing x by tx, letting $t\downarrow 0$ we get $\langle B(0)x,u\rangle = 0$ whenever $\langle Ax,u\rangle = 0$ and $\langle Ax,x\rangle = 0$. This implies that $B(0)x = \lambda(x)Ax$ for all x such that $\langle Ax,x\rangle = 0$. The set $\{x\colon \langle Ax,x\rangle = 0,\ |x| = 1\}$ is connected $(m>2)$ and on this set $\lambda(x) = \langle A^{-1}B(0)x,x\rangle$ is continuous. Since λ is finite valued, λ must be a constant and $B(0) = \lambda A$. Definition 4.3.1 becomes $v\otimes u\in(\acute{\otimes}^2TX)_0$ if $\langle B(0)u,v\rangle = 0$ as claimed.

5.5.2. Remark. The bundle $\acute{\otimes}TX$ in the above example does not satisfy 2.6(ii) at 0 (but satisfies 2.6(ii)). It suffices to notice that for $x\neq 0$, $x\in X$ one can find a pair $u,v\in\mathscr{R}^m$ such that $\langle Ax,u\rangle = 0$, $\langle Ax,v\rangle = 0$ but $\langle Au,v\rangle \neq 0$.

5.5.3. Remark. For X as in 5.5.1 the spaces $(\acute{\otimes}^k TX)_0$ and $\big((\otimes^k TX)\acute{\otimes} TX\big)_0$ are determined in a similar way. We have $u_1\otimes \dots \otimes u_k \in (\acute{\otimes}^k TX)_0$ if and only if $\langle Au_i, u_j\rangle = 0$, $i, j = 1, \dots, k$, $i \neq j$. Also $u_1\otimes \dots \otimes u_k \otimes v \in \big((\otimes^k TX)\acute{\otimes} TX\big)_0$ if and only if $\langle Au_i, v\rangle = 0$ for $i = 1, \dots, k$. In the process of verifying these one can notice that $N'_{\acute{\otimes}^k TX,0} = N_{\otimes^k TX,0}$.

5.5.4. If X is as in 5.5.1 except that $A \neq 0$ is singular, then we can represent X in the form $X = \ker A \times X_1$, where $X_1 = \{x \in (\ker A)^\perp : \langle A_1 x, x\rangle = 0\}$, $A_1 = A|_{(\ker A)^\perp}$ and a description of the bundles as in 5.5.3 can be obtained using Proposition 5.4.3.

5.5.5. For $m = 2$ the set X in 5.5.1 is a pair of lines and the bundles in question are easily determined by 5.3.5.

5.5.6. Remark. If $X = \{x \in \mathcal{R}^m;\ f(x) = 0\}$, where $f \in C^\infty(\mathcal{R}^m)$ and x is a non-degenerate critical point of f, then by Morse's lemma, in a suitable system of coordinates, X is of the form as in 5.5.1 and the above determination of $\otimes TX$ applies also in this case.

5.5.7. With X as in 5.5.1 it is easy to determine the form of an affine connection on X. Clearly at all points of X different from the vertex 0 remarks 5.2.3 can be applied. Also we know that $(x, u) \to (Ax, u)$ is in $N_{TX,0}$ and that this function generates $N_{TX,0}$. It follows that any affine connection $\gamma\colon TX \acute{\otimes} TX \to \mathcal{R}^m$ (see 4.4.3) satisfies $\langle Av, u\rangle + \langle Ax, \gamma(x, v\otimes u)\rangle = 0$ and for $x \in X$ and γ can be written in the form

$$\gamma(x, v\otimes u) = -\frac{\langle Av, u\rangle}{|Ax|^2} Ax + w(x), \qquad \text{where } w(x) \in T_x X,$$

i.e., $\langle Ax, w(x)\rangle = 0$. It follows that

$$|\gamma(x, v\otimes u)|^2 = \frac{\langle Av, u\rangle^2}{|Ax|^2} + |w(x)|^2.$$

For any $x \in X$ it is easy to find $v \in T_x X$ such that $v \perp x$ and $\langle Av, v\rangle \neq 0$. If $v_t = x + t^{1/4} v \in T_{tx} X$ for $t > 0$, then $v_t \otimes v_t \in (TX \acute{\otimes} TX)_{tx}$ and, for $t \downarrow 0$, $v_t \otimes v_t \to x\otimes x \in (TX \acute{\otimes} TX)_0$ since $\langle Ax, x\rangle = 0$. On the other hand

$$|\gamma(tx, v\otimes v)|^2 = \frac{t}{t^2 |Ax|^2} + |w(x)|^2$$

and it is evident that γ cannot be continuous at 0. For a fixed $v \in \mathcal{R}^m$, $x \to |Ax|^2 v - (Ax, v)Ax = v(x)$ is a C^∞ section of TX, but $\gamma\big(x, v(x)\otimes v(x)\big)$ is not in C^∞.

5.6. Example. Let $X = \{x \in \mathcal{R}^m \colon x_m^l = x_1^l + \dots + x_{m-1}^l\}$, $m > 2$, $l \geqslant 3$. By a calculation which we omit one can verify that $(\acute{\otimes}^2 TX)_0 = \mathcal{R}^m \otimes \mathcal{R}^m$. This with 5.5 shows that the vertices of the 2nd degree cone and of a higher degree cone are distinct from C^∞ point of view.

5.7. If Γ is a connection on X, then by analogy with the classical case it is natural to expect that Γ should give rise to a connection in the bundle $\otimes^k TX$ by the formula (in local coordinates)

$$\gamma_k\big((u_1\otimes \ldots \otimes u_k)\otimes v\big) = \sum_{l=1}^{k} u_1\otimes \ldots \otimes u_{l-1}\otimes \gamma(u_l\otimes v)\otimes \ldots \otimes u_k,$$

extended by linearity. The expression on the right-hand side is meaningful since $(u_1\otimes \ldots \otimes u_k)\otimes v \in \big((\otimes^k TX)\overset{\prime}{\otimes} TX\big)_x$ implies that $u_i\otimes v \in (TX\overset{\prime}{\otimes} TX)_x$.

In order for γ_k so defined to give a connection we must have for every $\omega \in \big((\otimes^k TX)\overset{\prime}{\otimes} TX\big)_x$, $\omega + \gamma_k(\omega) \in T^1\otimes^k TX$. A sufficient condition for this to hold is that every $a \in N_{\otimes^k TX}$ be a finite sum of functions of the form $a_1(x, u_1)\ldots a_k(x, u_k)$, where at least one of the a_i's is in N_{TX}. This is not true even in the case of polyhedral singularities: if $X = \{x \in \mathscr{R}^2 : x_1x_2(x_1 - x_2) = 0\}$, then $a(x, u, v) = x_1u^1v^2 - x_2u^2v^1$ is in $N_{TX\otimes TX,0}$ but is not in the span of products as above.

However, in the case of polyhedral singularities the zero connection gives rise to the zero connection in the bundles $\otimes^k TX$.

5.7.1. Example. If $X = \{(x_1, x_2) \in \mathscr{R}^2 : x_2(x_2 - x_1^2) = 0\}$, then $N_{TX,0}$ is generated by the functions $x_2u_1 - \frac{1}{2}xu$ and $(x_2 - x_1^2)u_2$. It follows that $u\otimes v + w \in T'X$ provided $u_1v_2 - \frac{1}{2}u_2v_1 + x_2w_1 - \frac{1}{2}x_1w_2 = 0$ and $u_2v_2 - 2x_1u_2v_1 + (x_2 - x_1^2)w_2 = 0$. When $x_2 = 0$ both equations imply that $w_2 = 0$ and impose no restriction on w_1. When $x_2 = x_1^2$, then $u_2 = 2x_1u_1$, $y_2 = 2x_1v_1$ and $2x_1(u_1v_1 - \frac{1}{2}u_1v_1) + x_1(x_1w_1 - \frac{1}{2}w_2) = 0$, i.e., $u_1v_1 + x_1w_1 - \frac{1}{2}w_2 = 0$. Note that by 5.3.4, $(TX\overset{\prime}{\otimes} TX) = [e_1\times e_1]$. The continuity of $(x, u\otimes v) \mapsto w$ at 0 would imply that $w_2 = 2u_1v_1$ which contradicts the preceding conclusion.

Note that in this example the condition in 5.7 is satisfied.

6. BUNDLES ASSOCIATED WITH HIGHER ORDER DERIVATIVES AND WITH THE EXTERIOR DERIVATIVE

6.1. The bundles $T'F$, $F\overset{\prime}{\otimes} TS$ etc. considered in the preceding sections are associated with the operation of the first order differentiation of arbitrary functions in $C^\infty L(F)$. In connection with differentiation of functions in various subspaces of $C^\infty L(F)$ or higher order differentiation of functions in $C^\infty(S)$ the procedure used in construction of $T'F$ yields with suitable modifications bundles over S which are better adapted to the purpose at hand. The modifications described in this section deal with the cases when $F = TX$, $X \subset \mathscr{R}^m$ and $N_{F,x}$ is replaced by $\tilde{N}_{TX,x}$ — the subspace of $N_{TX,x}$ consisting of all functions of the form $(y, u) \to f'(y; u)$, where $f \in N_{X,x}$ and with the case where $F = \otimes^k TX$ and $N_{F,x}$ is replaced by $N^a_{F,x}$ — the submodule of $N_{F,x}$ consisting of all skew symmetric functions.

6.2. For $x \in X \subset \mathscr{R}^m$ let

6.2.1. $T_x^2 X = [\{u \otimes v + w \in \mathscr{R}^m \otimes \mathscr{R}^m + \mathscr{R}^m\colon f''(x;\, u, v) + f'(x;\, w) = 0$ $\forall f \in N_{X,x}\}]$ and define the bundle $\tilde{T}^2 X$ by $\tilde{T}^2 X = \bigcup \{\{x\} \otimes T_x^2 X\colon x \in X\}$ and by the transformation law

$$(x, u \otimes v + w) \to \big(h(x), \tilde{h}'(x;\, u) \otimes \tilde{h}'(x;\, v) + (\tilde{h}'(x;\, w) + \tilde{h}''(x;\, u, v))\big),$$

where $\tilde{h}$ is any C^∞-extension to a neighborhood of x of the connecting homeomorphism $h\colon X \to X'$.

6.2.2. Remark. If $f \in N_{X,x}$, then, for any C^∞-function g defined in a neighborhood of x, $gf \in N_{X,x}$, and the condition in 6.2.1 becomes (since $f(x) = 0$) $g(x)\big(f''(x;\, u, v) + f'(x;\, w)\big) + g'(x;\, v) f'(x;\, u) + g'(x;\, u) f'(x;\, v)$ $= 0$ and since g is arbitrary this implies that $f'(x;\, u) = f'(x;\, v) = 0$. Hence $u \otimes v + w \in \tilde{T}_x^2 X$ implies that $u, v \in T_x X$.

6.2.3. Remark. It is easy to check that the transformation law 6.2.1 gives a mapping of $\tilde{T}_x^2 X$ onto $T_{h(x)}^2 h(X)$ which is independent of the choice of the extension $\tilde{h}$ of h.

6.2.4. Remark 6.2.1 and 6.2.3 show that the horizontal projection $u \otimes v + w \in \tilde{T}_x^2 X \to u \otimes v$ is well defined and that its image which we denote by $\tilde{\otimes}^2 TX$ is a subbundle of $\otimes^2 TX$ (with the induced structure). More explicitly $(\tilde{\otimes}^2 TX)_x = [\{u \otimes v\colon u \otimes v + w \in \tilde{T}^2 X$ for some $w \in \mathscr{R}^m\}]$.

6.2.5. In order to define $\tilde{T}^k X$ for $k > 2$ we denote $V_k = \otimes^k \mathscr{R}^m +$ $+ \otimes^{k-1} \mathscr{R}^m + \ldots + \mathscr{R}^m$ and for $\xi = \xi_k + \ldots + \xi_1 \in V_k$, $\eta = \eta_{k-1} + \ldots + \eta_1$ $\in V_{k-1}$ we write $\xi + \eta = \xi_k + (\xi_{k-1} + \eta_{k-1}) + \ldots + (\xi_1 + \eta_1) \in V_k$. Also, for $u \in \mathscr{R}^n$, $\xi \in V_k$, $\xi \otimes u = \xi_k \otimes u + \ldots + \xi_1 \otimes u + 0 \in V_{k+1}$.

Define $\tilde{T}_x^k X$ as the subspace of V_k spanned by all ellements of the form $\xi \otimes u + \eta$, where $\xi, \eta \in V_{k-1}$ are as above, $u \in \mathscr{R}^m$ and

6.2.6. $f^{(k)}(x;\, \xi_{k-1} \otimes u) + \sum_{l=2}^{k-1} f^{(l)}(x;\, \xi_{l-1} \otimes u + \eta_l) + f'(x;\, \eta_1) = 0$ for all $f \in N_{X,x}$.

The transformation law is easiest to write using induction. For $k = 2$ it is given in 6.2.1. Suppose that $(x, \xi) \to \big(h(x), H_{k-1}(x, \xi)\big)$ is the transformation law for $\tilde{T}^{k-1} X$. Then the law for $\tilde{T}^k X$ is given by

$$(x, \xi \otimes u + \eta) \to \big(h(x), H_{k-1}(x, \xi) \otimes h'(x, u) + H_{k-1}(x, \eta) + $$
$$+ H_{k-1x}(x, \xi;\, u)\big) = \big(h(x), H_k(x, \xi \otimes u + \eta)\big).$$

We note that all the H's are computed in terms of any extension h of the homeomorphism h and that they are independent of the particular choice of this extension. It is also easy to check that $H_k(x, \cdot)$ indeed maps $\tilde{T}_x^k X$ onto $\tilde{T}_{h(x)}^k h(X)$.

6.2.7. Remark. If $\xi = u_1 \otimes \ldots \otimes u_k + \xi_{k-1} + \ldots + \xi_1 \in \tilde{T}_x^k X$, then $u_1, \ldots, u_k \in T_x X$. This follows in the same way as Remark 6.2.1.

6.2.8. As in 4.3 there is a well defined horizontal projection $\xi_k + \ldots + \xi_1 \in \tilde{T}^k_x X \to \xi_k \in (\otimes^k TX)_x$ the image of which is a subbundle of $\otimes^k TX$ denoted by $\tilde{\otimes}^k TX$. $\tilde{\otimes}^k TX$ is symmetric: if $u_1 \otimes \ldots \otimes u_k \in \tilde{\otimes}^k TX_x$, then $u_{i_1} \otimes \ldots \otimes u_{i_k} \in \tilde{\otimes}^k TX_x$ for any permutation $(i_1, \ldots, i_k)$ of $(1, \ldots, k)$.

6.2.9. The bundle $\tilde{T}^k X$ provides a setting for the compatibility conditions for the derivatives of a function defined on X. If $\{f^{(l)}\}_{l=0}^k$ is the jet of derivatives of f at x, then the linear combinations $\sum_{l=0}^k f^{(l)}(x;\ \xi_k)$ with $\sum \xi_k \in \tilde{T}^k X$, and only those are determined by $f|_X$.

6.3. We consider some examples where the spaces $\tilde{T}^k X$ can be determined more or less explicitly.

If $X \subset \mathscr{R}^m$, if $x \in X$ is a point of homogeneity of X and if $\dim_x X = m$, then $\tilde{T}^k_x X = \otimes^k \mathscr{R}^m + \ldots + \mathscr{R}^m$.

6.3.1. Spaces of polyhedral type. Let X be as in 5.3.5, i.e., $X = \bigcup_{i=1}^N X_i$, where X_i's are closed convex sets and $x = 0 \in \bigcap_{i=1}^N X_i$. We can assume that $[X] = \mathscr{R}^m$. A function f is in $N_{X,0}$ if and only if $f(P_i x) = 0$, $i = 1, \ldots, N$, for every x in some neighborhood of 0 or in a convex set with a nonvoid interior, which contains 0, where P_i is the orthogonal projection of $\mathscr{R}^m$ onto $[X_i]$. This implies that $\underbrace{P_i \ldots P_i}_{l\text{-times}} \nabla^l f(0) = 0$ for $i = 1, \ldots, N$ and every l and that $\tilde{\otimes}^k TX_0$ is the orthogonal complement of the subspace of $\otimes^k \mathscr{R}^m$ consisting of all symmetric tensors A satisfying $P_i \ldots P_i A = 0$, $i = 1, \ldots, N$. It is immediate that if $\xi_l \in \tilde{\otimes}^l TX_0$, then $\xi_k + \xi_{k-1} + \ldots + \xi \in \tilde{T}^k_0 X$. Choosing $f \in N_{X,0}$ of the form $f(x) = \langle A, \underbrace{x \otimes x \otimes \ldots \otimes x}_{l\text{-times}} \rangle$, where $A \in \otimes^l \mathscr{R}^m$ is symmetric and satisfies $P_i \ldots P_i A = 0$ for $i = 1, \ldots, N$, we see that the converse is also true, i.e., $\tilde{T}^k_0 X = (\tilde{\otimes}^k TX + \tilde{\otimes}^{k-1} TX + \ldots + TX_0)$. This observation corresponds to the (well-known) fact that the set of compatibility conditions for derivatives up to order k of a function defined on a polyhedral set can be split into k sets, each involving only derivatives of the same order.

6.3.2. As the next example we consider X as in 5.6, i.e., $X = \{x \in \mathscr{R}^m : x_m^l = x_1^l + \ldots + x_{m-1}^l\}$, $l \geqslant 2$, $m \geqslant 3$. The only point of interest is $x = 0$. Suppose $f \in N_{X,0}$. Then by the preparation theorem of Malgrange

$$f(x) = g(x)(x_m^l - x_{m-1}^l - \ldots - x_1^l) + g_{l-1}(x')x_m^{l-1} + \ldots + g_1(x')x_m + g_0(x'),$$

where $x' = (x_1, \ldots, x_{m-1})$ and $g, g_{l-1}, \ldots, g_0$ are C^∞-functions defined in a neighborhood of 0. If $l = 2$ choosing $x_m = \pm |x'|$ we conclude that $g_1(x') = g_0(x') = 0$; for $l > 2$ we have, with

$$|x'|_l = (x_1^l + \ldots + x_{m-1}^l)^{1/l}, \quad g_{l-1}(x')|x'|_l^{l-1} + \ldots + g_1(x')|x'|_l + g_0(x') = 0$$

for all x in some neighborhood of 0. This implies by a straightforward argument that all the derivatives of $g_{m-1}, \ldots, g_0$ are 0 at 0 and it follows that the functions $g_0, \ldots, g_{l-1}$ do not appear in condition 6.2.3. The condition implies that $\tilde{\otimes}^k TX_0 = \otimes^k \mathscr{R}^m$ for $k < l$ and that $\tilde{\otimes}^l TX_0$ is spanned by the tensors of the form $u_1 \otimes \ldots \otimes u_l$, where $u_1^m \ldots u_l^m - u_1^{m-1} \ldots u_l^{m-1} - \ldots - u_1^1 \ldots u_l^1 = 0$, with $u_s = (u_s^1, \ldots, u_s^m)$, $s = 1, \ldots, l$. It is checked directly that if $u_{i_1} \otimes \ldots \otimes u_{i_l} \in \tilde{\otimes}^l TX_0$ for any choice $1 \leqslant i_1 < i_2 < \ldots < i_l \leqslant k$, then $u_1 \otimes \ldots \otimes u_k \in \tilde{\otimes}^k TX_0$, and that this condition is also necessary provided that $u_1, \ldots, u_k$ are linearly independent. This implies that for $l = 2$ $\tilde{\otimes}^2 TX_0 = \acute{\otimes}^2 TX_0$ but $\acute{\otimes}^k TX_0 \subsetneqq \tilde{\otimes}^k TX_0$ for $k > 2$.

The above remarks imply also that the invariant $\tilde{\otimes}^l TX_0$ distinguishes between the singularities of $X = X_l$ at 0 for different values of l (see 5.6).

6.4. We consider now the bundles associated with exterior derivatives.

For $x \in X \subset \mathscr{R}^m$ we denote by $N_{X,x}^{a,k}$ the space of germs at x of C^∞-k-forms in $\mathscr{R}^m$ vanishing on $\otimes^k TX$ in some neighborhood of x (i.e. of functions in $C^\infty L(\mathscr{R}^m \times (\otimes^k \mathscr{R}^m))$ which are skew symmetric. We define $(\tilde{\otimes}_a^k TX)_x$ as the subspace of $\tilde{\otimes}^k \mathscr{R}^m$ consisting of all tensors ξ for which $(d\omega)(x, \xi) = 0$ for every $\omega \in N_{X,x}^{a,k-1}$, where for $\xi = u_1 \otimes \ldots \otimes u_k$

$$(d\omega)(x, \xi) = \sum_{i=1}^{k} (-1)^{i-1} \omega_x'(x, u_1 \otimes \ldots \otimes u_{i-1} \otimes u_{i+1} \otimes \ldots \otimes u_k; u_i).$$

6.4.1. Remark. If $u_1 \otimes \ldots \otimes u_k \in (\tilde{\otimes}_a^k TX)_x$, then $u_1, \ldots, u_k \in T_x X$.

This is checked by choosing $\omega \in N_{X,x}^{a,k}$ of the form $\omega(x, \eta) = g(x)\langle \mathring{\eta}, \eta \rangle$, $\eta \in \otimes^{k-1} \mathscr{R}^m$, where $g \in N_{X,x}$ and $\mathring{\eta} \in \otimes^{k-1} \mathscr{R}^m$ are arbitrary.

6.4.2. Remark 6.4.1 implies that the bundle

$$\tilde{\otimes}_a^k TX = \bigcup \{\{x\}(\tilde{\otimes}_a^k TX)_x : x \in X\}$$

with the transformation law

$$(x, u_1 \otimes \ldots \otimes u_l) \to (h(x), h'(x; u_1) \otimes \ldots \otimes h'(x; u_l))$$

is a subbundle of $\otimes^k TX$. We denote by $\tilde{\bigwedge}^k TX$ the image of this bundle under the natural projection of $\otimes^k TX$ onto $\bigwedge^k TX$.

We remark that $\tilde{\bigwedge}^k TX$ could be defined directly by letting $(\tilde{\bigwedge}^k TX)_x$ to be the subspace of $\tilde{\bigwedge}^k \mathscr{R}^m$ spanned by all vectors of the form $u_1 \wedge \ldots \wedge u_k$ for which $d\omega(x, u_1 \wedge \ldots \wedge u_k) = 0$ for all $k-1$ forms ω which vanish on $\bigwedge^{k-1} TX$ in some neighborhood of x and by writing down the transformation law in obvious way.

6.4.3. It is clear from the preceding definition that if $\omega \in C^\infty L(\bigwedge^{k-1} TX)$ and if ω denotes any C^∞ extension of ω to a neighborhood of $x \in X$, then the formula $d\omega = d\tilde{\omega}|_{(x,(\tilde{\Lambda}^k TX)_x)}$ defines a function in $C^\infty L(\tilde{\bigwedge}^k TX)$ and that $\tilde{\bigwedge}^k TX$ is the largest subbundle of $\bigwedge^k TX$ for which this definition is meaningful.

If $\omega \in C^\infty L(\bigwedge^{k-1} TX)$ and $\tilde{\omega}$ is a local extension of ω, then $d\tilde{\omega}$ is a local extension of $d\omega$, also $d(d\tilde{\omega}) = 0$. In this sense one should understand the property that $d(d\omega) = 0$ (see [5]).

6.5. To consider the converse to the last statement of the preceding paragraph, recall that X is locally contractible to $x \in X$ provided that there is a neighborhood U of x in $\mathscr{R}^m$ and a C^∞-mapping $h\colon U \times [0,1] \to \mathscr{R}^m$ such that for every $y \in X \cap U$ we have $h(y,t) \in X$ for all $t \in [0,1]$, $h(y,1) = y$ and $h(y,0) = x$.

6.5.1. PROPOSITION (Poincaré's lemma). *If $\omega \in C^\infty L(\bigwedge^k TX)$, if X is locally contractible at x and if $d\omega = 0$ in some neighborhood U of x in X (i.e. as a function in $C^\infty L(\tilde{\bigwedge}^{k+1} TU)$), then there is a neighborhood U' of x in X and $\lambda \in C^\infty(\bigwedge^{k-1} TU')$ such that $\omega|_{\tilde{\Lambda}^k TU'} = d\lambda$.*

The proof follows the usual argument using the "homotopy" operator and depends on the remark that $T_{(x,t)}(X \times [0,1]) = T_x X + T_t\,[0,1]$: $:= T_x X + [e_0]$ and that $(u_1 + t_1 e_0) \otimes \ldots \otimes (u_k + t_k e_0) \in \big(\tilde{\otimes}_a^k T(X \times [0,1])\big)_{(x,t)}$ if and only if $u_1 \otimes \ldots \otimes u_k \in (\tilde{\otimes}_a^k TX)_x$ (see 5.4.3).

6.6. We consider some special cases where $\tilde{\bigwedge}^k TX$ can be calculated.

Observe that if x is a point of homogeneity of X, then $(\tilde{\bigwedge}^k TX)_x = \bigwedge^k T_x X$.

6.6.1. EXAMPLE. If X is a polyhedral set as in 6.3.1, then $u_1 \otimes \ldots \otimes u_k \in \otimes_a TX_0$ if and only if $\sum_{l=1}^{k} (-1)^{l-1} \langle A_l u_l, u_1 \otimes \ldots \otimes u_{l-1} \otimes u_{l+1} \otimes \ldots \otimes u_k \rangle = 0$ for all k-tuples of $A_l \in L(\mathscr{R}^m, \bigwedge^{k-1} \mathscr{R}^m)$ satisfying the condition $\overbrace{P_i \ldots P_i}^{k-1\text{-times}} A_l P_i = 0$ for $i = 1, \ldots, N$. This implies that $\sum_{i=1}^{N} \bigwedge^k T_0 X_i \subset \tilde{\bigwedge}^k TX_0$. By an argument similar to the one used in 5, it follows that actually $(\tilde{\bigwedge}^k TX)_0 = \sum \bigwedge^k T_0 X_i$.

6.6.2. EXAMPLE. Let $X = \{x \in \mathscr{R}^m\colon x_m^2 = x_1^2 + \ldots + x_{m-1}^2 = |x'|^2\}$. To compute $(\tilde{\bigwedge}^{k+1} TX)$, $k \geqslant 1$, let $f \in N_{TX}^{ak}$ and write $f(x, u_1 \otimes \ldots \otimes u_k) = \sum a_{i_1 \ldots i_k}(x) u_1^{i_1} \ldots u_k^{i_k}$ with the sum extended over the indices $1 \leqslant i_1 < \ldots \ldots < i_k \leqslant m$. If $u \in T_x X$ for some $x \in X$, $x \neq 0$, then $u^m = \sum_{j=1}^{m-1} x_m^{-1} x_j u^j$ and it follows that

$$a_{i_1 \ldots i_k}(x) + \sum_{j=1}^{k} (-1)^{j-k} a_{i_1 \ldots i_{j-1} i_{j+1} \ldots i_k m}(x)\, x_m^{-1} x_{i_j}$$

for $x_m^2 = |x'|^2$ and for all indices $1 \leqslant i_1 < \ldots < i_k < m$. Denoting $a = a_{i_1 \ldots i_k}$, $a_j = a_{i_1 \ldots i_{j-1} i_{j+1} \ldots i_k m}$ and taking the partial derivative of the above identity with respect to x_l (denoted by $,l$), $1 \leqslant l < m$, we get

$$a_{,l} + a_{,m} x_m^{-1} x_l + \sum_{j=1}^{k} (-1)^{j-k} \big(a_{j,l}\, x_m^{-1}\, x_{i_j} + a_{j,m}\, x_m^{-2}\, x_{i_j}\, x_l + a_j (x_m^{-1} x_{i_j})_{,l}\big)$$
$$= 0 \quad \text{for } x_m^2 = |x'|^2.$$

In calculation of df only the terms where $l \neq i_j$ appear, in which case $(x_m^{-1} x_{i_j})_{,l} = -x_m^{-3} x_{i_j} x_l$.

With $x_s = 0$ for all $s \neq l, m$ the last identity becomes $a_{,l} + a_{,m} = 0$ where $x_m = x_l$ and $a_{,l} - a_{,m} = 0$ where $x_m = -x_l$. At $x = 0$ we conclude that $a_{,l} = a_{,m} = 0$ and $df(0, u_1 \otimes \ldots \otimes u_{k+1}) = 0$ for every $u_1, \ldots, u_{k+1} \in \mathscr{R}^m$. It follows that $(\tilde{\bigwedge}^k TX)_0 = \bigwedge^k T_0 X$.

Bibliography

[1] N. Aronszajn and P. Szeptycki, *Theory of Bessel Potentials*, part IV, Ann. Inst. Fourier 25 (1975), p. 27–69.

[2] —, —, *Subcartesian Spaces*, J. Diff. Geom. 15 (1980), p. 393–416.

[3] M. Breuer and C. D. Marshall, *Banachian Differentiable Spaces*, Math. Ann. 237 (1978), p. 105–120.

[4] C. D. Marshall, *Calculus on Subcartesian Spaces*, J. of Differ. Geometry 10 (1975), p. 551–574.

[5] —, *De Rham Cohomology of Subcartesian Spaces*, ibidem 10 (1975), p. 575–588.

[6] K. Spallek, *Differenzierbare Räume*, Math. Ann. 180 (1969), p. 269–296.

[7] —, *Differentiable forms on differentiable spaces*, Rend. Mat. (2) Ser. VI, 4 (1971), p. 237–258.

[8] —, *Differentiable forms on differentiable spaces*, ibidem (2) Ser. VI, 5 (1972), p. 1–15.

[9] H. Whitney, *Tangents to analytic variety*, Ann. of Math. 81 (1964), p. 496–549.

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