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On the positivity of an invariant measure on open non-empty sets

by Antoni Leon Dawidowicz (Kraków)

Abstract. The paper is a continuation of the author's papers concerning the construction of the measures which are invariant with respect to a dynamical system on functional spaces.

Introduction. This paper is a continuation of other papers by the author on the construction of the measures invariant with respect to dynamical system on functional spaces. The start point is paper [1] where the author had constructed the measure invariant with respect to dynamical system generated by the partial equation. In the present paper, the author presents a new construction of the same measure. This construction is essentially different from the one presented in [1]. Basing on this new construction, the author can show that this measure is positive on the non-empty sets open in the uniform topology, which was not shown in [1]. This measure is different from the Rudnicki measure [3], because that one is concentrated on the space of C^1-functions and this one is concentrated on the space of functions having the points of non-differentiability.

1. Formulation of the theorem. Let V be the space of Lipschitz functions on the interval $[0;1]$ vanishing at zero point. Let us consider the differential equation

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = \lambda u, \tag{1}$$

when $\lambda > 1$, with the condition

$$u(0;x) = v(x), \qquad v \in V. \tag{2}$$

Let us define the semidynamical system on V by the formula

$$T_t v(x) = u(t;x). \tag{3}$$

This formula in the space V considered with the topology of uniform convergence generates the semidynamical system which can be described by the explicit formula

$$T_t v(x) = e^{\lambda t} v(xe^{-t}). \tag{4}$$

THEOREM. *On the space V, there exists the Borel measure μ satisfying the following conditions*:

(i) μ *is probabilistic*,
(ii) μ *is T_t-invariant*,
(iii) $\mu(U) > 0$ *for every open non-empty set* U,
(iv) μ *is ergodic*,
(v) $\mu(E_0) = 0$, *where* $E_0 = \{v \in V\colon \exists t > 0\colon T_t v = v\}$.

2. Construction of the measure μ. Let us consider the probabilistic space $(\Omega; \Sigma; P)$ constructed as follows. Let $\{p_i\}_{i=0}^{\infty}$ be a sequence of positive numbers such that

$$\sum_{i=0}^{\infty} p_i = 1 \tag{5}$$

and

$$\sum_{n=0}^{\infty} \Big(1 - \sum_{i=0}^{n} p_i\Big) < \infty. \tag{6}$$

Let us define the probability on the cylindrical subsets of cartesian product by the formula

$$P(k_i = \bar{k}_i\colon i = 1, \ldots, n) = \prod_{i=1}^{n} p_{\bar{k}_i}. \tag{7}$$

From the Kolmogorov theorem, P can be extended to the probabilistic measure on some σ-algebra Σ of the subsets of $\Omega = \boldsymbol{N}^{\boldsymbol{N}}$. From the estimation (6) and the Borel-Cantelli lemma follows the following.

LEMMA 1. $P(\exists n_0\colon \forall n \geqslant n_0 \; k_n \leqslant n) = 1$.

Now we shall construct the measure $\bar{\mu}$ satisfying the following conditions:

($\bar{\text{i}}$) $\bar{\mu}$ is probabilistic,
($\bar{\text{ii}}$) $\bar{\mu}$ is T-invariant, where $T = T_{\ln 2}$,
($\overline{\text{iii}}$) $\bar{\mu}(U) > 0$ for every open non-empty set,
($\overline{\text{iv}}$) $T^{-1}(E) = E \Rightarrow \bar{\mu}(E) \in \{0; 1\}$,
($\bar{\text{v}}$) $\bar{\mu}(E_0') = 0$, where $E_0' = \{v \in V\colon \exists s \geqslant 0,\ t > 0\colon T_{t+s} v = T_s v\}$.

Analogously to paper [1], having this measure, we shall define the measure needed by the formula

(8) $$\mu(E) = \frac{1}{\ln 2} \int_0^{\ln 2} \bar{\mu}(T_t^{-1}(E))\,dt.$$

In paper [1] the following lemma was proved.

LEMMA 2. *There exists a sequence of polynomials* $\{\sigma_n\}_{n=0}^{\infty}$ *dense in* V *such that the polynomials have a degree different from each other.*

Let us define the sequence of the right inverses to T by the formulae

$$S_n v(x) = \begin{cases} 2^{-\lambda} v(2x), & x \in [0; \tfrac{1}{2}], \\ 2^{-\lambda} v(1) + \sigma_n(2x-1), & x \in [\tfrac{1}{2}; 1]. \end{cases}$$

and consider for some sequence $\{k_i\}_{i=0}^{n}$ the set $S_{k_0} \dots S_{k_n}(V)$. All functions from this set are equal to each other up to additive constant on the interval $[2^{-n-1}; 1]$. Hence, for a fixed sequence $\{k_i\} \in \Omega$, the set $\bigcap_{n=0}^{\infty} S_{k_0} \dots S_{k_n}(V)$ has at most one element. From Lemma 1 and formula (22) in paper [1] it follows that $P(\text{card} \bigcap_{n=0}^{\infty} S_{k_0} \dots S_{k_n}(V) = 1) = 1$.

Let $\Omega_0 = \{\text{card} \bigcap_{n=0}^{\infty} S_{k_0} \dots S_{k_n}(V) = 1\}$ and let us define a map Φ: $\Omega_0 \to V$ such that for $k = \{k_i\}_{i=1}^{\infty}$, Φ_k is the unique function belonging to $\bigcap_{n=0}^{\infty} S_{k_0} \dots S_{k_n}(V)$. Let us now define the measure space $(V; \mathscr{B}; \bar{\mu})$:

$$\mathscr{B} = \{A \subset V;\ \Phi^{-1}(A) \in \Sigma\}; \qquad \bar{\mu}(A) = P(\Phi^{-1}(A)).$$

This is the same measure as that defined in paper [1]. Since we have the following result.

LEMMA 3. *The measure* $\bar{\mu}$ *satisfies conditions* $(\bar{\text{i}})$, $(\overline{\text{ii}})$, $(\overline{\text{iv}})$, $(\bar{\text{v}})$.

It is sufficient to show that the Borel sets (in the sense of uniform convergence) belong to $\mathscr{B}$ and $\bar{\mu}$ satisfies $(\overline{\text{iii}})$.

Let $U = U(v_0; \varepsilon) = \{v \in V:\ |v(x) - v_0(x)| < \varepsilon\ \ \forall x \in [0; 1]\}$.

LEMMA 4. *For every* $v_0 \in V$ *and* $\varepsilon > 0$, $U \in \mathscr{B}$.

Proof. Let us define three sequences of random variables:

(9) $$\xi_n = 2^{-\lambda n}[\sigma_{k_n}(2) - \sigma_{k_n}(1)] - [v_0(2^{-n}) - v_0(2^{-n-1})],$$

(10) $$\eta_n = \sup_{2^{-n-1} \leq x \leq 2^{-n}} \{2^{-\lambda n}[\sigma_{k_n}(2^{n+1}x - 1) - \sigma_{k_n}(1)] - [v_0(x) - v_0(2^{-n-1})]\},$$

$$(11)\qquad \zeta_n = \inf_{2^{-n-1} \leqslant x \leqslant 2^{-n}} \{2^{-\lambda n}[\sigma_{k_n}(2^{n+1}x-1)-\sigma_{k_n}(1)] - [v_0(x)-v_0(2^{-n-1})]\};$$

ξ_n, η_n, ζ_n as depending only on the one term of the sequence k are measurable. From this it follows that

$$\max_x |\Phi_k(x)-v_0(x)| = \max_{n=1}^{\infty} \max \{|\zeta_n + \sum_{m=1}^{n} \xi_m|; |\eta_n + \sum_{m=1}^{n} \xi_m|\}$$

is a measurable function of the variable $k \in \Omega_0$, and in consequence, U is measurable.

To complete the proof it is sufficient to claim that $\bar{\mu}(U) > 0$. This claim is based on the following lemma.

LEMMA 5. *Let* $G(n; \varepsilon) = \{v \in V: \forall x \in [0; 2^{-n}] |v(x)| < \varepsilon\}$. *For every* $\varepsilon > 0$ *and* $n \in N$, $\bar{\mu}(G(n; \varepsilon)) > 0$.

Proof. It is obvious that, for $\varepsilon < \varepsilon'$, $G(n; \varepsilon) \subset G(n; \varepsilon')$ and $\bigcup_{\varepsilon > 0} G(n; \varepsilon) = V$. Thus $\bar{\mu}(G(n; \varepsilon))$ is an increasing function of the argument ε with positive value for some $\varepsilon > 0$. Suppose that, for some $\varepsilon > 0$, $\bar{\mu}(G(n; \varepsilon)) = 0$. We can take an $\varepsilon > 0$ such that $\bar{\mu}(G(n; \varepsilon)) = 0$, $\bar{\mu}(G(n; 2^{\lambda}\varepsilon)) > 0$. From T-invariance of $\bar{\mu}$ it follows that

$$\bar{\mu}(G(n+1; \varepsilon)) = \bar{\mu}(T^{-1}(G(n; 2^{\lambda}\varepsilon))) = \bar{\mu}(G(n; 2^{\lambda}\varepsilon)) > 0.$$

Since $G(n+1; \varepsilon) = \bigcup_{\substack{\varepsilon' < \varepsilon \\ \varepsilon' \in Q}} G(n+1; \varepsilon')$, there exists an $\varepsilon' < \varepsilon$, such that $\bar{\mu}(G(n+1; \varepsilon')) > 0$.

From the density of $\{\sigma_n\}$ it follows that there exists an n' such that $\|\sigma_{n'}\| < \varepsilon - \varepsilon'$. Since that values of Φ_k on the interval $[0; 2^{-n-1}]$ depend only on k_i for $i \geqslant n+2$, the events $\{k_{n+1} = n'\}$ and $\{\Phi_k \in G(n+1; \varepsilon)\}$ are independent. Consequently,

$$0 = \bar{\mu}(G(n; \varepsilon)) = P(\Phi_k \in G(n; \varepsilon)) \geqslant P(\Phi_k \in G(n+1; \varepsilon'), k_{n+1} = n')$$
$$= \bar{\mu}(G(n+1; \varepsilon')) \cdot p_{n'}$$

which is impossible. Thus, in fact, $\bar{\mu}(G(n; \varepsilon)) > 0$.

3. Proof of the theorem. To finish the proof it is sufficient to prove that $U(v_0, \varepsilon)$ includes the subset whose measure $\bar{\mu}$ is positive. Let K_0 be the Lipschitz constant for v_0. Let us consider an n such that $K_0 2^{-n-1} < \frac{1}{4}\varepsilon$ and an $\varepsilon' > 0$ such that $\varepsilon' < \frac{1}{8}\varepsilon$. Let

$$A = \{v \in V: v \in G(n; \varepsilon'), \exists c \in \boldsymbol{R}: \forall x \geqslant 2^{-n-1} |v(x) - v_0(x) - c| < \varepsilon'\}.$$

Clearly,

$$A \supset G(n;\varepsilon') \cap \bigcap_{m=0}^{n-1} \{\|\Phi_k - v_0 - \Phi_k(2^{-m-1}) + v_0(2^{-m-1})\|_{[2^{-m-1};2^{-m}]} < \\ < \varepsilon'/(n+1)\}.$$

Since

$$\|\Phi_k - v_0 - \Phi_k(2^{-m-1}) + v_0(2^{-m-1})\|_{[2^{-m-1};2^{-m}]}$$

depends only on the mth term of sequence $\{k_n\}$, from the density of $\{\sigma_n\}$ and independence of the functions of different coordinates in Ω_0 follows that $\bar{\mu}(A) > 0$.

Now, it is sufficient to prove that $A \subset U(v_0;\varepsilon)$.

Let $v \in A$. For $x \leqslant 2^{-n-1}$

$$|v(x) - v_0(x)| \leqslant |v(x)| + |v_0(x)| \leqslant \varepsilon' + K_0 x \leqslant \varepsilon' + K_0 2^{-n-1} < \varepsilon.$$

For $x \geqslant 2^{-n-1}$ there exists a $c \in \boldsymbol{R}$ such that $|v(x) - v_0(x) - c| < \varepsilon'$. This c is common for all $x \geqslant 2^{-n-1}$. First, we shall estimate this c. It is obvious that $|v(2^{-n-1}) - v_0(2^{-n-1})| \leqslant \varepsilon' + \frac{1}{4}\varepsilon < \frac{3}{8}\varepsilon$, and from this it follows that $|c| < \frac{1}{2}\varepsilon$. Thus

$$|v(x) - v_0(x)| \leqslant |v(x) - v_0(x) - c| + |c| < \varepsilon' + \tfrac{1}{2}\varepsilon < \varepsilon$$

which completes the proof of the theorem.

4. Remarks. This result may be generalized to the case of the quasi-linear equation of the form

$$\frac{\partial u}{\partial t} + x\frac{\partial u}{\partial x} = F(u), \tag{12}$$

where F satisfies the same assumptions as in paper [2].

The most interesting is the case where F is a quadratic function. It is also interesting that the measure constructed in the present paper is concentrated on the functions whose derivatives vanish at zero. The Rudnicki measure [3] which is different from this one has the same property.

Remark. If $\{T_t\}_{t>0}$ is the semidynamical system defined on V by formula (1) and μ is the probabilistic measure on V invariant with respect to it, then μ is concentrated on the set of functions whose derivatives vanish at zero.

Proof. Let

$$\bar{P}v = \overline{\lim_{x\to 0}} \frac{v(x)}{x} \quad \text{for } v \in V.$$

Clearly, $\bar{P}$ is well defined. Moreover,

$$\bar{P}T_t v = \overline{\lim_{x\to 0}} \frac{e^{\lambda t} v(xe^{-t})}{x} = e^{(\lambda-1)t} \overline{\lim_{x\to 0}} \frac{v(xe^{-t})}{xe^{-t}} = e^{(\lambda-1)t} \bar{P}v.$$

Hence the diagram

$$\begin{array}{ccc} V & \xrightarrow{T_t} & V \\ \downarrow{\scriptstyle \bar{P}} & & \downarrow{\scriptstyle \bar{P}} \\ \boldsymbol{R} & \xrightarrow{e^{(\lambda-1)t}} & \boldsymbol{R} \end{array} \tag{13}$$

is commutative.

Moreover,

$$\bar{P}^{-1}\big((-\infty; \varepsilon]\big) = \Big\{v \in V\colon \overline{\lim_{x\to 0}} \frac{v(x)}{x} \leqslant \varepsilon\Big\}$$

$$= \bigcap_{\substack{\varepsilon' > \varepsilon \\ \varepsilon' \in Q}} \bigcup_{\substack{\delta > 0 \\ \delta \in Q}} \{v \in V\colon v(x) \leqslant \varepsilon' x \ \ \forall x \in [0; \delta]\} \in \mathscr{B}.$$

Thus the measure defined by the formula $\bar{\mu}(A) = \mu(\bar{P}^{-1}(A))$ is the probabilistic Borel measure on $\boldsymbol{R}$ invariant with respect to multiplicity operator for every positive multiplicator. The unique measure on $\boldsymbol{R}$ satisfying these properties is the Dirac measure δ_0. From this follows that $\mu(\{v\colon \bar{P}v = 0\}) = 1$. Analogously, it can be proved that

$$\mu\left(\left\{v\colon \underline{\lim}_{x\to 0} \frac{v(x)}{x} = 0\right\}\right) = 1$$

which completes the proof.

References

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INSTITUTE OF MATHEMATICS OF JAGIELLONIAN UNIVERSITY
KRAKÓW, POLAND

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