ANNALES
POLONICI MATHEMATICI
XXVIII (1973)

On a certain non-linear initial-boundary value problem for integro-differential equations of parabolic type

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Abstract. In this paper we consider a certain initial-boundary value problem for a system of parabolic equations with functional non-linear part and with functional boundary conditions. There is derived an a priori estimate of Friedman's type for solutions of the above problem. Using this estimate and applying the Leray–Schauder fixed point theorem we prove theorems on the existence and uniqueness of solutions of the problem considered.

The above-mentioned results involve as a particular case an initial-boundary value problem for a system of integro-differential equations with integro-differential boundary conditions. Moreover, there is proved a theorem on strong integro-differential inequalities.

In paper [6] some theorems were proved concerning a priori estimates and the existence of solutions of the first Fourier problem in a bounded domain for a system of parabolic equations with a linear main part and with a non-linear operator acting on unknown functions. These theorems involve a system of integro-differential equations as a particular case.

In this paper we extend the above-mentioned results to the following initial-boundary value problem in a bounded domain

$$(0.1)\qquad L^k u^k \equiv \sum_{i,j=1}^{n} a_{ij}^k(x,t)\,u_{x_i x_j}^k + \sum_{i=1}^{n} b_i^k(x,t)\,u_{x_i} - u_t^k = A^k u \quad \text{for } (x,t)\in D\times(0,T],$$

$$(0.2)\qquad \frac{\partial u^k(x,t)}{\partial \nu^k(x,t)} = B^k u, \quad (x,t)\in\Sigma, \quad (k=1,\dots,N)$$

$$(0.3)\qquad u^k(x,0) = \psi^k(x), \quad x\in\overline{D},$$

where A^k and B^k are some non-linear operators acting on unknown vector-function $u=(u^1,\dots,u^N)$; in particular,

$$(0.4)\qquad A^k u = f^k\Big(x,t,u,\Big\{\int_D u^i(y,t)\mu_1^i(x,t;dy)\Big\},\ \Big\{\int_0^t \mu_3^i(x,t;d\tau)\int_D u^i(y,\tau)\mu_2^i(x,t;dy)\Big\}\Big),$$

$$(0.5)\qquad B^k u = h^k\Big(x, t, u, \Big\{\int_D u^i(y,t)\nu_1^i(x,t;dy)\Big\},$$
$$\Big\{\int_0^t \nu_3^i(x,t;d\tau)\int_D u^i(y,\tau)\nu_2^i(x,t;dy)\Big\}\Big)\ (^1).$$

Some theorems on the existence of solutions of initial-boundary value problems for semilinear equations with various boundary conditions can be found in reference [2]. These theorems were obtained under more restrictive assumptions concerning the behaviour of the non-linear part of equations and boundary conditions with respect to unknown functions than those used in the present paper.

1. The estimate of the solution of the linear problem. Let D be a bounded open domain of the Euclidean space E_n of the variables $x = (x_1, \ldots, x_n)$ and $G = D\times(0, T]$, T being a positive constant. By Σ we denote the side surface of G, i. e. $\Sigma = S\times(0, T]$, where S is the boundary of the domain D.

We shall derive the estimate of Friedman's type for the solution of the problem

$$(1.1)\qquad Lu \equiv \sum_{i,j=1}^n a_{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^n b_i(x,t)u_{x_i} + c(x,t)u - u_t = f(x,t),$$
$$(x,t)\in G^\tau = D\times(0,\tau]\qquad (0<\tau\leqslant T),$$

$$(1.2)\qquad u(x,0) = 0,\qquad x\in\bar{D}\qquad (\bar{D} - \ \text{the closure of } D),$$

$$(1.3)\qquad \frac{\partial u(x,t)}{\partial\nu(x,t)} + g(x,t)u(x,t) = h(x,t),\qquad (x,t)\in\Sigma^\tau = S\times(0,\tau],$$

where $\partial u(x,t)/\partial\nu(x,t)$ is the conormal derivative (for the definition see [1], p. 137, 144).

The following assumptions will be needed:

(1.I) The coefficients of the operator L are defined in $\bar{G}$ and satisfy Hölder conditions:

$$|a_{ij}(x,t) - a_{ij}(x',t')| \leqslant M_0[|x-x'|^\alpha + |t-t'|^{\alpha/2}],$$
$$|b_i(x,t) - b_i(x',t)| \leqslant M_0|x-x'|^\alpha\qquad (0<\alpha\leqslant 1),$$
$$|c(x,t) - c(x',t)| \leqslant M_0|x-x'|^\alpha,$$

where

$$|x-x'| = \Big[\sum_{i=1}^n (x_i - x_i')^2\Big]^{1/2}.$$

(1) The meanings of the symbols used here are explained in Sections 1 and 2.

Furthermore the coefficients $b_i(x, t)$ and $c(x, t)$ are continuous with respect to the variable t.

(1.II) The operator L is parabolic in $\bar{G}$, i. e. the quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x, t)\,\xi_i \xi_j$$

is positive definite in $\bar{G}$.

(1.III) The surface S is of class $C^{1+\alpha}$ (see [1], p. 135).

(1.IV) The function $f(x, t)$ is continuous in the domain $\bar{G}$ and satisfies the local Hölder condition with exponent α_1 $(0 < \alpha_1 \leqslant 1)$ in $x \in D$, i. e. for any closed domain $D^* \subset D$

$$|f(x, t) - f(x', t)| \leqslant M(D^*)|x - x'|^{\alpha_1}$$

if $x, x' \in D^*$, $0 < t \leqslant T$, $M(D^*)$ being a constant depending on D^*.

(1.V) The functions $g(x, t)$ and $h(x, t)$ are continuous on the closed surface $\bar{\Sigma}$.

LEMMA 1. *If assumptions* (1.I)–(1.V) *are satisfied, then for any* $\tau \in (0, T]$ *problem* (1.1)–(1.3) *has a unique solution* $u(x, t)$ (²). *Moreover, for any* $0 < \beta < 1$ *we have* $u \in C_\beta(G)$ *and*

$$|u|_\beta^{G^\tau} \leqslant K(\beta)\tau^\gamma(|f|_0^{G^\tau} + |h|_0^{\Sigma^\tau}) \quad (^3), \tag{1.4}$$

where $\gamma = (1-\beta)/2$ *and the constant* $K(\beta)$ *depends only on the surface* S, *the operator* L *and* β.

Proof. The existence and uniqueness of solution of the problem in question are immediate consequences of Theorem 2 of [1] (p. 144). In order to prove estimate (1.4) we write (as in the proof of the above-mentioned Theorem 2) the solution $u(x, t)$ in the following form

$$\begin{aligned} u(x, t) &= \int_0^t \int_S \Gamma(x, t; \xi, \tau)\varphi(\xi, \tau)\, dS_\xi d\tau - \\ &\quad - \int_0^t \int_D \Gamma(x, t; \xi, \tau) f(\xi, \tau)\, d\xi d\tau \equiv \Phi(x, t) - F(x, t). \end{aligned} \tag{1.5}$$

(²) By a solution of problem (1.1)–(1.3) we understand a Σ_ν-regular solution, i. e. continuous in the domain $\overline{G^\tau}$, possessing in G^τ continuous derivatives appearing in Lu and continuous conormal derivative on Σ^τ.

(³) By $C_\beta(G)$ we denote the Banach space of all functions $u(x, t)$ with finite norm

$$|u|_\beta^G = |u|_0^G + \sup_G [|u(x, t) - u(x', t')|/(|x - x'|^\beta + |t - t'|^{\beta/2})],$$

where $|u|_0^G = \sup_G |u(x, t)|$.

Using the representation of function $\varphi(x, t)$, the estimates of function $M(x, t; \xi, \tau)$ (see [1], p. 145), the single-layer potential and the volume potential (see [2], p. 99, 101), we obtain

$$|\varphi|_0^{\Sigma^\tau} \leqslant K_1(|f|_0^{G^\tau} + |h|_0^{\Sigma^\tau}),$$

K_1 being a positive constant depending only on the surface S, on the operator L and on $|g|_0^{\Sigma}$. Hence, by Theorem 1 of [2] (p. 98, 99),

$$(1.6) \qquad |\Phi|_\beta^{G^\tau} \leqslant K_2(\beta)\tau^\gamma(|f|_0^{G^\tau} + |h|_0^{\Sigma^\tau}) \qquad (\gamma = (1-\beta)/2).$$

It follows from the proof of the estimate (228) of [2] (p. 102) that

$$(1.7) \quad |F(x, t) - F(x', t)| \leqslant K_3(\beta)\tau^\gamma |f|_0^{G^\tau}|x - x'|, \qquad x, x' \in D,\ 0 < t < \tau,$$

$$(1.8) \quad |F(x, t) - F(x, t')| \leqslant K_3(\beta)\tau^\gamma |f|_0^{G^\tau}|t - t'|^{\beta/2}, \qquad x \in D,\ t, t' \in (0, \tau).$$

Moreover,

$$(1.9) \qquad |F|_0^{G^\tau} \leqslant K_3(\beta)\tau^\gamma |f|_0^{G^\tau}.$$

Relations (1.5)–(1.9) imply estimate (1.4).

2. A priori estimates of solutions of the non-linear problem. Let $A^k(B^k)$, $k = 1, \ldots, N$, be an operator mapping the set $C_0^N(\bar{G})$ of all vector-functions $u(x, t) = (u^1(x, t), \ldots, u^N(x, t))$ continuous in $\bar{G}$ into the set of all functions continuous in $\bar{G}$ $(\bar{\Sigma})$.

In this section we derive an a priori estimate of solutions of the problem

$$(2.1) \qquad L^k u^k = A^k u, \qquad (x, t) \in G,$$

$$(2.2) \qquad u^k(x, 0) = 0, \qquad x \in \bar{D} \qquad (k = 1, \ldots, N),$$

$$(2.3) \qquad \frac{\partial u^k(x, t)}{\partial \nu^k(x, t)} = B^k u, \qquad (x, t) \in \Sigma.$$

By a solution $u(x, t) = (u^1(x, t), \ldots, u^N(x, t))$ of the above problem we shall always understand a Σ_ν-regular solution, i. e. every component $u^k(x, t)$ $(k = 1, \ldots, N)$ is a Σ_{ν^k}-regular function in $\bar{G}$ (see footnote ([2])).

The following assumptions are introduced:

(2.I) Operators L^k are parabolic in $\bar{G}$ and $a_{ij}^k \in C_\alpha(G)$ $(0 < \alpha \leqslant 1)$. The coefficients $b_i^k(x, t)$ are continuous in $\bar{G}$ and satisfy the uniform Hölder condition with respect to the variable x:

$$|b_i^k(x, t) - b_i^k(x', t)| \leqslant M_1|x - x'|^\alpha.$$

(2.II) There are positive constants M_2, M_3 such that for any vector-function $u \in \bigcup_{0<\varepsilon<1} C_\varepsilon^N(G)$ and any constants ν, τ $(0 \leqslant \nu < \tau \leqslant T)$ we have

$$|A^k u|_0^{G^{\nu,\tau}} \leqslant M_2 + M_3 |u|_0^{G^\tau} \quad (G^{\nu,\tau} = D \times (\nu, \tau]),$$
$$|B^k u|_0^{\Sigma^{\nu,\tau}} \leqslant M_2 + M_3 |u|_0^{G^\tau} \quad (\Sigma^{\nu,\tau} = S \times (\nu, \tau]),$$

where

$$|u|_0^G = \sum_{k=1}^{N} |u^k|_0^G,$$

and $C_\varepsilon^N(G)$ denotes the Banach space of all vector-functions $u = (u^1, \ldots, u^N)$ with finite norm

$$|u|_\varepsilon^G = \sum_{k=1}^{N} |u^k|_\varepsilon^G.$$

(2.III) If a vector-function $u(x, t) \in C_0^N(\bar{G})$ satisfies a local Hölder condition in $x \in D$ [4], then the functions $A^k u$ satisfy a local Hölder condition in $x \in D$ as well.

THEOREM 1. *Let assumptions* (1.III), (2.I)–(2.III) *be fulfilled and suppose that a vector-function $u(x, t)$ is a solution of problem* (2.1)–(2.3). *Under these assumptions if $u(x, t)$ satisfies a local Hölder condition in $x \in D$, then, for any $0 < \beta < 1$, $u \in C_\beta^N(G)$ and $|u|_\beta^G \leqslant M$, where M is a constant depending only on operators L^k, the boundary S and on the constants M_2, M_3, a and β.*

This theorem can be proved by similar considerations to those for Theorem 1 of [6], by making use of Lemma 1.

At present we shall consider the case operators A^k and B^k given by formulas (0.4) and (0.5).

Let us denote by $\mathfrak{M}_1 = \mathfrak{M}_2$ the σ-field of all Borel subsets of the domain $\bar{D}$ and by $\mathfrak{M}_3$ the σ-field of all Borel subsets of the interval $[0, T]$. By $\mu_i^k(x, t; \Omega)$ and $\nu_i^k(x, t; \Omega)$ ($k = 1, \ldots, N$; $i = 1, 2, 3$) we will denote finite non-negative measures defined on $\mathfrak{M}_i$ and depending on $(x, t) \in \bar{G}$ and $(x, t) \in \bar{\Sigma}$, respectively.

We make the following assumptions:

(2.IV) Measures $\mu_i^k(x, t; \Omega)$ are continuous in $(x, t) \in \bar{G}$, uniformly with respect to $\Omega \in \mathfrak{M}_i$; more precisely, there exist finite non-negative measures $\bar{\mu}_i$ (defined on $\mathfrak{M}_i$) with the following property: for any point $(x_0, t_0) \in \bar{G}$ and $\varepsilon > 0$ there is a number $\delta > 0$ such that if $(x, t) \in \bar{G}$ and $|x - x_0|^2 + |t - t_0| < \delta$, then for any $\Omega \in \mathfrak{M}_i$

$$|\mu_i^k(x, t; \ \Omega) - \mu_i^k(x_0, t_0; \ \Omega)| \leqslant \varepsilon \bar{\mu}_i(\Omega).$$

(2.V) For any closed domain $D^* \subset D$ there exist finite non-negative measures $\bar{\bar{\mu}}_i$ (defined on $\mathfrak{M}_i$) such that for any $\Omega \in \mathfrak{M}_i$, $x, x_0 \in D^*$, $0 < t \leqslant T$

[4] I. e. every component $u^k(x, t)$ ($k = 1, \ldots, N$) of the vector-function $u(x, t)$ satisfies a local Hölder condition in $x \in D$.

we have

$$|\mu_i^k(x, t;\ \Omega) - \mu_i^k(x_0, t;\ \Omega)| \leqslant \bar{\bar{\mu}}_i(\Omega)\,|x - x_0|^{\gamma_1},$$

where the constant γ_1 $(0 < \gamma_1 \leqslant 1)$ is independent of the domain D^*.

(2.VI) There is a constant $M_4 > 0$ such that for any $\Omega \in \mathfrak{M}_3$

$$\mu_3^k(x, t;\ \Omega) \leqslant M_4 m(\Omega) \qquad \text{if } (x, t) \in \bar{G}$$

and

$$\nu_3^k(x, t;\ \Omega) \leqslant M_4 m(\Omega) \qquad \text{if } (x, t) \in \bar{\Sigma},$$

$m(\Omega)$ being the Lebesgue measure of Ω.

(2.VII) Measures $\nu_i^k(x, t;\ \Omega)$ are continuous in $(x, t) \in \bar{\Sigma}$, uniformly with respect to $\Omega \in \mathfrak{M}_i$ in the sense of assumption (2.IV).

(2.VIII) Functions $f^k(x, t, p, q, r)$ are continuous in the domain $\bar{G} \times E_{3N}$ and fulfil the growth condition

$$|f^k(x, t, p, q, r)| \leqslant M_5 + M_6\,|(p, q, r)|, \qquad M_5, M_6 > 0,$$

where

$$|(p, q, r)| = \sum_{i=1}^{N} (|p^i| + |q^i| + |r^i|).$$

Moreover, the following local Hölder condition is satisfied: for any closed domain $D^* \subset D$ and any bounded domain $H \subset E_{3N}$ we have

$$|f^k(x, t, p, q, r) - f^k(x', t, p', q', r')| \\ \leqslant M_1(D^*)\,|x - x'|^{\gamma_2} + M_2(H)\,|(p - p', q - q', r - r')|^{\gamma_3}$$

if $x, x' \in D^*$, $0 < t \leqslant T$, $(p, q, r), (p', q', r') \in H$, where $\gamma_2, \gamma_3 \in (0, 1]$ are constants independent of domains D^* and H, while constants $M_1(D^*)$ and $M_2(H)$ may depend on D^* and H, respectively.

(2.IX) Functions $h^k(x, t, p, q, r)$ are continuous in the domain $\bar{\Sigma} \times E_{3N}$ and fulfil the inequalities

$$|h^k(x, t, p, q, r)| \leqslant M_7 + M_8\,|(p, q, r)|,$$

M_7 and M_8 being positive constants.

In order to prove Theorem 1 for the special case under consideration of operators A^k and B^k we need the following

LEMMA 2. *If assumptions* (2.IV)–(2.VII) *are fulfilled and if* $u(x, t)$ *is a continuous function in* $\bar{G}$, *then the functions*

$$v_1^k(x, t) = \int_0^t \mu_3^k(x, t;\ d\tau) \int_D u(y, \tau)\,\mu_2^k(x, t;\ dy),$$

$$v_2^k(x, t) = \int_D u(y, t)\,\mu_1^k(x, t;\ dy)$$

are continuous in $\bar{G}$, whereas the functions

$$w_1^k(x, t) = \int_0^t \nu_1^k(x, t;\ d\tau) \int_D u(y, \tau)\nu_2^k(x, t;\ dy),$$

$$w_2^k(x, t) = \int_D u(y, t)\nu_1^k(x, t; dy)$$

are continuous on $\bar{\Sigma}$. Moreover, the functions $v_i^k(x, t)$ satisfy the local Hölder condition with exponent γ_1 (occurring in assumption (2.V)) in $x \in D$.

This lemma can be proved by similar considerations to those for Lemma 4 of [4].

Now, by Lemma 2, Theorem 1 implies the following

THEOREM 2. *Let assumptions (1.III), (2.I), (2.IV)–(2.IX) be fulfilled. Suppose that a vector-function $u(x, t)$ is a solution of problem (2.1)–(2.3) in the case (0.4), (0.5) and assume that $u(x, t)$ satisfies a local Hölder condition in $x \in D$. Then the assertion of Theorem 1 holds true provided we replace the constants M_2, M_3 by $M_4, \ldots, M_9$, where $M_9 > 0$ is a constant bounding measures $\mu_i^k(x, t;\ \Omega)$ and $\nu_i^k(x, t;\ \Omega)$, $\Omega \in \mathfrak{M}_i$* (5).

3. Existence theorems for the non-linear problem. In this section we shall prove some theorems on the existence of solutions of problem (0.1)–(0.3), using an a priori estimate of those solutions which were derived in the previous section. The particular case (0.4), (0.5) will also be consider.

We make the following assumptions:

(3.I) For any β $(0 < \beta < 1)$ operators $A^k u$ and $B^k u$ $(k = 1, \ldots, N)$ are continuous in the space $C_\beta^N(G)$ in the following sense: if

$$u, u_m \in C_\beta^N(G) \quad \text{and} \quad \lim_{m\to\infty} |u_m - u|_\beta^G = 0,$$

then

$$\lim_{m\to\infty} |A^k u_m - A^k u|_0^G = 0 \quad \text{and} \quad \lim_{m\to\infty} |B^k u_m - B^k u|_0^\Sigma = 0.$$

(3.II) The functions $\psi^k(x)$ $(k = 1, \ldots, N)$ are continuous in $\bar{D}$ together with the derivatives $\psi_{x_i}^k$ and $\psi_{x_i x_j}^k$ and, moreover, $\psi_{x_i x_j}^k$ satisfy a local Hölder condition in D.

THEOREM 3. *If assumptions (1.III), (2.I), (2.III), (3.I) and (3.II) are satisfied, then there exists a solution $u(x, t)$ of problem (0.1)–(0.3); furthermore, $u \in C_\beta^N(G)$ for any β, $0 < \beta < 1$.*

Proof. We apply the method of Leray–Schauder. Namely, denote by Λ the set of all functions $v \in C_{\beta_0}^N(G)$ such that $v(x, 0) = 0$ for $x \in \bar{D}$, β_0 $(0 < \beta_0 < 1)$ being an arbitrary fixed number. Now we define on the

(5) The existence of such a constant follows from assumptions (2.IV) and (2.VII).

set $\Lambda \times [0, 1]$ a transformation Z putting $Z(v, \lambda) = w$, where w is a unique solution of the problem

$$L^k w^k = \lambda[A^k(v+\psi) - L^k\psi^k], \quad (x, t) \in G,$$
$$w^k(x, 0) = 0, \quad x \in \bar{D} \quad (k = 1, \ldots, N),$$
$$\frac{\partial w^k(x, t)}{\partial \nu^k(x, t)} = \lambda\left[B^k(v+\psi) - \frac{\partial \psi^k(x)}{\partial \nu^k(x, t)}\right], \quad (x, t) \in \Sigma$$

(existing by Lemma 1). Proceeding further as in the proof of Theorem 4 of [6] and using Lemma 1 and Theorem 1, one can show the existence of a fixed point v of the transformation $Z(v, 1)$ in the set Λ, i. e. $Z(v, 1) = v$. It is easy to observe that the function $u = v + \psi$ is a Σ_ν-regular solution of problem (0.1)–(0.3) and, moreover, $u \in C_\beta^N(G)$ for any $0 < \beta < 1$. This completes the proof.

As a corollary of Theorem 3 we obtain the following

THEOREM 4. *If assumptions* (1.III), (2.I), (2.IV)–(2.IX) *and* (3.II) *are satisfied, then the assertion of Theorem* 3 *is valid in the case* (0.4), (0.5).

At present we shall prove the stronger version of Theorem 3, ensuring the uniqueness as well. We retain all the assumptions of this theorem except (3.I), which is replaced by the following more restrictive condition:

(3.III) For any $0 < \beta < 1$ and any bounded subset $\Omega \in C_\beta^N(G)$ there exists a constant $M_{10} > 0$ such that for any ν, τ $(0 \leqslant \nu < \tau \leqslant T)$ the following inequalities are fulfilled:

$$|A^k u - A^k v|_0^{G^{\nu,\tau}} \leqslant M_{10}|u - v|_\beta^{G^\tau}, \qquad |B^k u - B^k v|_0^{\Sigma^{\nu,\tau}} \leqslant M_{10}|u - v|_\beta^{G^\tau}$$

if $u, v \in \Omega$.

THEOREM 5. *If assumptions* (1.III), (2.I)–(2.III), (3.II) *and* (3.III) *are fulfilled, then problem* (0.1)–(0.3) *has a unique solution* $u(x, t)$ *in the class* H *of all* Σ_ν*-regular functions in* $\bar{G}$ *satisfying a local Hölder condition in* $x \in D$ (6). *Moreover,* $u \in C_\beta^N(G)$ *for any* $0 < \beta < 1$.

Proof. It is sufficient to show the uniqueness of solutions. Suppose that functions u and v, satisfying a local Hölder condition in $x \in D$, are solutions of problem (0.1)–(0.3). It follows from Theorem 1 that $u, v \in C_\beta^N(G)$ for any $0 < \beta < 1$. In order to prove the identity $u \equiv v$ in $\bar{G}$ we proceed as in the proof of Theorem 5 of [6], making use of Lemma 1 and assumption (3.III).

Theorem 5 implies the following

THEOREM 6. *Let all the assumptions of Theorem* 4 *be fulfilled* (*with* $\gamma_3 = 1$ *in assumption* (2.VIII)). *Assume that for any bounded set* $H \subset E_{3N}$

(6) More precisely: a Σ_ν-regular function $v(x, t)$ in $\bar{G}$ belongs to H if and only if there exists a number $0 < \delta < 1$ such that $v(x, t)$ satisfies a local Hölder condition with exponent δ in $x \in D$.

there exists a constant $M_{11} > 0$ such that for any $(x, t) \in \Sigma$ and (p, q, r), $(\bar{p}, \bar{q}, \bar{r}) \in H$ we have

$$|h^k(x, t, p, q, r) - h^k(x, t, \bar{p}, \bar{q}, \bar{r})| \leqslant M_{11}|(p-\bar{p}, q-\bar{q}, r-\bar{r})|.$$

Under these assumptions the conclusion of Theorem 5 holds true in the case (0.4), (0.5).

4. On strong integro-differential inequalities. Now we prove the following theorem on strong integro-differential inequalities.

THEOREM 7. *We assume that operators L^k (defined by* (0.1)) *are parabolic in $\bar{G}$. Let $f^k(x, t, p, q, r)$ and $h^k(x, t, p, q, r)$ be functions defined on $\bar{G} \times E_{3N}$ and $\bar{\Sigma} \times E_{3N}$, respectively, and assume that these functions are non-increasing with respect to the variables $p^1, \ldots, p^{k-1}, p^{k+1}, \ldots, p^N, q, r$. Suppose that functions $u = (u^1, \ldots, u^N)$ and $v = (v^1, \ldots, v^N)$ are Σ_ν-regular in $\bar{G}$* (7) *and satisfy the following inequalities:*

$$L^k u^k - A^k u > L^k v^k - A^k v, \quad (x, t) \in G, \tag{4.1}$$

$$\frac{\partial u^k}{\partial \nu^k} - B^k u > \frac{\partial v^k}{\partial \nu^k} - B^k v, \quad (x, t) \in \Sigma \quad (k = 1, \ldots, N), \tag{4.2}$$

$$u^k(x, 0) < v^k(x, 0), \quad x \in \bar{D}, \tag{4.3}$$

where A^k and B^k are given by formulas (0.4) *and* (0.5), *respectively.*

Under these assumptions $u^k(x, t) < v^k(x, t)$ $(k = 1, \ldots, N)$ in $\bar{G}$.

Proof. The method of proving this theorem is the same as that used in the proofs of Theorems 63.1 and 63.3 of [3]. Namely, suppose that the assertion of the theorem is false. Then, by inequalities (4.3) and by the continuity of functions u^k and v^k, there exist an index k_0 $(1 \leqslant k_0 \leqslant N)$ and a point $(x_0, t_0) \in \bar{D} \times (0, T]$ such that

$$u^{k_0}(x_0, t_0) = v^{k_0}(x_0, t_0), \quad u^k(x, t) < v^k(x, t), \quad (x, t) \in \bar{D} \times [0, t_0) \quad (k = 1, \ldots, N). \tag{4.4}$$

Two cases are possible: $x_0 \in D$ or $x_0 \in S$. In the case where $x_0 \in D$ relations (4.4) imply that the function $u^{k_0}(x, t_0) - v^{k_0}(x, t_0)$ attains its maximum at the point x_0. Hence, using the parabolicity of L^k and the relation

$$u_t^{k_0}(x_0, t_0) - v_t^{k_0}(x_0, t_0) \geqslant 0$$

(which follows from (4.4)), we obtain

$$L^{k_0}(u^{k_0} - v^{k_0}) \leqslant 0 \quad \text{in } (x_0, t_0).$$

(7) The continuity of conormal derivatives of functions u^k and v^k on Σ is superfluous; instead of that it is enough to assume only the existence of these derivatives.

The monotonicity of the function f^{k_0} and relations (4.4) imply that

$$A^{k_0}u - A^{k_0}v \geqslant 0 \quad \text{in } (x_0, t_0).$$

In view of the last two inequalities we have

$$L^{k_0}u^{k_0} - L^{k_0}v^{k_0} \leqslant A^{k_0}u - A^{k_0}v \quad \text{in } (x_0, t_0),$$

which contradicts condition (4.1).

Now let us consider the case $x_0 \in S$. We denote by $\partial z(x, t)/\partial l^k(x, t)$ the directional derivative in the inward conormal direction with respect to the operator L^k (see [1], p. 137). It follows from (4.4) that

$$\frac{\partial(u^{k_0} - v^{k_0})}{\partial l^{k_0}}\bigg|_{(x_0, t_0)} \leqslant 0.$$

Hence, in view of the formula

$$\frac{\partial z(x, t)}{\partial \nu^k(x, t)} = |l^k(x, t)| \frac{\partial z(x, t)}{\partial l^k(x, t)},$$

we get

$$\frac{\partial(u^{k_0} - v^{k_0})}{\partial \nu^{k_0}}\bigg|_{(x_0, t_0)} \leqslant 0.$$

Relations (4.4) and the monotonicity of the function h^{k_0} imply that

$$B^{k_0}u - B^{k_0}v \geqslant 0 \quad \text{in } (x_0, t_0).$$

Combining the last two inequalities, we conclude that

$$\frac{\partial u^{k_0}}{\partial \nu^{k_0}} - \frac{\partial v^{k_0}}{\partial \nu^{k_0}} \leqslant B^{k_0}u - B^{k_0}v \quad \text{in } (x_0, t_0),$$

which contradicts condition (4.2).

Since in both possible cases we have obtained a contradiction, the theorem is proved.

Proceeding as in the proof of Theorem 2 of [5] and using Theorems 4, 2 and 7 one can prove a theorem on the existence of the maximum and minimum solutions of problem (0.1)–(0.3) in the case (0.4), (0.5) and a theorem on weak inequalities which is the counterpart of Theorem 11 of [5].

Remark. As in papers [4]–[6], all the theorems of this paper concerning the case (0.4), (0.5) remain true if we replace integrals by suitable functionals depending on parameters x, t.

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Reçu par la Rédaction le 10. 6. 1971