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A note on entire functions represented by Dirichlet series

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1. Introduction. This note attempts to supplement in two ways a recent paper by Ghosh and Srivastav ([2]) on entire functions $f(s)$, $s = \sigma + it$, represented by Dirichlet series. First, there are certain formulae stated in that paper (Theorem 2 of this note with $D = 0$) and proved in an earlier paper ([8], Theorem 2), involving ϱ and λ which are respectively the order of $f(s)$ according to Ritt ([6], p. 77) and the corresponding lower order. But the proofs of these formulae are defective, being based on a hypothesis less general than it need be (viz. $D = 0$ instead of $0 \leqslant D < \infty$ as in Theorem 2 *infra*), and also regardless of a relevant distinction between ϱ, λ on one hand and ϱ_*, λ_* on the other, ϱ_* and λ_* denoting respectively the order of $f(s)$ according to Sugimura ([10], p. 265) and the corresponding lower order. We seek to remove these defects in the treatment of Theorem 2 of this note. Secondly, Ghosh and Srivastav prove ([2], Lemma 2) certain formulae for ϱ and λ in terms of $M(\sigma)$ and $M'(\sigma)$ where $M(\sigma)$ is l.u.b. $|f(\sigma+it)|$ for $-\infty < t < \infty$ and $M'(\sigma)$ is the derivative of $M(\sigma)$. This note (in Theorem 3) establishes similar formulae for $M(\sigma)$ and $M^1(\sigma)$ in terms of ϱ and λ, where $M(\sigma)$ is defined as before and $M^1(\sigma)$ is defined for $f'(s)$ exactly as $M(\sigma)$ is for $f(s)$. These new formulae are analogues of certain well-known formulae due to S. M. Shah ([7], Theorems A, 1) for the order and the lower order of entire functions. The remaining principal result of this note (Theorem 1) gives a sufficient condition for the orders (or lower orders) of $f(s)$ in the senses of Ritt and Sugimura, and the corresponding orders of $f'(s)$ to be all equal to one another.

2. Notation. In the usual notation, adopted also by Ghosh and Srivastav ([2]), let

$$(1)\qquad f(s) = \sum_{1}^{\infty} a_n e^{s\lambda_n}, \quad s = \sigma + it, \quad 0 < \lambda_n < \lambda_{n+1} \ (n \geqslant 1), \quad \lambda_n \to \infty .$$

be an entire function in the specific sense that the Dirichlet series representing it is absolutely convergent for all finite s. Also, as usual, let

$$(2)\qquad M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(\sigma+it)|\,, \qquad \mu(\sigma) = \max_{n\geqslant 1} |a_n e^{(\sigma+it)\lambda_n}| = |a_\nu| e^{\sigma\lambda_\nu}\,,$$

where ν and hence λ_ν is a function of σ, i.e.

$$(3)\qquad \lambda_\nu = \lambda_{\nu(\sigma)} = \Lambda(\sigma)\,,$$

say. Furthermore, let the definitions of Ritt order ϱ and Ritt lower order λ, viz.

$$(4)\qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}\,,$$

be supplemented by the definitions of Sugimura order ϱ_* and Sugimura lower order λ_*:

$$(5)\qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log \mu(\sigma)}{\sigma} = \begin{matrix}\varrho_*\\ \lambda_*\end{matrix}\,.$$

Then $\varrho \geqslant \varrho_*$ and $\lambda \geqslant \lambda_*$ since, by Lemma 1 of the next section, $M(\sigma) \geqslant \mu(\sigma)$. To have $\varrho = \varrho_*$ and $\lambda = \lambda_*$ it is clearly enough if $\log M(\sigma) \sim \log \mu(\sigma)$. A sufficient condition for the last relation is $\limsup_{n\to\infty}(\log n/\log\lambda_n) < \infty$ when $\varrho < \infty$ ([11], p. 73; cf. [10], Satz 5), and is more stringent when $\varrho = \infty$ ([1], Theorem 2). In the assertion of Ghosh and Srivastav ([2], p. 93) that $\limsup_{n\to\infty}(\log n/\lambda_n) = 0$ is sufficient for $\log M(\sigma) \sim \log\mu(\sigma)$, the condition $\limsup_{n\to\infty}(\log n/\lambda_n) = 0$ should be corrected to $\limsup_{n\to\infty}(\log n/\log\lambda_n) < \infty$ in conformity with Yu's statement ([11], p. 73) to which they refer. However, what is relevant for their purpose is that $\limsup_{n\to\infty}(\log n/\lambda_n) < \infty$ is sufficient to make $\varrho = \varrho_*$ and $\lambda = \lambda_*$ (as shown by Theorem 1 *infra*) (1).

For any finite s, we may differentiate term by term the series in (1) and obtain a second absolutely convergent Dirichlet series:

$$(6)\qquad f'(s) = \sum_1^\infty a_n\lambda_n e^{s\lambda_n}\,, \qquad s = \sigma + it\,.$$

Therefore $f'(s)$ is an entire function in the same sense as $f(s)$ and we can define $M^1(\sigma)$, $\mu^1(\sigma)$, ϱ^1, λ^1, etc., for $f'(s)$ exactly like the corresponding concepts for $f(s)$ in (2)-(5).

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(1) When $\limsup_{n\to\infty}(\log n/\lambda_n) = \infty$, we may have $\varrho > \varrho_*$ and $\lambda > \lambda_*$. In fact, Sugimura has shown ([10], Satz 4) that, when $\log n/\lambda_n = O(\log\lambda_n)$, there is an entire Dirichlet series for which $\varrho > \varrho_*$.

3. Lemmas. The following lemmas, required for the theorems of this note, are for the most part known results. In these lemmas and in the work which follows, K, K', N, etc., and ε are strictly positive constants, not necessarily the same at each occurrence, and ε (according to convention) may be as small as we please.

LEMMA 1 (J. Hadamard). *For the entire function $f(s)$ represented by the Dirichlet series* (1),

$$a_n = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-(\sigma+it)\lambda_n} f(\sigma+it)\,dt \qquad (n \geqslant 1)\,.$$

Landau ([3], p. 788, Satz 35), proving Lemma 1 in a somewhat more general case, ascribes it to Hadamard.

LEMMA 2 (Sugimura [10], p. 267; Yu [11], p. 67). *With the definitions of $\mu(\sigma)$ and $\Lambda(\sigma)$ in* (2) *and* (3), *we have*:

$$\log\mu(\sigma) = \log\mu(\sigma_0) + \int_{\sigma_0}^{\sigma} \Lambda(x)\,dx\,.$$

LEMMA 3. *With the definitions of $\Lambda(\sigma)$, ϱ_* and λ_* in* (3) *and* (5), *we have*:

(a) $$\limsup_{\sigma\to\infty} \frac{\log\Lambda(\sigma)}{\sigma} = \varrho_* \quad \text{(Sugimura [10], Satz 1)}\,,$$

(b) $$\liminf_{\sigma\to\infty} \frac{\log\Lambda(\sigma)}{\sigma} = \lambda_* \quad \text{(Rahman [4], Theorem 1)}\,.$$

Rahman [4] states Lemma 3(b) for λ, assuming that $\limsup\limits_{n\to\infty}(\log n/\lambda_n) = 0$. But all that he makes out is that $\alpha \equiv \liminf\limits_{\sigma\to\infty}\big(\log\Lambda(\sigma)/\sigma\big) = \lambda_*$, since the justification for $\lambda_* = \lambda$, on his assumption, rests on Theorem 1 *infra*. Moreover, his proof that $\alpha \leqslant \lambda_*$ ([4], p. 205) is incorrect in details, while his proof that $\alpha \geqslant \lambda_*$ ([4], pp. 205-206) is needlessly long. Therefore we insert, for the sake of completeness, the following short proof of the two parts of Lemma 3 together.

By Lemma 2,

$$\log\mu(\sigma+1) = \log\mu(\sigma) + \int_{\sigma}^{\sigma+1} \Lambda(x)\,dx\,.$$

Since $\mu(\sigma)$ tends to ∞ with σ, we may confine ourselves to all large σ such that $\log\mu(\sigma) > 0$ and deduce from the last step the following inequalities in order, recalling first that $\Lambda(x)$ is a monotonic increasing function of x:

$$\Lambda(\sigma) \leqslant \int_{\sigma}^{\sigma+1} \Lambda(x)\,dx < \log\mu(\sigma+1)\,,$$

$$\text{(7)} \qquad \frac{\log\Lambda(\sigma)}{\sigma} < \frac{\log\log\mu(\sigma+1)}{\sigma+1}\cdot\frac{\sigma+1}{\sigma}\,,$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\Lambda(\sigma)}{\sigma} \leqslant \begin{matrix}\varrho_*\\ \lambda_*\end{matrix}\,,$$

where we use the definitions of (5). Next, supposing that $\sigma > \sigma_0$ in Lemma 2, we get successively:

$$\log\mu(\sigma) < \log\mu(\sigma_0) + (\sigma-\sigma_0)\Lambda(\sigma) \sim \sigma\Lambda(\sigma) \qquad (\sigma\to\infty)\,,$$

$$\text{(8)} \qquad \frac{\log\log\mu(\sigma)}{\sigma} < \frac{o(1)+\log\sigma}{\sigma} + \frac{\log\Lambda(\sigma)}{\sigma}\,,$$

$$\begin{matrix}\varrho_*\\ \lambda_*\end{matrix} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\Lambda(\sigma)}{\sigma}\,,$$

where we again use the definitions of (5). (7) and (8) together establish the two parts (a), (b) of Lemma 3.

Lemma 4. *If $M(\sigma)$ and $M^1(\sigma)$ are defined as in* (2), *for the functions $f(s)$ and $f'(s)$ in* (1) *and* (6) *respectively, then*

$$M^1(\sigma)/M(\sigma) \geqslant \log M(\sigma)/\sigma \qquad (\sigma > \sigma_1)\,.$$

Lemma 4 is given in a slightly more general form by Satya Narain Srivastava ([9], Lemma 1). It follows from the convexity of $\log M(\sigma)$ as a function of σ, proved by Doetsch and recalled by Yu ([11], p. 67).

Lemma 5. *If $M(\sigma)$, $M^1(\sigma)$ are defined as in Lemma* 4, *$\mu(\sigma)$ is defined as in* (2), *and if $\{\lambda_n\}$ is supposed to satisfy the additional condition*

$$\text{(1a)} \qquad \limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty\,,$$

then

(a) $\qquad M(\sigma) < K\mu(\sigma+D+\varepsilon) \qquad$ (Yu [11], p. 68),

(b) $\qquad M^1(\sigma) < K'\mu(\sigma+D+\varepsilon)\,,$

where K, K' depend on D and ε.

The proof of (b) is exactly like that of (a) given by Yu, but is given here for completeness. Corresponding to $\varepsilon > 0$, we can find a positive integer $N = N(\varepsilon)$ such that we have, for $n > N$,

$$\text{(9)} \qquad \lambda_n < e^{\varepsilon\lambda_n/2}\,, \qquad \log n < (D+\varepsilon/4)\lambda_n \quad \text{or} \quad n < e^{(D+\varepsilon/4)\lambda_n}\,,$$

where the second inequality is a consequence of (1a). Now, by definitions,

$$(10)\qquad M^1(\sigma) \leqslant \sum_{1}^{\infty} |a_n|\lambda_n e^{\sigma\lambda_n} = \Big(\sum_{1}^{N} + \sum_{N+1}^{\infty}\Big)|a_n| e^{(\sigma+D+\varepsilon)\lambda_n}\lambda_n e^{-(D+\varepsilon)\lambda_n}$$

$$< \mu(\sigma)N\lambda_N + \mu(\sigma+D+\varepsilon)\sum_{N+1}^{\infty}\lambda_n e^{-(D+\varepsilon)\lambda_n}\,.$$

The series on the right side of (10) is convergent since, by (9), its general term is subject to the restriction

$$\lambda_n e^{-(D+\varepsilon)\lambda_n} < e^{-(D+\varepsilon/2)\lambda_n} < n^{-(D+\varepsilon/2)/(D+\varepsilon/4)}\,.$$

Hence, from (10),

$$M^1(\sigma) < N\lambda_N\mu(\sigma) + K_1\mu(\sigma+D+\varepsilon) < (N\lambda_N + K_1)\mu(\sigma+D+\varepsilon)\,.$$

This completes the proof of Lemma 5(b).

4. Theorems. It will be observed that, in the proof of the theorems which follow, the concurrent use of Ritt and Sugimura orders cannot be avoided, even if we state Theorem 1 in the attenuated form $\varrho = \varrho^1$ and $\lambda = \lambda^1$ for Ritt orders alone ([2]).

THEOREM 1. *Let the entire function $f(s)$ and its derivative $f'(s)$ be given by Dirichlet series* (1) *and* (6) *respectively, with $\{\lambda_n\}$ satisfying the additional condition*

$$(1a)\qquad \limsup_{n\to\infty}\frac{\log n}{\lambda_n} = D < \infty\,.$$

Then the orders (or lower orders) of $f(s)$ defined by (4), (5), *and the corresponding orders (lower orders) of $f'(s)$ are all equal, i.e.*

$$\varrho = \varrho_* = \varrho^1 = \varrho^1_*\,,\qquad \lambda = \lambda_* = \lambda^1 = \lambda^1_*\,.$$

Proof. By Lemma 1 applied to $f'(s)$, we get for all $n \geqslant 1$,

$$|a_n| e^{\sigma\lambda_n}\lambda_n \leqslant M^1(\sigma)\,.$$

Here, choosing $n = \nu$ so that $|a_\nu| e^{\sigma\lambda_\nu} = \max|a_n| e^{\sigma\lambda_n}$ as in (2), (3), and then taking logarithms, we obtain

$$(12)\qquad \log\mu(\sigma) + \log\Lambda(\sigma) \equiv \log|a_\nu| e^{\sigma\lambda_\nu} + \log\lambda_\nu \leqslant \log M^1(\sigma)\,.$$

([2]) The referee points out that the part $\varrho = \varrho^1$ of Theorem 1 is proved by Q. I. Rahman, without the assumption (1a), in a note [5], accepted for publication earlier than this note.

Since $\lambda_* \geqslant 0$, it follows from Lemma 3(b) that, for all sufficiently large σ, $\log \Lambda(\sigma)/\sigma > -\varepsilon$ and hence (12) leads to the following inequalities in succession:

$$\log \mu(\sigma) - \varepsilon\sigma < \log M^1(\sigma) \qquad (\sigma > \sigma_0),$$

$$\log\log \mu(\sigma) < \log\log M^1(\sigma) + \log\left[1 + \frac{\varepsilon\sigma}{\log M^1(\sigma)}\right]$$

$$< \log\log M^1(\sigma) + \frac{\varepsilon\sigma}{\log M^1(\sigma)},$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log\mu(\sigma)}{\sigma} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M^1(\sigma)}{\sigma}, \qquad \text{or}$$

(13) $$\varrho_* \leqslant \varrho^1, \qquad \lambda_* \leqslant \lambda^1.$$

Next, Lemma 5(b) gives successively:

$$\log M^1(\sigma) < \log K' + \log\mu(\sigma + D + \varepsilon) \qquad (\sigma > \sigma_0)$$

$$\sim \log \mu(\sigma + D + \varepsilon) \qquad (\sigma \to \infty),$$

$$\log\log M^1(\sigma) < o(1) + \log\log\mu(\sigma + D + \varepsilon),$$

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M^1(\sigma)}{\sigma} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log\mu(\sigma + D + \varepsilon)}{\sigma + D + \varepsilon} \cdot \frac{\sigma + D + \varepsilon}{\sigma}, \qquad \text{or}$$

(14) $$\varrho^1 \leqslant \varrho_*, \qquad \lambda^1 \leqslant \lambda_*.$$

Lastly, exactly as Lemma 5(b) leads to (14), Lemma 5(a) leads to

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\sigma} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log\mu(\sigma + D + \varepsilon)}{\sigma + D + \varepsilon} \cdot \frac{\sigma + D + \varepsilon}{\varepsilon},$$

or

$$\varrho \leqslant \varrho_*, \qquad \lambda \leqslant \lambda_*,$$

while, from definitions, $\varrho_* \leqslant \varrho$ and $\lambda_* \leqslant \lambda$ universally. Hence

(15) $$\varrho = \varrho_*, \qquad \lambda = \lambda_*.$$

Combining (15) with (13) and (14), we obtain

$$\varrho = \varrho_* = \varrho^1, \qquad \lambda = \lambda_* = \lambda^1.$$

And from this result conclusion (11) is obvious, since we have also $\varrho^1 = \varrho^1_*$ and $\lambda^1 = \lambda^1_*$ by an application of (15) to $f'(s)$ instead of $f(s)$.

THEOREM 2. *Let entire functions $f(s)$ and $f'(s)$ be defined by Dirichlet series* (1) *and* (6) *as in Theorem* 1, *$\{\lambda_n\}$ satisfying the additional condition* (1a). *Let $\mu(\sigma)$ be defined for $f(s)$ as in* (2) *and $\mu^1(\sigma)$ correspondingly for $f'(s)$. Then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[\mu^1(\sigma)/\mu(\sigma)]}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}.$$

Proof. For $f'(s) = \sum_1^\infty \lambda_n a_n e^{s\lambda_n}$, let $\nu^1 = \nu^1(\sigma)$ and $\Lambda^1(\sigma) = \lambda_{\nu^1}$ be defined exactly as ν and $\Lambda(\sigma)$ for $f(s)$ in (2) and (3). Then

$$\mu^1(\sigma) = \lambda_{\nu^1}|a_{\nu^1}|e^{\sigma\lambda_{\nu^1}} \leqslant \Lambda^1(\sigma)\mu(\sigma) .$$

On the other hand,

$$\mu(\sigma) = |a_\nu|e^{\sigma\lambda_\nu} = \frac{1}{\lambda_\nu}\lambda_\nu|a_\nu|e^{\sigma\lambda_\nu} \leqslant \frac{1}{\Lambda(\sigma)}\mu^1(\sigma) .$$

Combining the last two steps, we have the inequality

$$\Lambda(\sigma) \leqslant \frac{\mu^1(\sigma)}{\mu(\sigma)} \leqslant \Lambda^1(\sigma) .$$

Taking logarithms of each member of this inequality, then dividing by σ, and finally letting $\sigma\to\infty$, we get by Lemma 3:

$$\frac{\varrho_*}{\lambda_*} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[\mu^1(\sigma)/\mu(\sigma)]}{\sigma} \leqslant \frac{\varrho_*^1}{\lambda_*^1} .$$

After this, the conclusion of Theorem 2 follows at once from that of Theorem 1.

THEOREM 3. *Let entire functions $f(s)$ and $f'(s)$ be given by Dirichlet series* (1) *and* (6) *respectively, with $\{\lambda_n\}$ satisfying the additional condition*

$$\lim_{n\to\infty} \frac{\log n}{\log \lambda_n} = 0 . \tag{1b}$$

Then, if $M(\sigma)$ and $M^1(\sigma)$ are defined for $f(s)$ and $f'(s)$ respectively as in (2), *we have*

$$\lim_{n\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[M^1(\sigma)/M(\sigma)]}{\sigma} = \frac{\varrho}{\lambda} .$$

Proof. Corresponding to any $\sigma > 0$, let R be a function of σ chosen in a manner to be indicated presently. Also, let $\max|a_n|e^{(\sigma+R)\lambda_n}$ for $n \geqslant 1$ correspond to $n = N$, so that, in the notation of (3), $\lambda_N = \Lambda(\sigma+R)$. Then, from the definitions,

$$\begin{aligned} M^1(\sigma) &\leqslant \sum_1^\infty |a_n|e^{(\sigma+R)\lambda_n}\lambda_n e^{-R\lambda_n} \\ &\leqslant |a_N|e^{\sigma\lambda_N}e^{R\lambda_N}\sum_1^\infty \lambda_n e^{-R\lambda_n} \\ &\leqslant \mu(\sigma)e^{R\lambda_N}\sum_1^\infty \lambda_n e^{-R\lambda_n} \\ &\leqslant M(\sigma)e^{R\Lambda(\sigma+R)}\sum_1^\infty \lambda_n e^{-R\lambda_n}, \end{aligned} \tag{16}$$

since (as has been remarked earlier) $\mu(\sigma) \leqslant M(\sigma)$ by Lemma 1. Now let $n(x)$ denote the number of λ_n's not exceeding x. Then (16) can be written as:

$$(17) \qquad M^1(\sigma)/M(\sigma) \leqslant e^{RA(\sigma+R)} \int_0^\infty x e^{-Rx} dn(x)$$

$$= e^{RA(\sigma+R)} \int_0^\infty -d(xe^{-Rx})\, n(x)$$

$$< e^{RA(\sigma+R)} \int_0^\infty (e^{-Rx} + xRe^{-Rx})\, n(x)\, dx$$

$$= e^{RA(\sigma+R)} \left[\int_0^X + \int_X^\infty \right] (e^{-Rx} + xRe^{-Rx})\, n(x)\, dx \,,$$

where X is chosen so that $n(x) < x^\varepsilon$ for $x \geqslant X$, such a choice being possible by condition (1b) in the form $\lim\{\log n(x)/\log x\} = 0$ as $x \to \infty$. In (17),

$$\left| \int_0^X \ldots \right| < n(X) \int_0^X (1 + e^{-1})\, dx \,,$$

and so (17), with our choice of X, gives us

$$M^1(\sigma)/M(\sigma) < Ke^{RA(\sigma+R)} + e^{RA(\sigma+R)} \int_X^\infty (e^{-Rx} + xRe^{-Rx}) x^\varepsilon dx$$

$$= Ke^{RA(\sigma+R)} + e^{RA(\sigma+R)} R^{-(1+\varepsilon)} \int_{RX}^\infty (e^{-u}u^\varepsilon + e^{-u}u^{1+\varepsilon})\, du$$

$$< Ke^{RA(\sigma+R)} + e^{RA(\sigma+R)} R^{-(1+\varepsilon)} \{\Gamma(1+\varepsilon) + \Gamma(2+\varepsilon)\}$$

$$< Ke^{RA(\sigma+R)} + K_1 e^{RA(\sigma+R)} R^{-(1+\varepsilon)} \,,$$

since we may suppose that $\varepsilon < 1$. If we now choose

$$R = \frac{1}{A(\sigma+1)} \,,$$

so that

$$R \to 0 \ (\sigma \to \infty) \,, \qquad RA(\sigma+R) \leqslant RA(\sigma+1) = 1 \quad (\sigma > \sigma_1) \,,$$

we immediately get

$$M^1(\sigma)/M(\sigma) < K' + K_1'[A(\sigma+1)]^{1+\varepsilon}$$

$$\sim K_1'[A(\sigma+1)]^{1+\varepsilon} \qquad (\sigma \to \infty) \,.$$

Taking logarithms of both sides of this inequality and dividing by σ, we get by Lemma 3:

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[M^1(\sigma)/M(\sigma)]}{\sigma} \leqslant \frac{\varrho_*}{\lambda_*}, \tag{18}$$

if we assume that $\varrho_* < \infty$ and recall that ε is arbitraily small. Since our present hypothesis (1b) implies hypothesis (1a) of Theorem 1 with $D = 0$, we have $\varrho_* = \varrho$ and $\lambda_* = \lambda$ by that theorem, and so (18) gives us

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[M^1(\sigma)/M(\sigma)]}{\sigma} \leqslant \frac{\varrho}{\lambda}. \tag{19}$$

On the other hand, Lemma 4 readily yields

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[M^1(\sigma)/M(\sigma)]}{\sigma} \geqslant \frac{\varrho}{\lambda}. \tag{20}$$

(19) and (20) together establish the conclusion of Theorem 3 when $\varrho_* < \infty$. The remaining two cases, $0 \leqslant \lambda_* < \varrho_* = \infty$ and $\lambda_* = \varrho_* = \infty$, are easily disposed of. The former case requires only the λ-inequalities of (19) and (20), while the latter case requires only (20).

Finally it may be observed that we may differentiate term by term, j times ($j \geqslant 1$), the absolutely convergent Dirichlet series (1) defining the entire function $f(s)$, and obtain another such series whose sum is the entire function $f^{(j)}(s)$. For the pair of functions $f(s)$ and $f^{(j)}(s)$, there are easy generalizations of Theorems 2 and 3 as follows, got by repeated use of the arguments which prove these theorems.

THEOREM 2′. *In the Dirichlet series* (1) *defining the entire function $f(s)$, let $\{\lambda_n\}$ satisfy condition* (1a) *of Theorem* 2. *Let $\mu(\sigma)$ be defined for $f(s)$ as in* (2) *and $\mu^j(\sigma)$ correspondingly for $f^{(j)}(s)$. Then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[\mu^j(\sigma)/\mu(\sigma)]^{1/j}}{\sigma} = \frac{\varrho}{\lambda}.$$

THEOREM 3′. *In the Dirichlet series* (1) *defining the entire function $f(s)$, let $\{\lambda_n\}$ be subject to restriction* (1b) *of Theorem* 3. *Let $M(\sigma)$ and $M^j(\sigma)$ be defined respectively for $f(s)$ and $f^{(j)}(s)$ as in* (2). *Then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log[M^j(\sigma)/M(\sigma)]^{1/j}}{\sigma} = \frac{\varrho}{\lambda}.$$

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