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The radius of univalence of certain analytic functions

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Abstract. The authors determine the radii of starlikeness, convexity and close-to-convexity of the functions $f(z) = \frac{1}{1+C} z^{1-C}[z^C F(z)]'$, $C \in N$, where $F(z)$ are starlike, convex or close-to-convex functions.

1. Introduction. Let S denote the class of regular and univalent functions $f(z)$ in $D = \{z: |z| < 1\}$ which are normalized by the conditions $f(0) = 0, f'(0) = 1$. For a fixed α, $0 \leqslant \alpha < 1$, let $C(\alpha)$ denote the subclass of S, consisting of all functions f satisfying the condition

$$\mathrm{Re}\left\{z\frac{f''(z)}{f'(z)}+1\right\} > \alpha \quad \text{for } z \in D. \tag{1.1}$$

Let $S^*(\alpha)$ denote the subclass of S consisting of all functions f satisfying the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad \text{for } z \in D. \tag{1.2}$$

Let $K(\alpha, \beta)$ denote the subclass of S formed by all functions f for which there exists some function $g(z) \in C(\beta)$ such that

$$\mathrm{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > \alpha, \quad 0 \leqslant \alpha < 1, 0 \leqslant \beta < 1; \text{ for } z \in D. \tag{1.3}$$

Functions in the classes $C(\alpha)$, $S^*(\alpha)$ and $K(\alpha, \beta)$ are known as convex functions of order α, starlike functions of order α and close-to-convex functions of order α and type β, respectively. We shall denote by S_n, $n = 1, 2, 3, \ldots$, the class of regular and univalent functions in D having a Taylor expansion of the form $f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$ It is clear that $S = S_1 \supset S_2 \supset S_3 \supset \ldots \supset S_n$.

In this paper we shall show that if $F(z)$ is in $(S^*(\beta))_n$, $(C(\beta))_n$ or $(K(\alpha, \beta))_n$, then $f(z) = (1/(1+C))z^{1-C}[z^C F(z)]'$ is starlike of order β, convex of order β or close-to-convex of order α and type β for $|z| < r_n^0$, respectively.

Also if $F(z) \in S_n$ and $\mathrm{Re}\{F'(z)\} > \beta$ in D, then $f(z) = (1/(1+C))z^{1-C}[z^C F(z)]'$ satisfies $\mathrm{Re}\{f'(z)\} > \beta$ in $\{|z| < r_n'\}$. All our results are sharp. For a suitable choice of C, β and n the results of S. K. Bajpai [1], S. D. Barnadi [2], A. E. Livingston [5] and K. S. Padmanabhan [6] follows as special cases of the results derived here.

2. We first give the following lemma.

LEMMA 1. *The function* $H(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots$ *is regular and satisfies* $\mathrm{Re}\{H(z)\} > a$ $(0 \leqslant a < 1)$ *for* $z \in D$, *iff*

$$H(z) = \frac{1+(2a-1)z^n \varphi(z)}{1+z^n \varphi(z)}, \tag{2.1}$$

where $\varphi(z)$ *is a regular function and satisfies* $|\varphi(z)| \leqslant 1$ *for* $z \in D$.

The proof of the lemma is simple.

Remark. The transformation

$$p(z) = \frac{1+\beta w(z)}{1+w(z)}, \qquad -1 \leqslant \beta \leqslant 1$$

maps the circle $|z| < r < 1$ into the circle

$$\left| p(z) - \frac{1-\beta r^{2n}}{1-r^{2n}} \right| \leqslant \frac{(1-\beta) r^n}{1-r^{2n}}. \tag{2.2}$$

THEOREM 1. *Let* $F(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots$ *be in* $(S^*(\beta))_n$, $f(z) = (1/(1+C))z^{1-C}[z^C F(z)]'$, $C = 1, 2, 3, \ldots$; *then* $f(z)$ *is starlike of order* β *for* $|z| < r_n^0$, *where*

$$r_n^0 = \begin{cases} \left\{ \dfrac{(n+1-\beta) + \sqrt{(n+1-\beta)^2 + (C+1)(C+2\beta-1)}}{C+2\beta-1} \right\}^{1/n} & \text{if } C+2\beta-1 \neq 0, \\ \left(\dfrac{1}{n+1} \right)^{1/n} & \text{if } C+2\beta-1 = 0. \end{cases}$$

This result is sharp.

Proof. Since $F(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots$ is in $(S^*(\beta))_n$, $\mathrm{Re}\ \{zF'(z)/F(z) > \beta$ for $z \in D$. Hence by Lemma 1 there exists a function $w(z) = z^n \varphi(z)$ with $|w(z)| \leqslant |z|^n$ and regular for $z \in D$, such that

$$\frac{zF'(z)}{F(z)} = \frac{z^C f(z) - C\int_0^z t^{C-1} f(t)\,dt}{\int_0^z t^{C-1} f(t)\,dt} = \frac{1+(2\beta-1)w(z)}{1+w(z)} \tag{2.3}$$

for $z \in D$. Solving (2.3) for $f(z)$, we have

$$f(z) = \frac{(C+1)-(C+2\beta-1)w(z)}{(1+w(z))z^C}\Big[\int_0^z t^{C-1}f(t)\,dt\Big]. \tag{2.4}$$

Differentiating (2.4) logarithmically and then using (2.4), we obtain

$$\begin{aligned}\frac{zf'(z)}{f(z)} &= \frac{z(C+2\beta-1)w'(z)}{(C+1)+(C+2\beta-1)w(z)} - \frac{zw'(z)}{1+w(z)} + \frac{1+(2\beta-1)w(z)}{1+w(z)} \\ &= -\frac{2(1-\beta)zw'(z)}{(1+w(z))(C+1+(C+2\beta-1)w(z))} + \frac{1+(2\beta-1)w(z)}{1+w(z)}.\end{aligned} \tag{2.5}$$

Therefore

$$\frac{zf'(z)}{f(z)} - \beta = (1-\beta)\left[\frac{1-w(z)}{1+w(z)} - \frac{2zw'(z)}{(1+w(z))(C+1+(C+2\beta-1)w(z))}\right]. \tag{2.6}$$

But

$$\operatorname{Re}\left\{\frac{1-w(z)}{1+w(z)}\right\} = \frac{1-|w(z)|^2}{|1+w(z)|^2}, \tag{2.7}$$

and

$$\begin{aligned}&\operatorname{Re}\left\{\frac{2zw'(z)}{(1+w(z))(C+1+(C+2\beta-1)w(z))}\right\} \\ &\leqslant \frac{2|z|\,|w'(z)|}{|(1+w(z))|\,|(C+1+(C+2\beta-1)w(z))|} \\ &\leqslant \frac{2n|z|^n(1-|w(z)|^2)}{(1-|z|^{2n})|(1+w(z))|\,|(C+1+(C+2\beta-1)w(z))|}.\end{aligned} \tag{2.8}$$

The last inequality has been obtained by using the known result [3], p. 290,

$$|w'(z)| \leqslant \frac{n|z|^{n-1}(1-|w(z)|^2)}{1-|z|^{2n}}. \tag{2.9}$$

Thus from (2.6) we note that $f(z)$ is starlike of order β if

$$\frac{2n|z|^n(1-|w(z)|^2)}{(1-|z|^{2n})|(1+w(z))|\,|(C+1+(C+2\beta-1)w(z))|} \leqslant \frac{1-|w(z)|^2}{|1+w(z)|^2}$$

or

$$\frac{2n|z|^n}{1-|z|^{2n}} \leqslant \left|\frac{C+1+(C+2\beta-1)w(z)}{1+w(z)}\right| = (C+1)\left|\frac{1+\dfrac{(C+2\beta-1)}{C+1}w(z)}{1+w(z)}\right|. \tag{2.10}$$

Since $|w(z)| \leqslant |z|^n = r^n < 1$ and $(C+2\beta-1)/(C+1) \leqslant 1$, we have from

$$\left|\frac{1+\dfrac{(C+2\beta-1)}{C+1}w(z)}{1+w(z)}\right| \geqslant \frac{1+\dfrac{C+2\beta-1}{C+1}|z|^n}{1+|z|^n}. \tag{2.11}$$

Hence, by (2.10) and (2.11), we infer that $f(z)$ is in $(S^*(\beta))_n$ if

$$\frac{2n|z|^n}{1-|z|^{2n}} \leqslant \frac{C+1+(C+2\beta-1)|z|^n}{1+|z|^n}$$

or

$$\frac{2n|z|^n}{1-|z|^n} \leqslant (C+1)+(C+2\beta-1)|z|^n. \tag{2.12}$$

From (2.12) we have

$$(1+C)-2(n+1-\beta)|z|^n-(C+2\beta-1)|z|^{2n} \geqslant 0.$$

Let

$$P(r) \equiv (1+C)-2(n+1-\beta)r^n-(C+2\beta-1)r^{2n}. \tag{2.13}$$

Thus $f(z)$ will be starlike of order β for $|z| < r_n^0$, where r_n^0 is the least positive root of the polynomial (2.13) given by

$$r_n^0 = \left\{\frac{-(n+1-\beta)+\sqrt{(n+1-\beta)^2+(C+1)(C+2\beta-1)}}{C+2\beta-1}\right\}^{1/n}.$$

To see that result is sharp for each C and n, consider the function

$$F(z) = \frac{z}{(1-z^n)^{(2/n)(1-\beta)}} \in (S^*(\beta))_n, \quad 0 \leqslant \beta < 1.$$

For this function, we have

$$f(z) = \frac{1}{1+C}z^{1-C}\left[\frac{z^{C+1}}{(1-z^n)^{(2/n)(1-\beta)}}\right]'$$

$$= \frac{1}{C+1}\left[\frac{z(1+C)-(C+2\beta-1)z^{n+1}}{(1-z^n)^{(2/n)(1-\beta)+1}}\right].$$

By direct computation, we obtain

$$\frac{zf'(z)}{f(z)} - \beta = (1-\beta)\left[\frac{(C+1)+2(n+1-\beta)z^n-(C+2\beta-1)z^{2n}}{(1-z^n)\left(C+1-(C+2\beta-1)z^n\right)}\right].$$

Thus

$$\frac{zf'(z)}{f(z)} - \beta = 0 \quad \text{for } z^n = -(r_n^0)^n.$$

Hence $f(z)$ is not starlike in any circle $|z| < r$ if $r > r_n^0$.

THEOREM 2. *Let* $F(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \ldots$ *be in* $(C(\beta))_n$, $f(z) = (1/(1+C))z^{1-C}[z^C F(z)]'$, $C = 1, 2, 3, \ldots$; *then* $f(z)$ *is convex of order* β *for* $|z| < r_n^0$, *where*

$$r_n^0 = \begin{cases} \left\{ \dfrac{-(n+1-\beta)+\sqrt{(n+1-\beta)^2+(C+1)(C+2\beta-1)}}{C+2\beta-1} \right\}^{1/n}, & \\ & \text{if } C+2\beta-1 \neq 0, \\ \left(\dfrac{1}{n+1} \right)^{1/n} & \text{if } C+2\beta-1 = 0, \end{cases}$$

This result is sharp.

Proof. We have

$$(1+C)f'(z) = (1+C)F'(z) + zF''(z)$$

or

$$(1+C)\operatorname{Re}\left\{ \frac{f'(z)}{F'(z)} \right\} = C + \operatorname{Re}\left\{ \frac{1+zF''(z)}{F'(z)} \right\}.$$

Since $F(z) \in (C(\beta))_n$, $\operatorname{Re}\{1+zF''(z)/F'(z) > \beta$ for $z \in D$, it is easy to verify that

$$\operatorname{Re}\left\{ \frac{f'(z)}{F'(z)} \right\} < \frac{C+\beta}{C+1} \quad \text{for } z \in D.$$

Hence $f(z)$ are close-to-convex functions of order $(C+\beta)/(C+1)$ with respect to $F(z)$ in D.

To show that $f(z)$ is convex of order β for $|z| < r_n^0$, we have

$$zf'(z) = \frac{1}{1+C} z^{1-C}\left[z^C\left(zF'(z)\right)\right]'.$$

Since $F(z) \in (C(\beta))_n$, $zF'(z) \in (S^*(\beta))_n$; hence $z\, f'(z)$ is starlike of order β for $|z| < r_n^0$ by Theorem 1. Therefore $f(z)$ is convex of order β for $|z| < r_n^0$.

To show that the result is sharp for each C and n, consider the function

$$F(z) = \int_0^z \frac{d\sigma}{(1-\sigma^n)^{(2/n)(1-\beta)}} \in (C(\beta))_n$$

as

$$zF'(z) = \frac{z}{(1-z^n)^{(2/n)(1-\beta)}} \in (S^*(\beta))_n.$$

For this function

$$f(z) = \frac{1}{1+C} z^{1-C}\left[z^C \int_0^z \frac{d\sigma}{(1-\sigma^n)^{(2/n)(1-\beta)}}\right]'$$

$$= \frac{1}{1+C}\left[\frac{z}{(1-z^n)^{(2/n)(1-\beta)}} + C\int_0^z \frac{d\sigma}{(1-\sigma^n)^{(2/n)(1-\beta)}}\right].$$

By direct computation, we obtain

$$1+\frac{zf''(z)}{f'(z)} - \beta = (1-\beta)\left[\frac{(1+C)+2(n+1-\beta)z^n-(C+2\beta-1)z^{2n}}{(1-z^n)(C+1)-(C+2\beta-1)z^n}\right].$$

Thus

$$1+\frac{zf''(z)}{f'(z)} - \beta = 0 \quad \text{for } z^n = -(r_n^0)^n;$$

hence $f(z)$ is not convex of order β for any circle $|z| < r$, if $r > r_n^0$.

3. In order to prove our next two theorems we require the following lemma:

LEMMA 2. *If* $H(z) = 1+c_n z^n + c_{n+1}z^{n+1}+\dots$ *is regular and* $\mathrm{Re}\{H(z)\} > a$ *for* $z \in D$, *then*

$$|H'(z)| \leqslant \frac{2n|z|^{n-1}\mathrm{Re}\{H(z)-a\}}{1-|z|^{2n}}. \tag{3.1}$$

Proof. Setting $h(z) = (H(z)-a)/(1-a)$, if we substitute $w(z) = (h(z)-1)/(h(z)+1)$ in (2.9), we obtain (3.1).

THEOREM 3. *Let* $F(z) = z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\dots$ *be in* $(K(a, \beta))_n$, $f(z) = (1/(1+C))z^{1-C}[z^C F(z)]'$, $C = 1, 2, 3, \dots$; *then* $f(z)$ *is close-to-convex of order* a *and type* β *for* $|z| < r_n^0$, *where*

$$r_n^0 = \begin{cases} \left\{\dfrac{-(n+1-\beta)+\sqrt{(n+-\beta)^2+(C+1)(C+2\beta-1)}}{C+2\beta-1}\right\}^{1/n}, & \\ & \text{if } C+2\beta-1 \neq 0, \\ \left\{\dfrac{1}{n+1}\right\}^{1/n}, & \text{if } C+2\beta-1 = 0. \end{cases}$$

This result is sharp.

Proof. Since $F(z) \in (K(a, \beta))_n$, there exists a function $G(z) \in (S^*(\beta))_n$ such that for $z \in D$

$$\mathrm{Re}\left\{\frac{zF'(z)}{G(z)}\right\} > a \quad \text{and} \quad \mathrm{Re}\left\{\frac{zG'(z)}{G(z)}\right\} > \beta.$$

Further, since $G(z)\epsilon\left(S^*(\beta)\right)$, from Theorem 1 we have

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\}>\beta \quad \text{for } |z|<r_n^0,$$

where

$$g(z)=\frac{1}{1+C}z^{1-C}[z^C G(z)]'.$$

Therefore

$$\frac{zF'(z)}{G(z)}=\frac{z^C f(z)-C\int\limits_0^z t^{C-1}f(t)\,dt}{\int\limits_0^z t^{C-1}g(t)\,dt}. \tag{3.2}$$

Let

$$\frac{zF'(z)}{G(z)}=P(z), \quad \text{where } P(z) \text{ is regular,} \quad P(0)=1$$

and $\mathrm{Re}\{P(z)\}>a$ for $z\epsilon D$.

Thus from (2.13) we have

$$z^C f(z)=C\int\limits_0^z f(t)t^{C-1}dt+P(z)\int\limits_0^z t^{C-1}g(t)\,dt. \tag{3.3}$$

Differentiating (3.3) with respect to z and then dividing by $g(z)$ throughout, we obtain

$$\frac{zf'(z)}{g(z)}=P(z)+P'(z)\frac{z^{1-C}\int\limits_0^z t^{C-1}g(t)\,dt}{g(z)}.$$

Therefore

$$\mathrm{Re}\left\{\frac{zf'(z)}{g(z)}-a\right\}\geqslant \mathrm{Re}\{P(z)-a\}-|P'(z)|\left|\frac{z^{1-C}\int\limits_0^z t^{C-1}g(t)\,dt}{g(z)}\right|. \tag{3.4}$$

But

$$\frac{1}{z^C g(z)}\int\limits_0^z t^{C-1}g(t)\,dt=\frac{G(z)}{CG(z)+zG'(z)}. \tag{3.5}$$

Since $G(z)\epsilon\left(S^*(\beta)\right)_n$, we have

$$\frac{zG'(z)}{G(z)}=\frac{1+(2\beta-1)w(z)}{1+w(z)}. \tag{3.6}$$

Therefore we have

$$(3.7)\qquad \left\{C+\frac{zG'(z)}{G(z)}\right\}^{-1}=\left\{C+\frac{1+(2\beta-1)w(z)}{1+w(z)}\right\}^{-1}$$
$$=\left|\frac{C+1+(C+2\beta-1)w(z)}{1+w(z)}\right|^{-1}.$$

Thus, using (3.7) in (3.4), we obtain

$$(3.8)\qquad \mathrm{Re}\left\{\frac{zf'(z)}{g(z)}-a\right\}$$
$$\geqslant\left[\mathrm{Re}\{P(z)-a\}-|zP'(z)|\left|\frac{1+w(z)}{C+1+(C+2\beta-1)w(z)}\right|\right]$$
$$\geqslant\mathrm{Re}\{P(z)-a\}\left\{1-\frac{2n|z|^n(1+|z|^n)}{(1-|z|^{2n})\big(C+1+(C+2\beta-1)|z|^n\big)}\right\}$$
$$=\mathrm{Re}\{P(z)-a\}\left\{\frac{(C+1)-2(n+1-\beta)|z|^n-(C+2\beta-1)|z|^n}{(1-|z|^n)\big(C+1+(C+2\beta-1)|z|^n\big)}\right\}.$$

The last inequality has been obtained by using lemma 2 and (2.11). Therefore $f(z)\in\big(K(a,\beta)\big)_n$ if $|z|<r_n^0$.

To show that the result is sharp, consider $F(z)=G(z)=z/(1-z^n)^{(2/n)(1-\beta)}$ $\in\big(S^*(\beta)\big)_n$; therefore $F(z)$ belongs to $\big(K(a,\beta)\big)_n$, where $a=\beta$.

THEOREM 4. *Let* $F(z)=z+a_{n+1}z^{n+1}+a_{n+2}z^{n+2}+\dots$ *be regular and have the property* $\mathrm{Re}\{F'(z)\}>\beta$ *for* $z\in D$, $f(z)=\big(1/(1-C)\big)z^{1-C}[z^CF(z)]'$, $C=1,2,3,\dots$; *then* $\mathrm{Re}\{f'(z)\}>\beta$ *for* $|z|<r'_n$, *where*

$$r'_n=\left\{\frac{-n+\sqrt{n^2+(C+1)^2}}{C+1}\right\}^{1/n}.$$

This result is sharp.

Proof. Let $F'(z)=P(z)$, where $P(0)=1$, and $\mathrm{Re}\{P(z)\}>\beta$ for $z\in D$. Then we have

$$(1+C)f'(z)=zF''(z)+(1+C)F'(z)=zP'(z)+(1+C)P(z).$$

Therefore

$$(3.9)\qquad (1+C)\,\mathrm{Re}\big(f'(z)-\beta\big)\geqslant(1+C)\,\mathrm{Re}\{P(z)-\beta\}-|zP'(z)|$$
$$\geqslant\mathrm{Re}\{P(z)-\beta\}\left\{1+C-\frac{2n|z|^n}{1-|z|^{2n}}\right\}$$
$$=\mathrm{Re}\{P(z)-\beta\}\left[\frac{(1+C)-2n|z|^n-(C+1)|z|^{2n}}{1-|z^{2n}}\right].$$

The last inequality has been obtained by using Lemma 2.

Therefore $\operatorname{Re} f'(z) > \beta$ for $|z| < r'_n$, where r'_n is the least positive root of the polynomial

$$(1+C)-2nr^n-(1+C)r^{2n} = 0. \tag{3.10}$$

To show that the result is sharp, consider

$$F(z) = \int_0^z \frac{1-(2\beta-1)\sigma^n}{1-\sigma^n}\,d\sigma.$$

It is clear that $\operatorname{Re}\{F'(z)\} > \beta$.
Therefore

$$f(z) = \frac{1}{1+C} z^{1-C}\left[z^C \int_0^z \frac{1-(2\beta-1)}{1-\sigma^n}\,d\sigma\right]'$$

$$= \frac{1}{1+C}\left[\frac{z-1-(2\beta-1)z^n}{1-z^n} + C\int_0^z \frac{1-(2\beta-1)^n}{1-\sigma^n}\,d\sigma\right].$$

By direct computation, we have

$$f'(z)-\beta = \frac{1-\beta}{1+C}\left[\frac{(1+C)+2nz^n-(C+1)z^{2n}}{(1-z^n)^2}\right].$$

Thus

$$f'(z)-\beta = 0 \quad \text{for } z^n = -(r'_n)^n,$$

and hence $\operatorname{Re}\{f'(z)\} \ngtr \beta$ in any circle $|z| < r$, if $r > r'_n$.

$C+2\beta-1 = 0$ only when $C = 1$, $\beta = 0$; the results of Susheel Chandra and Prem Singh [7] follow as special cases of the results derived here.

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