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On the Gross property*

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Abstract. If $f(z)$ is a function meromorphic and non-constant in the unit disk and if $f(z)$ has an asymptotic value a, finite or infinite, along a boundary path L whose end is the unit circumference, then we prove that the inverse function $z = \varphi(w)$ of $f(z)$ has the property that each regular functional element of $z = \varphi(w)$, with center at $w_0 \neq \infty$, can be continued analytically using only regular elements along each ray $\arg(w-w_0) = \theta$ up to the point $w = \infty$, except for at most a set of values θ of measure zero.

1. Introduction. Let $z = \varphi(w)$ be an analytic function whose domain is its Riemann surface Φ. If, for each $Q(w; w_0)$, $w_0 \neq \infty$, a regular functional element of $z = \varphi(w)$, $Q(w; w_0)$ can be continued analytically using only regular elements along each ray $\arg(w-w_0) = \theta$ up to the point $w = \infty$, except for at most a set of values θ of measure zero, then $z = \varphi(w)$ will be said to have the *Gross property*. Gross [2] proved that if $w = f(z)$ is a non-rational meromorphic function in $|z| < +\infty$ and $z = \varphi(w)$ is its inverse, then $z = \varphi(w)$ has the Gross property. More recently, Stebbins [7] proved that if $f(z)$ is non-constant and meromorphic in $|z| < 1$ and $f(z)$ has ∞ as an asymptotic value along a spiral, then the inverse $z = \varphi(w)$ of $f(z)$ has the Gross property. In this paper we prove that if $f(z)$ is a function of class (P^*), a larger class of functions than that considered by Stebbins, then the inverse $z = \varphi(w)$ of $f(z)$ has the Gross property.

2. Definitions and a lemma. Denote by D the unit disk in the complex plane and by C the unit circumference. By a boundary path of D we shall mean a simple continuous curve $S: z = s(t)$, $0 \leqslant t < 1$, in D such that $|s(t)| \to 1$ as $t \to 1$. In particular, if $\arg s(t) \to +\infty$ or $\arg s(t) \to -\infty$ as $t \to 1$, the boundary path S will be called a *spiral* in D. The end of a boundary path S, denoted by $E(S)$, will be the set of limit points of S on C.

Suppose that the function $f(z)$ is meromorphic in D and that S is a boundary path. We say that $f(z)$ tends to a complex value a, finite or

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infinite, along S, if

$$\lim f(z) = a \quad \text{for } |z| \to 1, \ z \in S,$$

and a will be said to be an *asymptotic value* of $f(z)$ along S.

DEFINITION 1. Let (P^*) be the class of functions non-constant and meromorphic in D which have an asymptotic value a, finite or infinite, along a boundary path S whose end $E(S)$ is C.

EXAMPLE. The example we give here is due to Bagemihl, Erdös, and Seidel [1].

Let $\{n_k\}$ be a sequence of increasing positive integers such that $\lim_{k\to+\infty} n_k/n_{k-1} = +\infty$, $n_1 > 1$. Then, let

$$g(z) = \prod_{j=1}^{\infty} \left\{1 - \left(\frac{z}{1-1/n_j}\right)^{n_j}\right\}.$$

The function $g(z)$ is holomorphic in D and it possesses n_j simple zeros on the circle $|z| = 1-1/n_j$ for $j = 1, 2, \ldots$ Let us, now, denote by γ_j the n_j circles of radius $1/j^2 n_j$ with center at the zeros of $g(z)$ on $|z| = 1-1/n_j$, and let Γ_j denote γ_j with its interior. Then there exists $j_0 > 0$ such that all the Γ_j, $j > j_0$, are disjoint. If $\Delta = D - \bigcup_{j=j_0}^{\infty} \Gamma_j$, then $g(z)$ has the property that it converges uniformly to ∞ in Δ as $|z| \to 1$ [1], p. 137. If we consider $f(z) = 1/g(z)$, then it is clear that (i) $f(z) \in (P^*)$, and, hence, $(P^*) \neq \emptyset$, and (ii) $w = 0$ is the only asymptotic value for $f(z)$, and, hence, $f(z)$ is not in the class of functions considered by Stebbins [7].

In the sequel $w = f(z)$ will be a function of class (P^*) and $z = \varphi(w)$ will be its inverse function with domain the Riemann surface Φ. The functional element $Q(w; w_0)$, which may be rational or algebraic, shall serve a double duty, not only representing a functional element of $z = \varphi(w)$, but also representing a point of the Riemann surface Φ. The projection of $Q(w; w_0)$ onto the w-plane is $w = w_0$.

The proof of the main result of this paper depends somewhat on the theory of transcendental singularities. Because of this, we give the following background.

DEFINITION 2. Let

$$\Lambda\colon\ w = Q\big(w; w(t)\big), \qquad 0 \leqslant t < 1,$$

with $\lim_{t\to 1} w(t) = \omega$, be a curve on the Riemann surface Φ of $z = \varphi(w)$. Then the curve Λ defines a transcendental singularity Ω of $z = \varphi(w)$ on Φ, with projection $w = \omega$, if

(i) for any positive number δ, $\delta < 1$, the system of functional elements $Q\big(w; w(t)\big)$, $0 \leqslant t \leqslant \delta$, defines an analytic continuation (possibly, of algebraic character), but

(ii) for any functional element $Q(w; w_0)$, rational or algebraic, with center at $w = \omega$, the system $Q(w; w(t))$, $0 \leqslant t \leqslant 1$, where $w(1) = \omega$, never defines an analytic continuation.

DEFINITION 3. Let $r > 0$. Suppose that

$$\Lambda\colon\; w = Q\big(w; w(t)\big), \quad 0 \leqslant t < 1,$$

with $\lim_{t\to\infty} w(t) = \omega$, defines a transcendental singularity Ω on Φ. Let t_r be the last value for t, $0 \leqslant t < 1$, such that $|w(t_r) - \omega| = r$, counting from $t = 0$. Then by an r-neighborhood of Ω, denoted by $U_r(\Omega)$, we mean all points $Q(w; c)$ of Φ such that $|c - \omega| < r$ and $Q(w; c)$ is an analytic continuation (possibly, of algebraic character) of $Q\big(w; w(t_r)\big)$ along a curve lying inside the disk $|w - \omega| < r$. If the transcendental singularity lies above the point $w = \infty$, then the circle we use to define $U_r(\Omega)$ is $|w| = r$ and the disk used is $|w| > r$.

LEMMA. *Let Λ: $w = Q\big(w; w(t)\big)$, $0 < t < 1$, with $\lim_{t\to 1} w(t) = \omega$, define a transcendental singularity Ω on Φ. Then*

$$\bigcap_{r>0} U_r(\Omega) = \emptyset.$$

Proof. Suppose that there exists a functional element $Q(w; w_0)$ such that $Q(w; w_0) \in \bigcap_{r>0} U_r(\Omega)$. Let $\{r_n\}$ be a sequence of positive numbers which decrease to zero as $n \to +\infty$. Then,

$$Q(w; w_0) \in \bigcap_{r>0} U_r(\Omega) \subseteq \bigcap_{n=1}^{\infty} U_{r_n}(\Omega).$$

Thus, $Q(w; w_0) \in U_{r_n}(\Omega)$ for every n. This implies that $|w_0 - \omega| < r_n$ for every n. Thus, $w_0 = \omega$, and $Q(w; w_0) = Q(w; \omega)$.

Suppose that p is the radius of convergence of $Q(w; \omega)$. There exists t_0, $0 < t_0 < 1$, such that $|w(t) - \omega| < p$ for all t, $t_0 < t < 1$. Thus, by definition 3, $Q\big(w; w(t)\big) \in U_p(\Omega)$ for all t, $t_0 < t < 1$. Also, $Q(w; \omega) \in U_p(\Omega)$. Hence, for each t, $t_0 < t < 1$, $Q\big(w; w(t)\big)$ is an analytic continuation of $Q(w; \omega)$ along a path lying in the circle of convergence of $Q(w; \omega)$. It follows, then, that $Q\big(w; w(t)\big)$, for $t_0 < t < 1$, is a direct analytic continuation of $Q(w; \omega)$. Thus, we may adjoin $Q(w; \omega)$ to Λ to complete the analytic continuation of Λ for $0 \leqslant t \leqslant 1$. This is a contradiction. Thus,

$$\bigcap_{r>0} U_r(\Omega) = \emptyset.$$

The importance of transcendental singularities is that there exists a one-to-one correspondence between the transcendental singularities of $z = \varphi(w)$ and the asymptotic boundary paths of $w = f(z)$, where $z = \varphi(w)$ is the inverse of $w = f(z)$. This result was first proved for $f(z)$

an entire function by Iversen [3]. Later, Noshiro [5] proved this result for $f(z)$ analytic on a Riemann surface F.

3. The main result.

THEOREM. *Let* $w = f(z) \epsilon (P^*)$ *and let* $z = \varphi(w)$ *be its inverse. Then* $z = \varphi(w)$ *has the Gross property.*

Proof. We remark that part of our proof is patterned after the proof of the Gross Star theorem given in Nevanlinna [4], p. 292–294.

We continue the regular functional element $Q(w; w_0)$ of $z = \varphi(w)$ analytically with regular elements along the ray $\arg(w-w_0) = \theta$ until either a singular point (algebraic or transcendental singularity) or the point $w = \infty$ is reached. Let S_0 be the resulting (simply-connected) star-shaped region in the w-plane made up of such segments.

Let M be the set of values θ in $0 \leqslant \theta \leqslant 2\pi$ such that the ray $\arg(w-w_0) = \theta$ of the star-shaped region S_0 terminates in a finite transcendental singularity at $w = w_\theta$ and this is the first singularity encountered on $\arg(w-w_0) = \theta$ as one continues $Q(w; w_0)$ analytically on $\arg(w-w_0) = \theta$ from $w = w_0$. Since all the zeros of $f'(z)$ are isolated points, we have that the set of algebraic singularities is countable. Thus, to prove our theorem it will suffice to show that $m^*(M) = 0$ ($m^*(M)$ denotes the outer Lebesgue measure of the set M).

Let $R > 0$. Let $M_R = \{\theta: \theta \epsilon M \text{ and } |w_\theta - w_0| < R\}$. Since $M_{R_1} \subseteq M_{R_2}$ for $R_1 < R_2$ and $M = \bigcup_{n=1}^{\infty} M_n$, $m^*(M) = \lim_{n\to+\infty} m^*(M_n)$. Thus, it suffices to show that $m^*(M_n) = 0$ for an arbitrary integer n.

Since $f(z) \epsilon (P^*)$, $f(z) \to a$ on a boundary path S: $z = s(t)$, $0 \leqslant t < 1$, with $E(S) = C$. In order to prove our theorem we must consider the following three cases: (i) a is finite and $w_0 \neq a$, (ii) a is finite and $w_0 = a$, and (iii) a is infinite.

Let a be finite and, further, let $a \neq w_0$. Let p be the radius of convergence of the regular element $Q(w; w_0)$. Choose an integer n sufficiently large so that $|a-w_0| < n$. Let $p_0 = \frac{1}{2}\min(|a-w_0|, p)$. For each integer $m > 1$ we define the set W_m as follows:

$$W_m = \{w: |\arg(w-w_0) - \arg(a-w_0)| \leqslant \pi/m \text{ and } p_0 \leqslant |w-w_0| \leqslant n\}.$$

Let $M_{n,m} = \{\theta: \theta \epsilon M_n \text{ and } w_\theta \notin W_m\}$. Since $M_{n,m+1} \supseteq M_{n,m}$ and $M_n - \{\arg(a-w_0)\} = \bigcup_{m=2}^{\infty} M_{n,m}$, we have $m^*(M_n) = \lim_{m\to+\infty} m^*(M_{n,m})$. Thus, it suffices to show that $m^*(M_{n,m}) = 0$ for an arbitrary integer m.

Let us, now, assume that m is an arbitrary, but fixed, integer. Suppose, also, that $M_{n,m} \neq \emptyset$. Since $f(z) \to a$ on S, there exists $t_0, 0 < t_0 < 1$, such that $f(s(t)) \epsilon W_m$ for all t, $t_0 \leqslant t < 1$. Let $S^1: z = s(t)$, $t_0 \leqslant t < 1$. Clearly, $E(S^1) = E(S) = C$. We map the simply-connected region $D - S^1$ in a one-to-one conformal manner by $\zeta = \zeta(z)$ onto the disk $|\zeta| < 1$ in such a way that the unique prime end P of $D - S^1$ whose impression is C corresponds to $\zeta = 1$ under $\zeta = \zeta(z)$.

Let $S^* = [S_0 \cap \{|w - w_0| < n\}] - W_m$. We consider the function $F(w) = \zeta(\varphi(w))$ on S^*. Then, $F(w)$ maps S^* in a one-to-one conformal manner onto a simply-connected subregion B of $|\zeta| < 1$. Each ray of S^* which terminates at a transcendental singularity of S^* is mapped by $\varphi(w)$ onto a boundary path L in D disjoint from S^1, and L is mapped by $\zeta = \zeta(z)$ onto a path in B which terminates at $\zeta = 1$ (since $E(S^1) = E(S) = C$).

At this point, we remark that the remainder of the proof of case (i), which we are about to give, will apply also to case (ii) and (iii).

Consider the function

$$t(w) = \frac{F(w) - 1}{F(w) - F(w_0)}$$

on S^*. The function $t(w)$ is a one-to-one conformal map of S^* onto a schlicht region S_t. Since w_0 is an interior point of S^*, $t = \infty$ is an interior point of S_t. The point $t = 0$ is an exterior point or a boundary point of S_t. Since

$$|F(w) - F(w_0)| \leqslant |F(w)| + |F(w_0)| \leqslant 2,$$

$t(w) \to 0$ if and only if $F(w) \to 1$. Since we have assumed that $M_{n,m} \neq \emptyset$ there exists a ray $\arg(w - w_0) = \theta$ of S^* which terminates at a transcendental singularity at $w = w_\theta$. Along this ray $F(w)$ tends to 1 as w tends to w_θ. Thus, $t(w)$ tends to 0 as w tends to w_θ along $\arg(w - w_0) = \theta$. Hence,, $t = 0$ is a boundary point of S_t.

Let $r > 0$. Let $\Delta_t(r)$ be the finite or countable collection of component arcs of $|t| = r$ which fall into the region S_t. These arcs separate the boundary point $t = 0$ from the interior point $t = \infty$ of S_t.

Let $\Delta_w(r)$ be the image arcs of the arcs of $\Delta_t(r)$ under the map $w(t)$ which is the inverse of $t(w)$. The arcs of $\Delta_w(r)$ are crosscuts of S^* which separate the transcendental singularities of S^* from $w = w_0$. Therefore, $m^*(M_{n,m})$ is less than or equal to the sum of the variations of $\arg(w - w_0)$ on the crosscuts $\Delta_w(r)$. But, the sum of the variations of $\arg(w - w_0)$ on the crosscuts $\Delta_w(r)$ is less than or equal to $s(r)/d_r$, where $s(r)$ is the sum of the lengths of the crosscuts $\Delta_w(r)$ and d_r is the shortest distance from $w = w_0$ to the crosscuts $\Delta_w(r)$.

We now fix r_0, $r_0 > 0$. Since $w_0 \in S^*$ corresponds to $\infty \in S_t$ under $t(w)$ and since r_0 is fixed and finite, $d_r > 0$. Then, for $0 < r < r_0$, $d_{r_0} < d_r$. Hence,

$$m^*(M_{n,m}) \leqslant s(r)/d_r \leqslant s(r)/d_{r_0}.$$

Thus, it suffices to show that $s(r)$ can be made arbitrarily small by a suitable selection of r.

Let $z = z(\zeta)$ be the inverse of $\zeta = \zeta(z)$. Let $t = re^{i\theta}$. Let

$$G(t) = f\left(z\left(\frac{tF(w_0)-1}{t-1}\right)\right).$$

Then $G(t)$ maps S_t onto S^*. By the Schwarz inequality

$$(s(r))^2 = \Big(\int_{\Delta_t(r)} |G'(t)|\,|dt|\Big)^2 \leqslant \Big(\int_{\Delta_t(r)} |G'(t)|^2|dt|\Big)\Big(\int_{\Delta_t(r)} |dt|\Big) \leqslant 2\pi r \int_{\Delta_t(r)} |G'(t)|^2 r d\theta.$$

But

$$\int_{\Delta_t(r)} |G'(t)|^2 r d\theta = -\frac{dA}{dr},$$

where $A(r)$ denotes the area of that subregion of S^* which contains the point $w = w_0$ and which is bounded by the crosscuts $\Delta_w(r)$. Integrating between r and r_0 $(r_0 > r)$, we get

$$(1) \qquad \int_r^{r_0} \frac{(s(r))^2}{r} dr \leqslant -\int_r^{r_0} \frac{2\pi r}{r}\frac{dA}{dr} dr = -2\pi\big(A(r_0) - A(r)\big) \leqslant 2\pi^2 n^2.$$

This inequality holds for all r, $0 < r < r_0$.

Let $\delta > 0$. Suppose $s(r) \geqslant \delta$ for all r, $0 < r < r_0$. Then

$$\int_r^{r_0} \frac{(s(r))^2}{r} dr \geqslant \delta^2 \log\frac{r_0}{r} \to +\infty$$

as $r \to 0$. This contradicts inequality (1) above. Thus, there exists a number r, $0 < r < r_0$, such that $s(r) < \delta$. Therefore, $m^*(M_{n,m}) < \delta/d_{r_0}$. Since δ is an arbitrary positive number, $m^*(M_{n,m}) = 0$. This completes the proof for case (i).

We consider the case with a finite and $w_0 = a$. By the lemma above, there exists $p_0 > 0$ such that the functional element $Q(w; w_0) = Q(w; a)$ $\notin U_{p_0}(\Omega)$, where Ω is the transcendental singularity of $z = \varphi(w)$ which corresponds to the asymptotic boundary path S. Let $p^* = \min(p_0, p)$, where p is the radius of convergence of $Q(w, a)$. There exists t_0, $0 < t_0 < 1$, such that $|f(s(t)) - a| < p^*$ for all t, $t_0 \leqslant t < 1$. Let S^1: $z = s(t)$, $t_0 \leqslant t < 1$. We map the simply-connected region $D - S^1$ onto $|\zeta| < 1$ in the same manner as before by the one-to-one conformal map $\zeta = \zeta(z)$.

Let n be any fixed integer with $n > p_0$. Let $S^* = S_0 \cap \{|w-a| < n\}$. The function $F(w) = \zeta(\varphi(w))$ maps S^* in a one-to-one conformal manner onto a simply-connected subregion B of $|\zeta| < 1$. The image of S^1 under $w = f(z)$ lies in $U_{p^*}(\Omega)$. Each ray of S^* which terminates at a transcendental singularity is mapped by $z = \varphi(w)$ onto a boundary path L in D. The path L is disjoint from S^1 in D. Indeed, if not, then any point common to S^1 and L is mapped by $f(z)$ onto a point $Q(w; b)$ with $|b-a| < p^*$. Then $Q(w; b) \in U_{p^*}(\Omega)$ and $Q(w; b)$ is a direct analytic continuation of $Q(w; a)$. Hence, $Q(w; a) \in U_{p^*}(\Omega)$. But this is impossible. Thus, L is disjoint from S^1 in D and, hence, L is mapped by $\zeta = \zeta(z)$ onto a path in B which terminates at $\zeta = 1$.

We now consider the function $t(w)$ on S^* of case (i) and we use the analysis of case (i) to show $m^*(M_n) = 0$.

For our last case we begin by assuming $a = \infty$. Then, for a fixed integer n, the exists t_0, $0 < t_0 < 1$, such that $|f(s(t))| > |w_0| + n$ for all t, $t_0 \leqslant t < 1$. Let S^1: $z = s(t)$, $t_0 \leqslant t < 1$. Let $\zeta = \zeta(z)$ be the map of case (i) which maps $D - S^1$ onto $|\zeta| < 1$. Let $S^* = S_0 \cap \{|w - w_0| < n\}$. We consider the one-to-one conformal map $F(w) = \zeta(\varphi(w))$ on S^* which maps S^* onto a subregion B of $|\zeta| < 1$. Each ray of S^* which terminates at a transcendental singularity is mapped by $\varphi(w)$ onto a boundary path L in D disjoint from S^1, and L is mapped by $\zeta = \zeta(z)$ onto a path in B which terminates at $\zeta = 1$.

We can now form the function $t(w)$ on S^* as in case (i) and once again, apply the analysis of case (i) to prove $m^*(M_n) = 0$. This completes the proof of our theorem.

Remark. Although the Gross property is a necessary condition for the inverse function of a function of class (P^*), it is not a sufficient condition. The inverse of the elliptic modular function $w = v(z)$ for the unit disk has the Gross property, but $v(z) \notin (P^*)$.

COROLLARY 1. *Let $w = f(z) \in (P^*)$ and let $z = \varphi(w)$ be its inverse function with Riemann surface Φ. Let $Q(w; w_0)$ be an arbitrary functional element (regular or algebraic) of $z = \varphi(w)$ with center w_0 lying in $|w-a| < r$. Then a continuous path L can be found lying inside $|w-a| < r$, starting at $w = w_0$ and terminating at $w = a$, such that there exists an analytic continuation of $Q(w; w_0)$ of algebraic character on Φ above L except perhaps at the endpoint $w = a$ of L* (Iversen's property).

COROLLARY 2. *Let $f(z) \in (P^*)$. If a is a complex value, finite or infinite, which is taken by $f(z)$ only finitely many times in D, then a is an asymptotic value of $f(z)$ along some boundary path L of D.*

Proof. The proof is the same as that given in Noshiro [6], p. 4–5.

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