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# Complete differentials of higher order in linear field modules

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**Abstract.** Complete differentials of higher order in linear field modules are defined. A certain necessary condition for the existence of a complete differential of higher order is given. It is proved that this condition holds in a broad class of linear field modules, which contains differential modules and modules of vector fields on differential spaces of class $\mathcal{D}_0$. Jet-field modules of a linear field module are constructed. The exactness of the sequence of jet-modules is examined. A one-to-one correspondence between complete differentials of higher order and splittings of jet-module sequences is established. An example of a differential space of class $\mathcal{D}_0$ is given in which

1° the module of vector fields over that space in every neighbourhood of a certain point does not possess any vector basis, i.e. it is not differential,

2° a covariant derivative, i.e. a complete differential of the first order, exists in the module of vector fields.

**Introduction.** We consider a manifold $M$ and a vector bundle $\xi$ over $M$ and we denote (as usual) by $J^k(\xi)$ the vector bundle of holonomic $k$-order jets of local sections of $\xi$. The exact sequence of vector bundles

$$0 \to L_s^k(TM, \xi) \to J^k(\xi) \to J^{k-1}(\xi) \to 0$$

called the *sequence of jet-bundles* is well known from the works by R. Palais ([6]), N. V. Que ([9]), D. Spencer ([13], [14]) and others. A differential operator of order $k$, corresponding to a splitting of the sequence, is termed by Palais a complete differential of order $k$ in a bundle $\xi$. In the case $k = 1$ it is simply a covariant derivative.

In the present work we consider an arbitrary differential space $(M, \mathscr{C})$ ([5], [10]) instead of a manifold $M$ and an arbitrary linear field module $\mathscr{W}$ instead of the module of sections of a vector bundle $\xi$. Differential spaces have been examined in the works by R. Sikorski ([11], [12]), W. Waliszewski ([18], [19]) as well as in the works by P. Walczak ([15]–[17]), K. Cegiełka ([2], [3]), M. Pustelnik ([8]) and others. Linear field modules defined on differential spaces were introduced by R. Sikorski ([11]).

In the present work we shall construct a linear field module $J^k(\mathscr{W})$ and an exact sequence of jet-modules analogous to the sequence of jet-bundles and we shall prove the equivalence between the definition of a complete differential as a certain differential operator of order $k$ and as a splitting of the jet-module sequence. The construction of the module $J^k(\mathscr{W})$ will be possible under certain assumptions about the module $\mathscr{W}$; the exactness of the jet-module sequence will occur in certain conditions. These assumptions and conditions will be examined more precisely for a class of pseudo-differential modules, which contains differential modules ([11], [12]) and modules of vector fields on a differential space of the class $\mathscr{D}_0$ ([16], [17]). K. Cegiełka in [2] showed that if a linear field module $\mathscr{W}$ on a differential space $(M, \mathscr{C})$ is differential and if it is possible to subordinate a smooth partition of unity to every open covering of the space $(M, \tau_{\mathscr{C}})$, then there exists in $\mathscr{W}$ a scalar product, and so a covariant derivative also exists. It turns out that the existence of a scalar product does not imply the existence of a local basis in the module under consideration. An adequate example will be given at the end of section 3.

**1. Preliminaries.** Differential spaces discussed in this paper as well as the notions of a tangent vector, tangent space, smooth mapping, tangent mapping, smooth vector field and the denotations $\tau_{\mathscr{C}}$ and $\mathscr{C}_A$ have been adopted from the works by R. Sikorski [10], [12]. A $\mathscr{C}$-module of smooth vector fields on a differential space $(M, \mathscr{C})$ will be denoted by $\mathscr{X}(M, \mathscr{C})$ and the vector subspace of the tangent space $(M, \mathscr{C})_p$, $p \in M$, consisting of these vectors which are values of a smooth vector field will be denoted by $(M, \mathscr{C})'_p$.

In a differential space $(M, \mathscr{C})$ whose topology is paracompact and locally compact, for any open covering there exists a smooth partition of unity subordinated to this covering; this fact has been proved by K. Cegiełka ([2]), M. Pustelnik in [8] proved that the assumption of local compactness may be replaced by $\mathscr{C}$-normality. It is easy to show that the assumptions of $\mathscr{C}$-normality is weaker than that of local compactness (assuming paracompactness) and equivalent to the existence of a smooth partition of unity subordinate to an arbitrary open covering.

**1.1. Differential spaces of class** $\mathscr{D}_0$. The existence and specification of the widest class of differential spaces in which the theorem on a diffeomorphism holds was a problem raised by Waliszewski and solved by Walczak in his paper [16]. Paper [17] was devoted to the investigation of that class.

THEOREM 1.1.1. *If $(M, \mathscr{C})$ is a differential space of class $\mathscr{D}_0$, then the set $M'$ of all points $p \in M$ for which*

$$(M, \mathscr{C})_p = (M, \mathscr{C})'_p$$

*is open and dense in topology $\tau_{\mathscr{C}}$.*

Proof. The openness of $M'$ is evident from the definition of this set. For a non-negative integer $n$, let $M_n$ be the set of all points $p \in M$ for which $\dim(M, \mathscr{C})_p = n$. It is easy to see that

$$M' = \bigcup_n \operatorname{Int} M_n.$$

To prove that $\overline{M'} = M$ we shall show that every point $p \in M$ has a neighbourhood $U \in \tau_{\mathscr{C}}$ such that

$$U \subset \overline{M' \cap U}. \tag{1.1.1}$$

We take a set $U$ covering $p$ such that $\dim(M, \mathscr{C})_q \leqslant \dim(M, \mathscr{C})_p$ for $q \in U$. Obviously, if $n = \dim(M, \mathscr{C})_p$, then

$$M' \cap U = \bigcup_{k=0}^{n} ((\operatorname{Int} M_k) \cap U) = \bigcup_{k=0}^{n} \operatorname{Int}(M_k \cap U).$$

Let $A_k = (M_k \cap U) \setminus \operatorname{Int}(M_k \cap U)$, $k = 0, 1, \ldots, n$. Since

$$U \setminus (M' \cap U) = \bigcup_{k=0}^{n} A_k,$$

to show inclusion (1.1.1) it suffices to prove the equality

$$\operatorname{Int}\Big(\bigcup_{k=0}^{r} A_k\Big) = \emptyset, \qquad r = 0, 1, \ldots, n. \tag{1.1.2}$$

We apply induction on $r$. Since $M_0 \cap U$ is open, equality (1.1.2) is satisfied for $r = 0$. Assume that (1.1.2) is satisfied for an integer $r < n$. From the openness of the set $U \cap (M_0 \cup \ldots \cup M_r)$ and the equality $\overline{A_{r+1}} \cap \cap (M_0 \cup \ldots \cup M_r) \cap U = \emptyset$ results

$$\operatorname{Int}\Big(\bigcup_{k=0}^{r+1} A_k\Big) = \Big(\operatorname{Int} \bigcup_{k=0}^{r} A_k\Big) \cup \operatorname{Int} A_{r+1} = \emptyset. \qquad \text{q.e.d.}$$

The above theorem states that, in general, there are "many" vector fields in a differential space of class $\mathscr{D}_0$.

**1.2. Examples of differential spaces.**

**1.2.1.** Let $M$ and $N$ be manifolds of class $C^\infty$ and let $f$: $M \to N$; denote by $\mathscr{T}(M)$ and $\mathscr{T}(N)$ the rings of smooth functions on $M$ and $N$; then the differential spaces $(f[M], \mathscr{T}(N)_{f[M]})$ and $(f^{-1}[\{a\}], \mathscr{T}(M)_{f^{-1}[\{a\}]})$, where $a \in N$ are not in general submanifolds.

**1.2.2.** The differential space $(M \times N, \mathscr{T}(M) \times \mathscr{T}(N))$ ([13]) is not a manifold if $M$ and $N$ are manifolds with a boundary.

**1.2.3.** Let $N$, $N'$ be submanifolds of $M$. The differential spaces $(N \cap N', \mathscr{T}(M)_{N \cap N'})$ and $(N \cup N', \mathscr{T}(M)_{N \cup N'})$ need not be submanifolds.

**1.2.4.** On a manifold $M$, an arbitrary collection of vector fields

$X_1, \ldots, X_k$ defines several subspaces of the space $(M, \mathcal{T}(M))$ of the form $(A, \mathcal{T}(M)_A)$, where, for example,

(a) $A = \{p \in M;\ X_1(p) = \ldots = X_k(p) = 0\}$,

(b) $A = \{p \in M;$ the vectors $X_1(p), \ldots, X_k(p)$ are lineary independent$\}$.

**1.2.5.** Let $K$ be a solid in $\boldsymbol{R}^n$. A differential space $(K, C^\infty(\boldsymbol{R}^n)_K)$ and the $k$-dimensional skeletons of this solid with the differential structure induced from $\boldsymbol{R}^n$ need not be manifolds. However, the solid may be a union (in the sense of example 3) of manifolds with a boundary.

**1.2.6.** Let $(M, g)$ be a Riemannian manifold. Let us fix point $p \in M$ and denote by $C(p)$ the set of vectors $v \in M_p$ for which the differential $(d\ \mathrm{Exp}_p)_v$ is not an isomorphism. The corresponding differential subspaces $C(p)$ and $\mathrm{Exp}_p[C(p)]$ of the spaces $M_p$ and $M$ need not be submanifolds.

**1.2.7.** We define a structure $\mathscr{C}$ on the set $R$ of real numbers by the formula

$$\mathscr{C} = (S_C C_0)_R, \quad \text{where } C_0 = \{R \ni t \mapsto |t-s| \in R;\ s \in R\}.$$

Then $\dim(R, C)_t = 2$ and $\dim(R, C)'_t = 0$ for any point $t \in R$.

The spaces in examples 1–6 are obviously of class $\mathscr{D}_0$, while in the last example the space $(R, \mathscr{C})$ is not of class $\mathscr{D}_0$, according to Theorem 1.1.1.

**1.3. Linear field modules.**

DEFINITION 1.3.1. A *linear field module* is a triple $\mathscr{W} = ((M, \mathscr{C}), \Phi, \mathscr{W})$, where $(M, \mathscr{C})$ is a differential space, $\Phi$ is a function assigning vector spaces $\Phi(p)$ to points $p \in M$ and $\mathscr{W}$ is a certain $\mathscr{C}$-module of linear $\Phi$-fields satisfying the condition:

If $W$ is a linear $\Phi$-field such that for any point $p \in M$ there exist a neighbourhood $U \in \tau_{\mathscr{C}}$ of this point and a field $V \in \mathscr{W}$ such that $W|U = V|U$, then $W \in \mathscr{W}$.

A module $\mathscr{W}$ satisfying the last condition is said to be closed with respect to localization.

We shall denote by $\Phi_{\mathscr{W}}(p)$ the vector space consisting of vectors $v \in \Phi(p)$ which are the values of fields from the module $\mathscr{W}$.

Suppose that with every point $p \in M$ there is associated a linear mapping $L(p)\colon \Phi_{\mathscr{W}}(p) \to \Psi_{\mathscr{V}}(p)$ satisfying the condition

$$L(W) = (M \ni p \mapsto L(p)(W(p))) \in \mathscr{V} \quad \text{for } W \in \mathscr{W};$$

then $L$ is called a *homomorphism of the linear field module* $((M, \mathscr{C}), \Phi, \mathscr{W})$ *into the linear field module* $((M, \mathscr{C}), \Psi, \mathscr{V})$. Then $L\colon \mathscr{W} \to \mathscr{V}$ is a homomorphism of $\mathscr{C}$-modules.

A homomorphism of $\mathscr{C}$-modules $L\colon \mathscr{W} \to \mathscr{V}$ induces a homomorphism of linear field modules if and only if it satisfies the following condition:

if $W \in \mathscr{W}$ and $W(p) = 0$, then $L(W)(p) = 0$;

if $\mathscr{V}$ and $\mathscr{W}$ are modules of $\Phi$ and $\theta$-linear fields on a differential space $(M, \mathscr{C})$, then we denote by $L_s^k(\mathscr{V}, \mathscr{W})$ the module of all linear $\Psi$-fields $L$, where $\Psi(p) = L_s^k(\Phi_{\mathscr{V}}(p); \theta_{\mathscr{W}}(p))$, $p \in M$, such that $L(V_1, \ldots, V_k) \in \mathscr{W}$ for $V_1, \ldots, V_k \in \mathscr{V}$. The module $L(\mathscr{V}, \mathscr{C})$ will be denoted by $\mathscr{V}^*$.

An example of a differential space $(M, \mathscr{C})$, a linear field module $\mathscr{W}$ and a $\mathscr{C}$-linear mapping from $\mathscr{X}(M, \mathscr{C})$ into $\mathscr{W}$ which is not a linear $\Psi$-field will be given at the end of section 3. However, if every vector field $V \in \mathscr{X}(M, \mathscr{C})$ equal 0 at $p$ is of the form $V = \sum_{i=1}^{n} f^i W_i$ for some functions $f^i \in \mathscr{C}$ such that $f^i(p) = 0$ and fields $W_i \in \mathscr{X}(M, \mathscr{C})$, $i = 1, \ldots, n$, then every $\mathscr{C}$-multilinear mapping from the module $\mathscr{X}(M, \mathscr{C})$ into $\mathscr{W}$ is a linear $\Psi$-field.

**1.4. Pseudo-differential modules.**

Definition 1.4.1. A linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is called a *pseudo-differential module* if for any point $q \in M$ there exist a neighbourhood $U \in \tau_{\mathscr{C}}$ of this point and a differential module $((U, \mathscr{C}_U), \Psi, \mathscr{V})$ such that $\Phi(p) \subset \Psi(p)$ for $p \in U$ and

(1.4.1) if $V \in \mathscr{V}$ and $V(p) \in \Phi_{\mathscr{W}}(p)$ for any point $p \in U$, then $V \in \mathscr{W}_U$.

Differential modules and modules of smooth vector fields on a differential space of class $\mathscr{D}_0$ are examples of pseudo-differential modules. Basic properties of pseudo-differential modules are given underneath:

Theorem 1.4.1. *If $((M, \mathscr{C}), \Phi, \mathscr{W})$ is a pseudo-differential module, then:*

(1) *$\Psi_{\mathscr{W}^*}(p) = (\Phi_{\mathscr{W}}(p))^*$, where $\Psi(p) = (\Phi_{\mathscr{W}}(p))^*$, $p \in M$; i.e. for any linear mapping $\tau\colon \Phi_{\mathscr{W}}(p) \to R$ there exists a field $h \in \mathscr{W}^*$ such that $h(p) = \tau$;*

(2) *if $W$ is a $\Phi_{\mathscr{W}}$-linear field such that for any field $h \in \mathscr{W}^*$ the function $h \circ W$ belongs to the ring $\mathscr{C}$, then $W \in \mathscr{W}$;*

(3) *this module is reflexive, i.e. the mapping $H_{\mathscr{W}}\colon \mathscr{W} \to \mathscr{W}^{**}$ defined by the formula $H_{\mathscr{W}}(W) = (\mathscr{W}^* \ni h \mapsto h \circ W \in \mathscr{C})$, $W \in \mathscr{W}$, is a linear field module isomorphism.*

Remark. Actually, it will be proved that the relations $1 \Rightarrow (2 \Leftrightarrow 3)$ hold for any linear field modules.

(a) $(1 \wedge 2) \Rightarrow 3$. It suffices to prove that $\ker H_{\mathscr{W}} = 0$ and $\operatorname{im} H_{\mathscr{W}} = \mathscr{W}^{**}$. If $H_{\mathscr{W}}(W) = 0$ for a certain field $W \in \mathscr{W}$, then $h(p)(W(p)) = 0$ for every field $h \in \mathscr{W}^*$. From condition (1) follows the equality $W(p) = 0$. Now consider a field $L \in \mathscr{W}^{**}$. From assumption (1) it follows that for any point $p \in M$ there is exactly one element $v \in \Phi_{\mathscr{W}}(p)$ such that $L(p)(\tau) = \tau(v)$ for $\tau \in \Psi_{\mathscr{W}^*}(p)$. This defines a certain linear $\Phi_{\mathscr{W}}$-field $W$ for which $h \circ W = (M \ni p \mapsto h(p)(W(p)))$ $= (M \ni p \mapsto L(p)(h(p))) = L(h) \in \mathscr{C}$ for every field $h \in \mathscr{W}^*$. From the assumption (2) it follows that $W \in \mathscr{W}$.

(b) $(1 \wedge 3) \Rightarrow 2$. If $W$ is an arbitrary linear $\Phi_{\mathscr{W}}$-field such that $h \circ W \in \mathscr{C}$ for any field $h \in \mathscr{W}^*$, then $(\mathscr{W}^* \ni h \mapsto h \circ W \in \mathscr{C}) \in \mathscr{W}^{**}$. Hence there exists exactly one field $W' \in \mathscr{W}$ such that $h \circ W = h \circ W'$ for any field $h \in \mathscr{W}^*$. In view of condition (1) we have the equality $W = W'$.

Proof of the theorem. Obviously, it suffices to check that a pseudo-differential module fulfils conditions (1) and (2). Let us take a point $q \in M$, a neighbourhood $U \in \tau_{\mathscr{C}}$ of $q$ and a differential module $((U, \mathscr{C}_U), \theta, \mathscr{V})$ such that $\Phi(p) \subset \theta(p)$ for $p \in U$ and condition (1.4.2) is fulfilled.

(1) Let $\tau$: $\Phi_{\mathscr{W}}(q) \to R$ be an arbitrary linear mapping and let $\varrho$: $\theta(q) \to R$ be a certain linear extension of it. Let us take an arbitrary field $F \in \mathscr{V}^*$ such that $F(q) = \varrho$. Obviously, the field $F' = F|\Phi_{\mathscr{W}}$ defined by the formula $(F|\Phi_{\mathscr{W}})(p) = F(p)|\Phi_{\mathscr{W}}(p)$, $p \in U$, is an element of the module $(\mathscr{W}_U)^*$ and has the property: $F'(q) = \tau$. Taking into account the $\mathscr{C}$-regularity of the space $(M, \tau_{\mathscr{C}})$ ([14]), we see that condition (1) is fulfilled.

(2) Let $W$ be an arbitrary linear $\Phi_{\mathscr{W}}$-field such that $h \circ W \in \mathscr{C}$ for any field $h \in \mathscr{W}^*$. In particular, $F \circ (W|U) = F|\Phi_{\mathscr{W}} \circ (W|U) \in \mathscr{C}$ for any field $F \in \mathscr{V}^*$. Therefore $W|U \in \mathscr{V}$, and further, from assumption (1.4.1), it follows that $W|U \in \mathscr{W}_U$. Hence $W \in \mathscr{W}$. q.e.d.

**1.5. Examples of linear field modules.**

**1.5.1.** Let $\xi$ and $\eta$ be vector bundles over manifolds $M$ and $N$, respectively, and let $\alpha$: $\xi \to \eta$ be a morphism of vector bundles, i.e. a smooth mapping such that $\alpha_p = \alpha|\xi_p$: $\xi_p \to \eta_{f(p)}$, $p \in M$ is a linear mapping, where $f$: $M \to N$. Let $\mathscr{W}$ be a submodule of the module $C^\infty(\xi)$ consisting of sections $\sigma$ for which $\sigma(p) \in \ker \alpha_p$, $p \in M$, and $\mathscr{V}$ a submodule of $C^\infty(f^*\eta)$ consisting of fields $\sigma$ for which $\sigma(p) \in \alpha_p[\xi_p]$, $p \in M$. The linear field modules

$$(M, (M \ni p \mapsto \ker \alpha_p), \mathscr{W}), \quad (M, (M \ni p \mapsto \operatorname{im} \alpha_p), \mathscr{V})$$

are not, in general, differential modules (i.e. $\bigcup_{p \in M} \ker \alpha_p$ and $\bigcup_{p \in M} \operatorname{im} \alpha_p$ generally are not subbundles of $\xi$ and $f^*\eta$, respectively).

**1.5.2.** Let $\xi$ and $\eta$ be vector bundles over a manifold $M$ and $\Phi$ a differential operator of order $k$ from the bundle $\xi$ into $\eta$. Following Spencer ([13], [14]), we denote by $\varphi$ the corresponding morphism of the vector bundle $J^k(\xi)$ into $\eta$, by $P_l(\varphi)$ its $l$-th extension $P_l(\varphi)$: $J^{k+l}(\xi) \to J^l(\eta)$ and by $\sigma_l(\varphi)$ the unique linear morphism $\sigma_l(\varphi)$: $S^{k+l}T^* \otimes \xi \to S^l T^* \otimes \xi$, $l \geqslant 0$, such that the following diagram is commutative:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
S^{k+l}T^* \otimes \xi & \xrightarrow{\sigma_l(\varphi)} & S^l T^* \otimes \xi \\
\downarrow & & \downarrow \\
J^{k+l}(\xi) & \xrightarrow{P_l(\varphi)} & J^l(\xi) \\
\downarrow & & \downarrow \\
J^{k+l-1}(\xi) & \xrightarrow{P_{l-1}(\varphi)} & J^{l-1}(\xi) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Let $g_{k+l} = \ker \sigma_l(\varphi) \subset S^{k+l} T^* \otimes \xi$ and $R_{k+l} = \ker P_l(\varphi) \subset J^{k+l}(\xi)$. Adequate linear field modules (constructed accordingly to the scheme from the former example) with values in $g_{k+l}$ and $R_{k+l}$, respectively, are not differential, in general.

**1.5.3.** Let $M$ be a manifold. An arbitrary collection of smooth vector fields $X_1, \ldots, X_k$ defines a linear field module in which $\Phi(p)$, $p \in M$, is the vector space spanned by the vectors $X_1(p), \ldots, X_k(p)$.

**1.5.4.** Let us consider a curve $f$: $(a, b) \to \mathbf{R}^n$ of class $C^\infty$ and at every point $p \in (a, b)$ the osculating space of order $k$ to $f$ in the sense of E. Cartan ([1]), i.e. the plane in $\mathbf{R}^n$ spanned by the points: $f(p)$, $f(p)+f'(p)$, $f(p)+f''(p), \ldots, f(p)+f^{(k)}(p)$. Let us produce a linear field module in which $\Phi(p)$, $p \in (a, b)$, will be the osculating space of order $k$ to the curve $f$ and all mappings $V$: $(a, b) \to \mathbf{R}^n$ of class $C^\infty$, such that $V(p) \in \Phi(p)$, $p \in (a, b)$, will form a linear $\Phi$-field module. The generated linear field module need not be differential. A generalization of the above definition of the osculating space to a curve in the case of a realization $f$ of a manifold $M$ in the space $\mathbf{R}^n, f$: $M \to \mathbf{R}^n$, was given by W. Pohl ([7]). Proceeding as above, we can again define a linear field module which, in general is not differential.

## 2. Ideals $I_p^{(k)}(M, \mathscr{C})$.

DEFINITION 2.1. For an arbitrary differential space $(M, \mathscr{C})$ and a point $p \in M$ we define by induction the sets $I_p^{(k)}(M, \mathscr{C})$, $k \in N$, in the following way:

(a) $I_p^{(1)}(M, \mathscr{C}) = I_p(M, \mathscr{C})$ equals the set of functions $f \in \mathscr{C}$ for which $f(p) = 0$;

(b) $f \in I_p^{(k+1)}(M, \mathscr{C})$ if and only if $f \in I_p^{(k)}(M, \mathscr{C})$ and for any collection of vector fields $X_1, \ldots, X_k \in \mathscr{X}(M, \mathscr{C})$ the equality

$$[(X_1, \ldots, X_k) f](p) = 0$$

holds.

Note that:

(2.1) The sets $I_p^{(k)}(M, \mathscr{C})$, $k \in N$, are ideals in the ring $\mathscr{C}$,

(2.2) If $f \in I_p^{(k+1)}(M, \mathscr{C})$, $1 \leqslant r \leqslant k$, $X_1, \ldots, X_r \in \mathscr{X}(M, \mathscr{C})$, then $(X_1, \ldots, X_r) f \in I_p^{(k+1-r)}(M, \mathscr{C})$,

(2.3) $[((I_p^{(1)}(M, \mathscr{C}))^k]_M \subset I_p^{(k)}(M, \mathscr{C})$.

As a rule, inclusion (2.3) cannot be replaced by an equality.

EXAMPLE 2.1. Let $A \subset R^2$ be the set of points $(x, y)$ for which $x = 0$ or $y = 0$ and let $D = C^\infty(R^2)_A$. Obviously, the dimension of the space $(A, D)_{(0,0)}$ is equal 2; moreover, since every smooth vector field on $(A, D)$ is equal 0 at the point $(0, 0)$, the dimension of the space $(A, D)'_{(0,0)}$ is equal 0. Consequently $I_p^{(k)}(A, D) = I_p^{(1)}(A, D)$ for $k \geqslant 1$. There exists a function

$\alpha \in I_p^{(1)}(A, D)$ such that $(d\alpha)_{(0,0)} \neq 0$; so $\alpha \notin \left((I_p^{(1)}(A, D))^2\right)_A$ and also $\alpha \notin \left((I_p^{(1)}(A, D))^k\right)_A$.

THEOREM 2.1. *For any differential space* $(M, \mathscr{C})$, *any point* $p \in M$ *and any positive integer* $k$ *there exists exactly one linear mapping*

$$d_p^{(k)}\colon I_p^{(k)}(M, \mathscr{C}) \to L_s^k\left((M, \mathscr{C})_p', R\right)$$

*such that for vector fields* $X_1, \ldots, X_k \in \mathscr{X}(M, \mathscr{C})$ *and functions* $f \in I_p^{(k)}(M, \mathscr{C})$ *the equality*

$$(d_p^{(k)} f)\left(X_1(p), \ldots, X_k(p)\right) = [(X_1, \ldots, X_k) f](p)$$

*holds. Moreover, the sequence*

$$(2.4) \qquad 0 \to I_p^{(k+1)}(M, \mathscr{C}) \hookrightarrow I_p^{(k)}(M, \mathscr{C}) \xrightarrow{d_p^{(k)}} L_s^k\left((M, \mathscr{C})_p', R\right) \to 0$$

*is exact if* $\dim(M, \mathscr{C})_p' < \infty$.

Proof. The existence of the mapping $d_p^{(k)}$, its uniqueness and linearity may be checked just as in the case when $(M, \mathscr{C})$ is a manifold ([6]). To prove the exactness of the sequence (2.4) it suffices to show the surjectivity of the mapping $d_p^{(k)}$ in the case when $\dim(M, \mathscr{C})_p' > 0$. Let $\alpha\colon \otimes^k\left((M, \mathscr{C})_p'\right)^* \to L^k\left((M, \mathscr{C})_p', R\right)$ be the natural linear isomorphism. Let us fix a basis $v_1, \ldots, v_n$ of the space $(M, \mathscr{C})_p'$ and take arbitrary vector fields $X_1, \ldots, X_k \in \mathscr{X}(M, \mathscr{C})$ such that $X_i(p) = v_i$, $i = 1, \ldots, n$. There exist functions $\beta_1, \ldots, \beta_n \in \mathscr{C}$ such that $\beta_j(p) = 0$ and $X_i(\beta_j) = \delta_{ij}$, $i, j \leqslant n$ ([12]). An arbitrary element $\tau$ of the space $\otimes^k\left((M, \mathscr{C})_p'\right)^*$ is of the form

$$\tau = \sum_{i_1, \ldots, i_k = 1} a_{i_1, \ldots, i_k} d_p^{(1)} \beta_{i_1} \otimes \ldots \otimes d_p^{(1)} \beta_{i_k}$$

with uniquely determined numbers $a_{i_1, \ldots, i_k} \in R$. $\alpha(\tau)$ is a symmetric mapping if and only if the matrix

$$[a_{i_1, \ldots, i_k};\ 1 \leqslant i_1, \ldots, i_k \leqslant n]$$

is symmetric.

Let now $\alpha(\tau)$ be an arbitrary element of the space $L_s^k\left((M, \mathscr{C})_p', R\right)$. Let

$$f = \sum_{\substack{\alpha_1 + \ldots + \alpha_n = k \\ 0 \leqslant \alpha_1, \ldots, \alpha_n \leqslant k}} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} \beta_1^{\alpha_1} \cdot \ldots \cdot \beta_n^{\alpha_n} a_{(\alpha_1, \ldots, \alpha_n)},$$

where the number $a_{(\alpha_1, \ldots, \alpha_n)}$ is equal $a_{i_1, \ldots, i_k}$ for the sequence $i_1, \ldots, i_k$ constructed in the following way: at the beginning the number 1 appears $\alpha_1$ times, then the number 2 is repeated $\alpha_2$ times etc., the number $n$ occurs $\alpha_n$

times. It is clear that for a sequence $\alpha'_1, \ldots, \alpha'_n$ such that $\alpha'_1 + \ldots + \alpha'_n = k$ and $0 \leqslant \alpha'_1, \ldots, \alpha'_n \leqslant k$

$$(d_p^{(k)} f)\big(\underbrace{X_1(p), \ldots, X_1(p)}_{\alpha'_1 \text{ times}}, \ldots, \underbrace{X_n(p), \ldots, X_n(p)}_{\alpha'_n \text{ times}}\big)$$

$$= (d_p^{(k)} f)\big(X_1^{\alpha'_1}(p), \ldots, X_n^{\alpha'_n}(p)\big)$$

$$= \sum_{\substack{\alpha_1 + \ldots + \alpha_n = k \\ \alpha_1, \ldots, \alpha_n \geqslant 0}} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} a_{(\alpha_1, \ldots, \alpha_n)} (X_1^{\alpha'_1}, \ldots, X_n^{\alpha'_n})(\beta_1^{\alpha_1}, \ldots, \beta_n^{\alpha_n})(p)$$

$$= \sum_{\substack{\alpha_1 + \ldots + \alpha_n = k \\ \alpha_1, \ldots, \alpha_n \geqslant 0}} \frac{1}{\alpha_1! \cdot \ldots \cdot \alpha_n!} a_{(\alpha_1, \ldots, \alpha_n)} \alpha_1! \cdot \ldots \cdot \alpha_n! \, \delta_{\alpha_1}^{\alpha'_1} \cdot \ldots \cdot \delta_{\alpha_n}^{\alpha'_n}$$

$$= a_{(\alpha'_1, \ldots, \alpha'_n)} = \alpha(\tau)\big(\underbrace{X_1(p), \ldots, X_1(p)}_{\alpha'_1 \text{ times}}, \ldots, \underbrace{X_n(p), \ldots, X_n(p)}_{\alpha'_n \text{ times}}\big)$$

q.e.d.

There exists a differential space $(M, \mathscr{C})$ and a point $p \in M$ at which $\dim(M, \mathscr{C})_p = \infty$ and $\dim(M, \mathscr{C})'_p < \infty$.

EXAMPLE 2.2. Let $(A, D)$ be a differential space from Example 2.1. Let us take $(M, \mathscr{C}) = \bigtimes_{m \in N} (A_m, D_m)$, where $(A_m, D_m) = (A, D)$, $m = 1, 2, \ldots$, and a point $p \in M$ such that $pr_n(p) = (0, 0)$. It can be proved that $\dim(M, \mathscr{C})_p = \infty$ and $\dim(M, \mathscr{C})'_p = 0$.

LEMMA 2.1. *If functions $f^1, \ldots, f^n$ belong to the ideal $I_p^{(k)}(M, \mathscr{C})$ and $g_1, \ldots, g_n$ are arbitrary functions of class $\mathscr{C}$, then*

$$d_p^{(k)}\Big(\sum_{i=1}^{n} f^i g_i\Big) = \sum_{i=1}^{n} (d_p^{(k)} f^i) g_i(p).$$

Proof. The proof will be inductive on $k$. By the linearity of $d_p^{(k)}$ it suffices to prove the equality for $n = 1$. Let $f \in I_p^{(k)}(M, \mathscr{C})$ and $g \in \mathscr{C}$. When $k = 1$ the proof is evident. Let $k > 1$,

$$\begin{aligned}
d_p^{(k)}(f \cdot g)(X_1(p), \ldots, X_k(p)) &= [(X_1, \ldots, X_k)(f \cdot g)](p) \\
&= \big[(X_1, \ldots, X_{k-1})(X_k(f \cdot g))\big](p) \\
&= \big[(X_1, \ldots, X_{k-1})((X_k f) g + f(X_k g))\big](p) \\
&= \big[(X_1, \ldots, X_{k-1})((X_k f) g)\big](p) + \big[(X_1, \ldots, X_{k-1})(f(X_k g))\big](p) \\
&= d_p^{(k-1)}((X_k f) g)(X_1(p), \ldots, X_{k-1}(p)) + 0 \\
&= [d_p^{(k-1)}(X_k f) g(p)](X_1(p), \ldots, X_{k-1}(p)) \\
&= d_p^{(k-1)}(X_k f)(X_1(p), \ldots, X_{k-1}(p)) \cdot g(p)
\end{aligned}$$

$$= [(X_1, \ldots, X_{k-1})(X_k f)](p) \cdot g(p)$$
$$= [(X_1, \ldots, X_k) f](p) \cdot g(p)$$
$$= [(d_p^{(k)} f) \cdot g(p)](X_1(p), \ldots, X_k(p)). \quad \text{q.e.d.}$$

## 3. Modules of jets. An exact sequence of jet-modules.

**3.1. Opening remarks.** For an arbitrary linear field module $\mathscr{W} = ((M, \mathscr{C}), \Phi, \mathscr{W})$ we shall look for the possibly weakest conditions under which a linear field module $J^k(\mathscr{W})$, called the *module of jets of order* $k$ of the module $\mathscr{W}$, can be rationally defined.

The notion of jet appeared in Ch. Ehresmann's work [4]. In the same series of articles we can find also the notion of a holonomic extension of order $k$ of a bundle $\xi$. In the case of linear bundles this notion was introduced in a way different but equivalent and more useful for us by R. Palais ([16]) in the course of presenting the theory of differential operators.

The definition of the jet field module $J^k(\mathscr{W})$ in the case of a linear field module is a generalization of this construction.

**3.2. Definition of a complete differential of higher order in a linear field module. Examples.**

DEFINITION 3.2.1. A *complete differential of order $k$ in a linear $\Phi$-field module $\mathscr{W}$ over a differential space $(M, \mathscr{C})$* is defined as an $R$-linear mapping

$$D^k\colon \mathscr{W} \to L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$$

satisfying the condition

$$(3.2.1) \qquad (D^k(f \cdot W))(p) = d_p^{(k)}(f - f(p)) \otimes W(p) + f(p)(D^k W)(p)$$

for fields $W \in \mathscr{W}$, points $p \in M$ and functions $f \in \mathscr{C}$ such that

$$f - f(p) \in I_p^{(k)}(M, \mathscr{C}).$$

For $k = 1$ we have the ordinary definition of a covariant derivative. We shall further denote $(D^k W)(p)$ by $D_p^k(W)$.

EXAMPLE 3.2.1. A fundamental example of a complete differential of order $k$ is the mapping

$$d^k\colon C^\infty(R^n) \to L_s^k(\mathscr{X}(R^n, C^\infty(R^n)), C^r(R^n))$$

define by the formula

$$(d^k f)(X_1, \ldots, X_k)(p) = \sum_{i_1, \ldots, i_k = 1} X_1(p)(pr_{i_1}) \cdot \ldots \cdot X_k(p)(pr_{i_k}) f_{|i_1 \ldots i_k}(p)$$

for $X_1, \ldots, X_k$ smooth vector fields on $R^n$ and $pr_j\colon R^n \to R$, $j = 1, \ldots, n$, the natural projections.

Thus, in order to evaluate $(d^k f)(X_1, \ldots, X_k)(p)$, the vectors

$X_2(p), \ldots, X_k(p)$ should be extended to vector fields $Y_2, \ldots, Y_k$, constant with respect to the natural covariant derivative in the module $\mathscr{X}(\boldsymbol{R}^n, C^\infty(\boldsymbol{R}^n))$ and the following quantity should be computed:

$$(d^k f)(X_1, \ldots, X_k)(p) = X_1(p)[(Y_2, \ldots, Y_k) f].$$

EXAMPLE 3.2.2. Let us consider a vector bundle $\xi$ over a manifold $M$, a covariant derivative $\tilde{\nabla}$ in the tangent bundle $TM$ with vanishing curvature tensor and a covariant derivative $\nabla$ in $\xi$ such that, whenever $\bar{X}$ and $\bar{Y}$ are $\tilde{\nabla}$-constant fields defined on an open set $U \subset M$, the curvature tensor of $\nabla$ satisfies

$$R_{\bar{X},\bar{Y}}\sigma = -\nabla_{[\bar{X},\bar{Y}]}\sigma,$$

$\sigma$ being any section of $\xi$ over $U$. For vector field $X$ on the manifold $M$ and a point $p \in M$ we denote by $\bar{X}^p$ the $\tilde{\nabla}$-constant field defined in a certain neighbourhood of $p$ such that $X(p) = \bar{X}^p(p)$. Let

$$(D_{X_1,\ldots,X_k}\sigma)(p) = \big(\nabla_{\bar{X}_1^p}(\ldots(\nabla_{\bar{X}_k^p}\sigma)\ldots)\big)(p).$$

The operator $D$ defined in this way is a complete differential of order $k$.

**3.3. The modules $Z_p^{(k)}(\mathscr{W})$ and $Z_p^k(\mathscr{W})$.** Let us consider a certain vector bundle $\xi$ over a manifold $M$. R. Palais [6] has defined, for an arbitrary point $p \in M$ and an integer $k \geqslant 0$, a submodule $Z_p^k(\xi)$ of $C^\infty(\xi)$ (the module of global sections of $\xi$) to be equal $I_p^k(M)C^\infty(\xi)$. It corresponds to these global sections whose holonomic $k$-jet at $p$ (in the terminology of Ehresmann) is equal to 0. If $D^k$ is a complete differential of order $k$ in the module $C^\infty(\xi)$, then $Z_p^k(\xi)$ consists of these sections $\sigma \in Z_p^{k-1}(\xi)$ for which $D_p^k(\sigma) = 0$.

DEFINITION 3.3.1. Assume, for an arbitrary linear field module $\mathscr{W} = ((M, \mathscr{C}), \Phi, \mathscr{W})$ and a point $p \in M$, that

(a) $Z_p^{(k)}(\mathscr{W}) = I_p^{(k+1)}(M, \mathscr{C})\mathscr{W}$, $k = 0, 1, 2, \ldots$

(b) $Z_p^0(\mathscr{W}) = Z_p^{(0)}(\mathscr{W})$ and $Z_p^k(\mathscr{W})$, $k = 1, 2, \ldots$, is equal to the submodule of $\mathscr{W}$ containing these and only these fields $W \in Z_p^{(k-1)}(\mathscr{W})$ which can be written in the form $W = \sum_{i=1}^{n} f^i W_i$, $f^1, \ldots, f^n \in I_p^{(k)}(M, \mathscr{C})$, $W_1, \ldots, W_n \in \mathscr{W}$, such that

$$\sum_{i=1}^{n} (d_p^{(k)} f^i) \otimes W_i(p) = 0.$$

The modules $Z_p^{(k)}(\mathscr{W})$ and $Z_p^k(\mathscr{W})$ are closed with respect to localization. It is easy to see that if $D^k$ is a complete differential of order $k$ in a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$, then $Z_p^k(\mathscr{W})$ contains those and only those fields $W \in Z_p^{(k-1)}(\mathscr{W})$ for which $D_p^k(W) = 0$. For an arbitrary open set $U \in \tau_{\mathscr{C}}$ the following equalities hold:

(3.3.1) $$(Z_p^{(k)}(\mathscr{W}))_U = Z_p^{(k)}(\mathscr{W}_U), \quad (Z_p^k(\mathscr{W}))_U = Z_p^k(\mathscr{W}_U).$$

The inclusion

(3.3.2) $$Z_p^{(k)}(\mathscr{W}) \subset Z_p^k(\mathscr{W}), \quad k \in N,$$

cannot, in general, be replaced by an equality.

EXAMPLE 3.3.1. Consider a differential space $(\boldsymbol{R}, C^\infty(\boldsymbol{R}))$, a positive integer $r$ and an assignement $\Phi$ defined by the formula:

$$\Phi(p) = \begin{cases} \boldsymbol{R}^r, & p \neq 0, \\ \boldsymbol{R}^{r-1} \times \{0\}, & p = 0. \end{cases}$$

Let us include into the module $\mathscr{W}$ those and only those fields $(f^1, \ldots, f^r)$ for which $f^1, \ldots, f^{r-1} \in C^\infty(\boldsymbol{R})$ and $f^r \in I_0(\boldsymbol{R})$. Clearly,

$$\begin{aligned} Z_0^{(k)}(\mathscr{W}) &= \{(f^1, \ldots, f^r);\ f^1, \ldots, f^{r-1} \in I_0^{k+1}, f^r \in I_0^{k+2}\} \\ &\subsetneq \{(f^1, \ldots, f^r):\ f^1, \ldots, f^r \in I_0^{k+1}\} = Z_0^k(\mathscr{W}). \end{aligned}$$

If the manifold $M$ has a positive dimensions, then for any natural number $k$ we have

$$Z_p^k(C^\infty(\xi)) \subsetneq Z_p^{(k-1)}(C^\infty(\xi)) = Z_p^{k-1}(C^\infty(\xi)).$$

In general, the equality on the right does not hold in pseudo-differential modules (Example 3.3.1) but it holds in differential modules.

THEOREM 3.3.1. *If a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is a differential module, then $Z_p^{(k)}(\mathscr{W}) = Z_p^k(\mathscr{W})$, $k \in N$, $p \in M$.*

Proof. Every field $W \in Z_p^k(\mathscr{W})$ is of the form $\sum_{i=1}^n f^i \cdot W_i$ with functions $f^i \in I_p^{(k)}(M, \mathscr{C})$, $i = 1, \ldots, n$, satisfying the condition $\sum_{i=1}^n d_p^{(k)} f^i \otimes W_i(p) = 0$. There exist a neighbourhood $U$ of $p$ and fields $V_1, \ldots, V_r \in \mathscr{W}$ such that the fields $V_1|U, \ldots, V_r|U$ are a vector basis for the module $\mathscr{W}_U$ and $W_i|U = (\sum_{j=1}^r \lambda_i^j V_j)|U$, $i = 1, \ldots, n$, for certain functions $\lambda_i^j \in \mathscr{C}$. Thus by Lemma 2.1

$$\begin{aligned} 0 &= \sum_{i=1}^n d_p^{(k)} f^i \otimes W_i(p) = \sum_{i=1}^n d_p^{(k)} f^i \otimes \sum_{j=1}^r \lambda_i^j(p) \cdot V_j(p) \\ &= \sum_{j=1}^r \Big(\sum_{i=1}^n (d_p^{(k)} f^i) \lambda_i^j(p)\Big) \otimes V_j(p) = \sum_{j=1}^r d_p^{(k)}\Big(\sum_{i=1}^n f^i \lambda_i^j\Big) \otimes V_j(p). \end{aligned}$$

From the fact that the vectors $V_1(p), \ldots, V_r(p)$ are linearly independent we obtain the equalities $d_p^{(k)}(\sum_{i=1}^n f^i \lambda_i^j) = 0$, $j = 1, \ldots, r$, and from Theorem 2.1

we get the relation $\Psi^j = \sum_{i=1}^{n} f^i \lambda_i^j \in I_p^{(k+1)}(M, \mathscr{C}), j = 1, \ldots, r$. Thus $W|U = (\sum_{i=1}^{n} f^i W_i)|U = (\sum_{j=1}^{r} \Psi^j V_j)|U$, which means that $W \in Z_p^{(k)}(\mathscr{W})$. q.e.d.

THEOREM 3.3.2. *If a differential space* $(M, \mathscr{C})$ *is of class* $\mathscr{D}_0$ *and if we have* $(M, \mathscr{C})_p = (M, \mathscr{C})'_p$ *at a point* $p \in M$, *then* $Z_p^{(k)}(\mathscr{X}(M, \mathscr{C})) = Z_p^k(\mathscr{X}(M, \mathscr{C}))$, $k \in N$. *Consequently the set of points* $p \in M$ *for which the two modules are equal is dense in* $\tau_{\mathscr{C}}$ *and covers the set* $M'$.

Proof. From Theorem 1.1.1 follows the existence of a neighbourhood $U$ of $p$ such that, for any $q \in U$, $\dim(M, \mathscr{C})_q = \dim(M, \mathscr{C})_p$ and $(M, \mathscr{C})'_q = (M, \mathscr{C})_q$. Therefore the module $\mathscr{X}(U, \mathscr{C}_U)$ is differential. From the preceding theorem follows the equality

$$Z_p^{(k)}(\mathscr{X}(U, \mathscr{C}_U)) = Z_p^k(\mathscr{X}(U, \mathscr{C}_U)).$$

It is easy to prove the present theorem applying equalities (3.3.1). q.e.d.

**3.4. The mapping $d_p^{(k)}$ for linear field modules. Condition $*k$).** In ([6]) R. Palais has proved the existence and uniqueness of an $R$-linear mapping $d_p^k$: $Z_p^{k-1}(C^\infty(\xi)) \to L_s^k(M_p, \xi_p)$, $k \geqslant 1$, such that if $W \in Z_p^{k-1}(C^\infty(\xi))$ and $h \in C^\infty(\xi^*)$, then

$$d_p^k(h \circ W) = h(p) \circ d_p^k(W). \tag{3.4.1}$$

Note that $h \circ W \in I_p^k(M)$. If $W \in Z_p^{k-1}(C^\infty(\xi))$ and $W = \sum_{i=1}^{n} f^i W_i$, where $f^i \in I_p^k(M)$, $i = 1, \ldots, n$, then

$$d_p^k(W) = \sum_{i=1}^{n} d_p^k f^i \otimes W_i(p). \tag{3.4.2}$$

Indeed, let us consider a field $h \in C^\infty(\xi^*)$ and vectors $v_1, \ldots, v_k \in M_p$. From Lemma 2.1 follows

$$\begin{aligned} d_p^k(h \circ W)(v_1, \ldots, v_k) &= \sum_{i=1}^{n} d_p^k(f^i(h \circ W_i))(v_1, \ldots, v_k) \\ &= \sum_{i=1}^{n} (d_p^k f^i)(h \circ W_i)(p)(v_1, \ldots, v_k) \\ &= \sum_{i=1}^{n} (d_p^k f^i)(v_1, \ldots, v_k)(h \circ W_i)(p) \\ &= h(p)(\sum_{i=1}^{n} (d_p^k f^i)(v_1, \ldots, v_k) W_i(p)) \\ &= h(p)\big((\sum_{i=1}^{n} d_p^k f^i \otimes W_i(p))(v_1, \ldots, v_k)\big). \end{aligned}$$

Applying the formula analogous to (3.4.2) we define the mapping $d_p^{(k)}$ for linear field modules. Let $((M, \mathscr{C}), \Phi, \mathscr{W})$ be a linear field module.

DEFINITION 3.4.1. We denote by $d_p^{(k)}$, $p \in M$, $k \in N$, an $R$-linear mapping

$$d_p^{(k)}\colon Z_p^{(k-1)}(\mathscr{W}) \to L_s^k((M, \mathscr{C})_p', \Phi_{\mathscr{W}}(p))$$

such that $d_p^{(k)}(f \cdot W) = d_p^{(k)} f \otimes W(p)$ for $f \in I_p^{(k)}(M, \mathscr{C})$, $W \in \mathscr{W}$.

THEOREM 3.4.1. *A mapping $d_p^{(k)}$ exists if and only if the following condition is satisfied*:

$*$k) *if* $\sum_{i=1}^{n} f^i W_i = 0$, *where* $f^i \in I_p^{(k)}(M, \mathscr{C})$, $W_i \in \mathscr{W}$, $i = 1, \ldots, n$, $n \in N$, *then* $\sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p) = 0$.

*There exists at most one mapping* $d_p^{(k)}$.

Proof. If $d_p^{(k)}$ exists and if $\sum_{i=1}^{n} f^i W_i = 0$ for $f^i \in I_p^{(k)}(M, \mathscr{C})$, $W_i \in \mathscr{W}$, then $\sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p) = \sum_{i=1}^{n} d_p^{(k)}(f^i W_i) = d_p^{(k)}(\sum_{i=1}^{n} f^i W_i) = 0$, so that condition $*$k) is satisfied. The existence and uniqueness of the mapping $d_p^{(k)}$ under condition $*$k) is a consequence of the property that any field $W \in Z_p^{(k-1)}(\mathscr{W})$ is of the form $\sum_{i=1}^{n} f^i W_i$, $f^i \in I_p^{(k)}(M, \mathscr{C})$, and that $d_p^{(k)}(W) = \sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p)$ does not depend on the representation of the field $W$ in this form. Thus the last formula defines the desired $R$-linear mapping. q.e.d.

It follows directly from the definition of a complete differential of order $k$ that if a complete differential exists in a linear field module, then condition $*$k) is fulfilled at every point of the underlying space. Condition $*$k) need not be satisfied in every linear field module.

EXAMPLE 3.4.1. Consider a differential space $(\boldsymbol{R}, C^\infty(\boldsymbol{R}))$ and the assignement $\Phi$ defined as follows: $\Phi(p) = 0$ for $p \neq 0$ and $\Phi(0) = R$. Let $\mathscr{W}$ be the module of the all linear $\Phi$-fields. For an arbitrary function $f \in I_0^k \setminus I_0^{k+1}$ and the field $W \in \mathscr{W}$ equal 1 at the point 0 we have

$$f \cdot W = 0 \quad \text{and} \quad d_0^{(k)} f \otimes W(p) \neq 0.$$

Remark. Let $((M, \mathscr{C}), \Phi, \mathscr{W})$ be a linear field module. For arbitrary fields $W \in Z_p^{(k-1)}(\mathscr{W})$ and $h \in \mathscr{W}^*$ we have

$$h \circ W \in I_p^{(k)}(M, \mathscr{C}) \quad \text{and} \quad d_p^{(k)}(h \circ W) = h(p) \circ d_p^{(k)}(W).$$

Proof. Assume that the field $W$ is of the form

$$\sum_{i=1}^{n} f^i W_i \quad \text{for } f^i \in I_p^{(k)}(M, \mathscr{C}),\ i = 1, \ldots, n.$$

For any vectors $v_1, \ldots, v_k \in (M, \mathscr{C})'_p$

$$d_p^{(k)}(h \circ W)(v_1, \ldots, v_k) = h(p) \circ \Big(\sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p)\Big)(v_1, \ldots, v_k)$$

$$= h(p)\Big(d_p^{(k)}\Big(\sum_{i=1}^{n} f^i W_i\Big)\Big)(v_1, \ldots, v_k)$$

$$= h(p) \circ d_p^{(k)}(W)(v_1, \ldots, v_k). \quad \text{q.e.d.}$$

Condition $*k)$ is satisfied in a fairly broad class of linear field modules (see Theorem 1.4.1).

THEOREM 3.4.2. *Let $((M, \mathscr{C}), \Phi, \mathscr{W})$ be a linear field module. If this module satisfies at a point $p$ the conditions*:

(a) $\dim \Phi_{\mathscr{W}}(p) < \infty$,

(b) $\Psi_{\mathscr{W}^*}(p) = (\Phi_{\mathscr{W}}(p))^*$, *where* $\Psi(q) = (\Phi_{\mathscr{W}}(q))^*$, $q \in M$,

*then for* $k \in N$

(A) *there exists exactly one $R$-linear mapping*

$$d_p^{[k]}\colon Z_p^{(k-1)}(\mathscr{W}) \to L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p))$$

*satisfying the equality*

(3.4.3) $\quad d_p^{(k)}(h \circ W) = h(p) \circ d_p^{[k]}(W) \quad$ *for* $W \in Z_p^{(k-1)}(\mathscr{W})$ *and* $h \in \mathscr{W}^*$;

(B) *the module satisfies condition $*k)$ at the point $p$ and* $d_p^{[k]} = d_p^{(k)}$.

Proof. Assume that conditions (a) and (b) are satisfied at a point $p \in M$. For an arbitrarily fixed field $W \in Z_p^{(k-1)}(\mathscr{W})$ there exists the $R$-linear mapping $\Psi_{\mathscr{W}^*}(p) \ni w \mapsto d_p^{(k)}(h \circ W)$, where $h \in \mathscr{W}^*$ and $h(p) = w$; and for any collection of vectors $v_1, \ldots, v_k$ from $(M, \mathscr{C})'_p$ there exists exactly one element $d_p^{[k]}(W)(v_1, \ldots, v_k) \in \Phi_{\mathscr{W}}(p)$ such that

$$d_p^{(k)}(h \circ W)(v_1, \ldots, v_k) = h(p)\big(d_p^{[k]}(W)(v_1, \ldots, v_k)\big)$$

for $h \in \mathscr{W}^*$. The mapping

$$d_p^{[k]}(W) = \Big(\bigtimes_k (M, \mathscr{C})'_p \ni (v_1, \ldots, v_k) \mapsto d_p^{[k]}(W)(v_1, \ldots, v_k) \in \Phi_{\mathscr{W}}(p)\Big)$$

is symmetric and $k$-linear; it defines a linear mapping

$$d_p^{[k]}\colon Z_p^{(k-1)}(\mathscr{W}) \to L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p)).$$

This is the only mapping which has property (3.4.3) and we have $d_p^{(k)} = d_p^{[k]}$.

Now we show that condition $*k)$ is fulfilled at the point $p \in M$. Let us consider any functions $f^1, \ldots, f^n \in I_p^{(k)}(M, \mathscr{C})$ and fields $W_1, \ldots, W_n \in \mathscr{W}$ such that $\sum_{i=1}^{n} f^i W_i = 0$. For any field $h \in \mathscr{W}^*$ and vectors $v_1, \ldots, v_k \in (M, \mathscr{C})'_p$

$$0 = h(p)\big(d_p^{(k)}(\sum_{i=1}^{n} f^i W_i)(v_1, \ldots, v_k)\big)$$

$$= h(p)(\sum_{i=1}^{n} (d_p^{(k)} f^i) \otimes W_i(p))(v_1, \ldots, v_k).$$

From assumption (a) and (b) follows

$$\sum_{i=1}^{n} (d_p^{(k)} f^i) \otimes W_i(p) = 0. \quad \text{q.e.d.}$$

THEOREM 3.4.3. *If a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ satisfies at $p \in M$ the following conditions:*

(a) $\dim (M, \mathscr{C})'_p < \infty$,

(b) $\dim \Phi_{\mathscr{W}}(p) < \infty$,

(c) $*k$),

*then the following sequence is exact:*

$$(3.4.4) \qquad 0 \to Z_p^k(\mathscr{W}) \hookrightarrow Z_p^{(k-1)}(\mathscr{W}) \xrightarrow{d_p^{(k)}} L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p)) \to 0.$$

Proof. It suffices to show the surjectivity of the mapping $d_p^{(k)}$ in the case when $\dim \Phi_{\mathscr{W}}(p) > 0$. Let us take an arbitrary element $\tau \in L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p))$ and a basis $v_1, \ldots, v_k$ of the space $\Phi_{\mathscr{W}}(p)$. There exist elements $\tau^1, \ldots, \tau^r \in L_s^k((M, \mathscr{C})'_p, R)$ such that $\tau = \sum_{i=1}^{r} \tau^i \otimes v_i$. From Theorem 2.1 we conclude that there exist functions $f^1, \ldots, f^r \in I_p^{(k)}(M, \mathscr{C})$ such that $d_p^{(k)} f^i = \tau^i$, $i = 1, \ldots, r$. For any fields $W_1, \ldots, W_r \in \mathscr{W}$ such that $W_i(p) = v_i$, $i = 1, \ldots, r$, the equality $d_p^{(k)}(\sum_{i=1}^{n} f^i W_i) = \tau$ is satisfied. q.e.d.

In what follows we assume that all linear field modules under consideration satisfy the assumptions of the last theorem.

From the definition of the mapping $d_p^{(k)}$ follows the equality: $Z_p^k(\mathscr{W}) = \ker d_p^{(k)}$. Therefore there exists a linear isomorphism

$$\varrho_p^k \colon Z_p^{(k-1)}(\mathscr{W})/Z_p^k(\mathscr{W}) \to L_s^k((M, C)'_p, \Phi_{\mathscr{W}}(p))$$

with the property $\varrho_p^k(W + Z_p^k(\mathscr{W})) = d_p^{(k)}(W)$ for $W \in Z_p^{(k-1)}(\mathscr{W})$. The inverse isomorphism will be denoted by $i_p^k$; it will be considered as an injective linear mapping

$$i_p^k \colon L_s^k((M, \mathscr{C})'_p, \Phi_{\mathscr{W}}(p)) \to \mathscr{W}/Z_p^k(\mathscr{W}).$$

Since $Z_p^k(\mathscr{W}) \subset Z_p^{(k-1)}(\mathscr{W})$, there exists the canonical surjective linear mapping

$$r_p^{k,(k-1)}\colon \mathscr{W}/Z_p^k(\mathscr{W}) \to \mathscr{W}/Z_p^{(k-1)}(\mathscr{W})$$

with the kernel $Z_p^{(k-1)}(\mathscr{W})/Z_p^k(\mathscr{W})$ (equal to im $i_p^k$). Hence the following sequence is exact:

$$0 \to L_s^k((M, \mathscr{C})_p', \Phi_{\mathscr{W}}(p)) \xrightarrow{i_p^k} \mathscr{W}/Z_p^k(\mathscr{W}) \xrightarrow{r_p^{k,(k-1)}} \mathscr{W}/Z_p^{(k-1)}(\mathscr{W}) \to 0. \tag{3.4.5}$$

DEFINITION 3.4.2. Consider an arbitrary non-negative number $k$, a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ and a point $p \in M$. Denote by

$$j_p^k\colon \mathscr{W} \to \mathscr{W}/Z_p^k(\mathscr{W}) \quad \text{and} \quad j_p^{(k)}\colon \mathscr{W} \to \mathscr{W}/Z_p^{(k)}(\mathscr{W})$$

the canonical linear mappings. The spaces

$$J_p^k(\mathscr{W}) = \mathscr{W}/Z_p^k(\mathscr{W}) \quad \text{and} \quad J_p^{(k)}(\mathscr{W}) = \mathscr{W}/Z_p^{(k)}(\mathscr{W})$$

will be called the *jet spaces*, of order $k$ and $(k)$, respectively, at the point $p$.

For any field $W \in Z_p^{(k-1)}(\mathscr{W})$

$$j_p^k(W) = i_p^k(d_p^{(k)}(W)). \tag{3.4.6}$$

Indeed, $i_p^k(d_p^{(k)}(W)) = i_p^k(\varrho_p^k(W + Z_p^k(\mathscr{W}))) = i_p^k(\varrho_p^k(j_p^k(W))) = j_p^k(W)$.

LEMMA 3.4.1. *If $D^k$ is a complete differential of order $k$ in a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$, then for any point $p \in M$ there exists exactly one $R$-linear mapping*

$$T_p\colon J_p^k(\mathscr{W}) \to L_s^k((M, \mathscr{C})_p', \Phi_{\mathscr{W}}(p))$$

*such that $D_p^k = T_p \circ j_p^k$. It will be called the mapping linearizing the complete differential $D^k$ at the point $p$. It satisfies the condition: $T_p \circ i_p^k = \mathrm{id}$.*

Proof. If there exists a mapping linearizing the complete differential $D^k$ at a point $p$, then it is defined by the formula

$$T_p(j_p^k(W)) = D_p^k(W); \tag{3.4.7}$$

therefore there exists at most one such mapping.

Consider the mapping $T_p$ defined by formula (3.4.7). The formula defines the mapping $T_p$ correctly because if $j_p^k(W) = j_p^k(W')$, then $(W - W') \in Z_p^k(\mathscr{W})$, which implies $D_p^k(W - W') = 0$. The equality $T_p \circ i_p^k = \mathrm{id}$ follows from $D_p^k(W) = d_p^{(k)}(W)$ for $W \in Z_p^{(k-1)}(\mathscr{W})$. q.e.d.

To conclude this subsection we prove one more important fact.

THEOREM 3.4.4. *If $D^k$ is a complete differential of order $k$ in a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ and $T_p$ is the mapping linearizing $D^k$ at a point $p$, then*

$$\ker r_p^{k,(k-1)} \cap \ker T_p = 0.$$

Proof. Let us consider any field $W \in \mathscr{W}$ such that

$$j_p^k(W) \in \ker r_p^{k,(k-1)} \cap \ker T_p.$$

Then $W \in Z_p^{(k-1)}(\mathscr{W})$ and so $W = \sum_{i=1}^{n} f^i W_i$ for certain functions $f^1, \ldots, f^n \in I_p^{(k)}(M, \mathscr{C})$ and fields $W_1, \ldots, W_n \in \mathscr{W}$. Besides,

$$0 = T_p(j_p^k(W)) = D_p^k(W) = D_p^k\Big(\sum_{i=1}^{n} f^i W_i\Big) = \sum_{i=1}^{n} d_p^{(k)} f^i \otimes W_i(p);$$

hence $W \in Z_p^k(\mathscr{W})$ and consequently $j_p^k(W) = 0$. q.e.d.

**3.5. Jet field module of order $k$ and $(k)$. An exact sequence of jet-modules.**

DEFINITION 3.5.1. (a) The $(k)$-order jet field module, $k = 0, 1, \ldots$, of a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is the least linear $(M \ni p \mapsto J_p^{(k)}(\mathscr{W}))$-field module closed with respect to localization, containing all fields of the form:

$$M \ni p \mapsto j_p^{(k)}(W) \in J_p^{(k)}(\mathscr{W}) \quad \text{for } W \in \mathscr{W}.$$

(b) The $k$-order jet field module, $k = 0, 1, 2, \ldots$, of a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is the least $(M \ni p \mapsto J_p^k(\mathscr{W}))$-field module closed with respect to localization, containing all fields of the form:

(i) $$M \ni p \mapsto j_p^k(W) \in J_p^k(\mathscr{W}) \quad \text{for } W \in \mathscr{W},$$

(ii) $$M \ni p \mapsto i_p^k(S_p) \in J_p^k(\mathscr{W}) \quad \text{for } S \in L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W}).$$

It is clear that for any jet fields $S$ of order $k$ the field $(M \ni p \mapsto r_p^{k,(k-1)}(S_p))$ is a jet field of order $(k-1)$. Moreover, the mappings $i^k$: $L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W}) \to J^k(\mathscr{W})$ and $r^{k,(k-1)}$: $J^k(\mathscr{W}) \to J^{(k-1)}(\mathscr{W})$ defined by the formula $i^k(L)(p) = i_p^k(L_p)$ for $L \in L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$ and $p \in M$, $r^{k,(k-1)}(L)(p) = r_p^{k,(k-1)}(L_p)$ for $L \in J^k(\mathscr{W})$ and $p \in M$, are homomorphisms of linear field modules. The following natural mappings are $R$-linear,

$$j^k\colon \mathscr{W} \to J^k(\mathscr{W}) \quad \text{and} \quad j^{(k)}\colon \mathscr{W} \to J^{(k)}(\mathscr{W}).$$

Notice also that $j^0$: $\mathscr{W} \to J^0(\mathscr{W})$ is a $\mathscr{C}$-linear mapping.

In the sequel of this section we shall examine the sequence

$$(3.5.1) \qquad 0 \to L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W}) \xrightarrow{i^k} J^k(\mathscr{W}) \xrightarrow{r^{k,(k-1)}} J^{(k-1)}(\mathscr{W}) \to 0,$$

called the *jet-module sequence*.

THEOREM 3.5.1. *If a differential space $(M, \mathscr{C})$ is paracompact and $C$-normal, then the mapping $r^{k,(k-1)}$ in sequence* (3.5.1) *is a surjection.*

Proof. Let us consider an arbitrary field $W \in J^{(k-1)}(\mathscr{W})$. For any point $p \in M$ there exists a neighbourhood $U^p \in \tau_{\mathscr{C}}$ of $p$ such that $W|U^p = \big(\sum_{i=1}^{n} f^i j^{(k-1)} W_i\big)|U^p$ for a certain positive integer $n$, functions $f^i \in \mathscr{C}$ and fields $W^i \in \mathscr{W}$, $i = 1, 2, \ldots, n$. According to paracompactness, we subordinate a locally finite family $(V_t, t \in T)$ to the family $(U^p, p \in M)$, and applying $\mathscr{C}$-normality we choose a smooth partition of unity $(\varphi_t)_{t\in T}$ subordinate to this covering. We define fields $\theta_t \in J^k(\mathscr{W})$, $t \in T$, by the formula $\theta_t = \sum_{i=1}^{n} f^i j^k(W_i)$ and we put $\theta = \sum_{t\in T} \varphi_t \theta_t$. Obviously $\theta \in J^k(\mathscr{W})$, and since $r_p^{k,(k-1)}(\theta_t(p)) = W(p)$ for $p \in V_t$, we have $r^{k,(k-1)}(\theta) = W$. q.e.d.

The exactness of sequence (3.5.1) at the term "$J^k(\mathscr{W})$" in the case $k = 1$ will be proved without additional assumptions about the module $\mathscr{W}$. In the general case it will be proved for a broad class of linear field modules containing pseudo-differential modules.

To show exactness let us consider an arbitrary field $S \in \ker r^{k,(k-1)}$ and notice that there exists exactly one field $L$ such that $i_p^k(L_p) = S_p$ for any point $p \in M$. We shall check that $L \in L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$. From the definition of the module $J^k(\mathscr{W})$ it follows that in a certain neighbourhood $U$ of $p \in M$ the field $S$ is of the form

$$S|U = i^k(\L)|U + \left(\sum_{j=1}^{n} f^j j^k(W_j)\right)|U$$

for some field $\L \in L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W})$, a positive integer $n$, functions $f^1, \ldots, f^n \in \mathscr{C}$ and fields $W_1, \ldots, W_n \in \mathscr{W}$. Since for any point $q \in U$,

$$\begin{aligned} 0 = r_q^{k,(k-1)}(S_q) &= r_q^{k,(k-1)}\left(i_p^k(\L_q) + \sum_{j=1}^{n} f^j(q) j_q^k(W_j)\right) \\ &= j_q^{(k-1)}\left(\sum_{j=1}^{n} f^j(q)\, W_j\right), \end{aligned}$$

then $\sum_{j=1}^{n} f^j(q)\, W_j \in Z_q^{(k-1)}(\mathscr{W})$.

From equality (3.4.6) one can easily derive the equality

$$\begin{aligned} i_q^k(L_q) = S_q = i_q^k(\L_q) + j_q^k\left(\sum_{j=1}^{n} f^j(q)\, W_j\right) &= i_q^k(\L_q) + i_q^k\left(d_q^{(k)}\left(\sum_{j=1}^{n} f^j(q)\, W_j\right)\right) \\ &= i_q^k\left(\L_q + d_q^{(k)}\left(\sum_{j=1}^{n} f^j(q)\, W_j\right)\right). \end{aligned}$$

As $i_q^k$ is an injection, we have

$$L_q = \L_q + d_q^{(k)}\left(\sum_{j=1}^{n} f^j(q)\, W_j\right) \quad \text{for } q \in U. \tag{3.5.2}$$

To prove exactness it suffices to show that

$$\big(U \ni q \mapsto d_q^{(k)}\big(\sum_{j=1}^{n} f^j(q)\, W_j\big)(V_1(q), \ldots, V_k(q))\big) \in \mathscr{W}_U$$

for $V_1, \ldots, V_k \in \mathscr{X}(U, \mathscr{C}_U)$.

THEOREM 3.5.2. *The sequence*

$$0 \to L\big(\mathscr{X}(M, \mathscr{C}), \mathscr{W}\big) \xrightarrow{i^1} J^1(\mathscr{W}) \xrightarrow{r^{1,0}} J^0(\mathscr{W})$$

*is exact.*

Proof. Since $\sum_{j=1}^{n} f^j(q)\, W_j \in Z_q^0(\mathscr{W})$ for $q \in U$, then in particular $(\sum_{j=1}^{n} f^j W_j)|U = 0$, and so $\sum_{j=1}^{n} (f^j - f^j(q))\, W_j \in Z_q^0(\mathscr{W})$. Hence

$$0 = d_q^{(1)}\big(\sum_{j=1}^{n} f^j W_j\big) = d_q^{(1)}\big(\sum_{j=1}^{n} (f^j - f^j(q))\, W_j\big) + d_q^{(1)}\big(\sum_{j=1}^{n} f^j(q)\, W_j\big),$$

and this produces the equalities:

$$\begin{aligned} d_q^{(1)}\big(\sum_{j=1}^{n} f^j(q)\, W_j\big)(V(q)) &= -d_q^{(1)}\big(\sum_{j=1}^{n} (f^j - f^j(q))\, W_j\big)(V(q)) \\ &= -\sum_{j=1}^{n} d_q^{(1)}(f^j - f^j(q)) \otimes W_j(q)(V(q)) \\ &= -\sum_{j=1}^{n} d_q^{(1)}(f^j - f^j(q))(V(q))\, W_j(q) \\ &= -\sum_{j=1}^{n} V(q)(f^j - f^j(q))\, W_j(q) \\ &= -\sum_{j=1}^{n} V(q)(f^j)\, W_j(q) = -\big(\sum_{j=1}^{n} V(f^j)\, W_j\big)(q). \qquad \text{q.e.d.} \end{aligned}$$

THEOREM 3.5.3. *If a linear field module* $((M, \mathscr{C}), \Phi, \mathscr{W})$ *satisfies condition*:

*whenewer $W$ is a linear $\Phi_W$-field such that, for any field $h \in \mathscr{W}^*$, the function $h \circ W$ is from the ring $\mathscr{C}$, then $W \in \mathscr{W}$,*

*then the sequence*

$$0 \to L_s^k\big(\mathscr{X}(M, \mathscr{C}), \mathscr{W}\big) \xrightarrow{i^k} J^k(\mathscr{W}) \xrightarrow{r^{k,(k-1)}} J^{(k-1)}(\mathscr{W})$$

*is exact.*

Proof. We shall show that every point $p\in U$ has a neighbourhood $V\subset U$ such that

$$\Big(V\ni q\mapsto d_q^{(k)}\big(\sum_{j=1}^{n} f^j(q)\,W_j(V_1(q),\,\ldots,\,V_k(q))\big)\Big)\in\mathscr{W}_V=(\mathscr{W}_U)_V$$

for $V_1,\ldots,V_k\in\mathscr{X}(V,\mathscr{C}_V)$.

Let us consider a function $\gamma\in\mathscr{C}$ separating the point $p$ in the set $U$, i.e. a function $\gamma$ such that $\gamma|B_0=1$ for some neighbourhood $B_0$ of $p$ and $\gamma|U_0=0$ for an open set $U_0$ such that $U_0\cup U=M$. Obviously,

$$\gamma\cdot\sum_{j=1}^{n} f^j(q)\,W_j\in Z_q^{(k-1)}(\mathscr{W})$$

for any $q\in M$. We put $V=B_0$. Then for $q\in V$ we have

$$d_q^{(k)}\big(\gamma\cdot\sum_{j=1}^{n} f^j(q)\,W_j\big)=d_q^{(k)}\big(\sum_{j=1}^{n} f^j(q)\,W_j\big).$$

It suffices to show that

$$\Big(M\ni q\mapsto d_q^{(k)}\big(\gamma\cdot\sum_{j=1}^{n} f^j(q)\,W_j\big)\big(V_1(q),\,\ldots,\,V_k(q)\big)\Big)\in\mathscr{W}. \tag{3.5.3}$$

For an arbitrary field $h\in\mathscr{W}^*$ it follows from Theorem 3.4.2 that

$$\begin{aligned}
&h(q)\Big(d_q^{(k)}\big(\gamma\cdot\sum_{j=1}^{n} f^j(q)\,W_j\big)\big(V_1(q),\,\ldots,\,V_k(q)\big)\Big)\\
&\qquad=d_q^{(k)}\Big(h\circ\big(\gamma\cdot\sum_{j=1}^{n} f^j(q)\,W_j\big)\big(V_1(q),\,\ldots,\,V_k(q)\big)\Big)\\
&\qquad=d_q^{(k)}\big(\sum_{j=1}^{n}\gamma\cdot f^j(q)\,h\circ W_j\big)\big(V_1(q),\,\ldots,\,V_k(q)\big)\\
&\qquad=\Big((V_1,\,\ldots,\,V_k)\big(\sum_{j=1}^{n}\gamma\cdot f^j(q)\,h\circ W_j\big)\Big)(q)\\
&\qquad=\sum_{j=1}^{n} f^j(q)\,[(V_1,\,\ldots,\,V_k)(\gamma\cdot h\circ W_j)](q)\\
&\qquad=\big(\sum_{j=1}^{n} f^j\,[(V_1,\,\ldots,\,V_k)(\gamma\cdot h\circ W_j)]\big)(q).\qquad\text{q.e.d.}
\end{aligned}$$

Now we present the announced example of a linear field module $\big((M,\mathscr{C}),\,\Phi,\,\mathscr{W}\big)$ for which there exists a $\mathscr{C}$-linear mapping $L\colon\mathscr{X}(M,\mathscr{C})\to\mathscr{W}$ which is not a linear $\Psi$-field for $\Psi=\big(M\ni q\mapsto L\big((M,\mathscr{C})'_p,\,\Phi_{\mathscr{W}}(p)\big)\big)$.

EXAMPLE 3.5.1. Consider the differential space

$$(R, \mathscr{C}) = \big(R, \big(S_C(\{\mathrm{id}_R, (R \ni x \mapsto |x|)\})\big)_R\big).$$

This space is of class $\mathscr{D}_0$. Let $e_x \in (R, C^\infty(R))_x$ for $x \neq 0$ be unitary vector, i.e. such that $e_x(\mathrm{id}_R) = 1$. The tangent space $(R, \mathscr{C})_0$ is 2-dimensional, having as a basis the vectors $e_0$ and $\omega$ defined by the formulas:

$$e_0(\mathrm{id}_R) = 1, \quad e_0(|\cdot|) = 0; \quad \omega(\mathrm{id}_R) = 0, \quad \omega(|\cdot|) = 1.$$

The vector field $V = (R \ni x \mapsto x e_x \in (R, \mathscr{C})_x)$ is smooth because $V(\mathrm{id}_R) = \mathrm{id}_R$ and $V(|\cdot|) = |\cdot|$. It cannot be written in the form $\sum_{i=1}^{n} f^i W_i$ for any numbers $n \in N$, functions $f^1, \ldots, f^n \in I_0^{(1)}(R, \mathscr{C})$ and fields $W_1, \ldots, W_n \in \mathscr{X}(R, \mathscr{C})$. Every vector field $W \in \mathscr{X}(R, \mathscr{C})$ is equal 0 at the point 0 and so, if $V = \sum_{i=1}^{n} f^i W_i$ for functions $f^i$ with $f^i(p) = 0$, $i = 1, \ldots, n$, then $V(\mathrm{id}_R) = \sum_{i=1}^{n} f^i W_i(\mathrm{id}_R)$. We thus would get the equality $\mathrm{id}_R = \sum_{i=1}^{n} f^i g_i$ for certain functions $f^i, g_i$, $i = 1, \ldots, n$, from the ideal $I_0(R, \mathscr{C})$, and this produces a contradiction:

$$1 = e_0(\mathrm{id}_R) = \sum_{i=1}^{n} e_0(f^i g_i) = 0.$$

Now take a jet field module of order 0 of the initial module and a $\mathscr{C}$-linear mapping $j^0\colon \mathscr{X}(R, \mathscr{C}) \to J^0(\mathscr{X}(M, \mathscr{C}))$. $j^0$ is not a linear $\Psi$-field because for the vector field $V$ we have

$$V(0) = 0 \quad \text{and} \quad j^0(V)(0) = j_0^0(V) \neq 0.$$

A scalar product in a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ is a linear field $G \in L_s^2(W, \mathscr{C})$ such that $G(p)(v, v) > 0$ for $0 \neq v \in \Phi_{\mathscr{W}}(p)$ and $G^*\colon \mathscr{W} \to \mathscr{W}^*$ defined by the formula $G^*(V)(W) = G(V, W)$ is an isomorphism of linear field modules.

EXAMPLE 3.5.2. In the space $(R, \mathscr{C})$ from the preceding example every smooth vector field is of the form $f \cdot e$, where $f \in \mathscr{C}$ is a function such that $f(0) = 0$. Every linear field $h \in \mathscr{W}^*$ is of the form $f \cdot e^*$, where $f\colon R \to R$ is a function such that $f(0) = 0$, $f \cdot \mathrm{id}_R \in \mathscr{C}$ and $f \cdot |\cdot| \in \mathscr{C}$. The function $f$ defined by the formula $f(x) = 1$ when $x \neq 0$ and $f(0) = 0$ can serve as example. We shall construct a scalar product $G$ in the module $\mathscr{X}(R, \mathscr{C})$. We put $G(x)(e_x, e_x) = 1/x$ for $x \neq 0$ and, of course, $G(0) = 0$. As every function $f \in \mathscr{C}$ equal 0 at the point 0 is of the form $f(x) = x \cdot f_1(x) + |x| \cdot f_2(x)$, $x \in R$, where $f_1, f_2 \in \mathscr{C}$, we see that $G(V, W) \in \mathscr{C}$ for $V, W \in \mathscr{X}(R, \mathscr{C})$. Let us take the vector field $V = f \cdot \mathrm{id}_R \cdot e$ for any field $h \in \mathscr{W}^*$ of the form $f \cdot e^*$. Then $V \in \mathscr{X}(R, \mathscr{C})$

and $G(V, W) = h(W)$. It is clear that the form $G(x)$, $x \in R$, is positive, and so $G$ is a scalar product.

In the module $\mathscr{X}(R, \mathscr{C})$ there exists a symmetric covariant derivative determined by the scalar product just constructed. Notice that if $f, g \in \mathscr{C}$ and $f(0) = g(0) = 0$, then the function $g$ is differentiable except at zero and the function $h$ defined by the formula

$$h(x) = f(x)g'(x) \quad \text{for } x \neq 0 \text{ and } h(0) = 0$$

is from the ring $\mathscr{C}$. It is easy to prove that the following formula defines the generated covariant derivative:

$$(\nabla_{f \cdot e} g \cdot e)(x) = \begin{cases} (f \cdot g' - g \cdot f/(2 \cdot \mathrm{id}_R))(x)e_x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

**4. Complete differentials of higher order in relation to splittings of a sequence of jet-modules.**

DEFINITION 4.1. *A splitting of the exact sequence of jet-modules*

$$0 \to L_s^k(\mathscr{X}(M, \mathscr{C}), \mathscr{W}) \to J^k(\mathscr{W}) \to J^{(k-1)}(\mathscr{W}) \to 0$$

(also a *connection* in the case $k = 1$) is an assignment

$$M \ni p \mapsto \mathscr{T}_p \subset J_p^k(\mathscr{W})$$

satisfying the conditions:

(i) $\mathscr{T}_p$ is a linear subspace of the space $J_p^k(\mathscr{W})$,

(ii) $J_p^k(\mathscr{W}) = \mathscr{T}_p \oplus \ker r_p^{k,(k-1)}$, $p \in M$,

(iii) if $P_p$: $J_p^k(\mathscr{W}) \to \ker r_p^{k,(k-1)}$ is the projection defined by the above direct sum then for $S \in J^k(\mathscr{W})$ the field

$$P(S) = (M \ni p \mapsto P_p(S_p))$$

belongs to the module $J^k(\mathscr{W})$.

THEOREM 4.1. *If $D^k$ is a complete differential of order $k$ in a linear field module $((M, \mathscr{C}), \Phi, \mathscr{W})$ and $T_p$ is a mapping linearizing this differential at a point $p \in M$, then the assignment $M \ni p \mapsto \ker T_p$ is a splitting of the exact jet-module sequence of order $k$.*

Proof. Theorem 3.4.4 states that $\ker T_p \cap \ker r_p^{k,(k-1)} = 0$. For any element $j_p^k(W) \in J_p^k(\mathscr{W})$, $W \in \mathscr{W}$, we have $j_p^k(W) = i_p^k(D_p^k(W)) + (i_p^k(-D_p^k(W)) + j_p^k(W))$ and $i_p^k(D_p^k(W)) \in \ker r_p^{k,(k-1)}$ and, by Lemma 3.4.1, $T_p(i_p^k(-D_p^k(W)) + j_p^k(W)) = -D_p^k(W) + D_p^k(W) = 0$; thus condition (ii) is fulfilled. Now consider an arbitrary field $S \in J^k(\mathscr{W})$; in a certain neighbourhood $U$ of $p$ the field $S$ is of the form $S|U = i^k(L)|U + (\sum_{j=1}^{n} f^j j^k(W_j))|U$ for a certain field

$\L \in \L^k_s(\mathcal{X}(M, \mathcal{C}), \mathcal{W})$, a number $n \in N$, functions $f^1, \ldots, f^n \in \mathcal{C}$ and fields $W_1, \ldots, W_n \in \mathcal{W}$. Hence

$$\begin{aligned} P(S)|U &= P\big(i^k(\L) + \sum_{j=1}^{n} f^j j^k(W_j)\big)|U \\ &= i^k(\L)|U + \sum_{j=1}^{n} f^j|U \cdot P\big(j^k(W_j)\big)|U \\ &= i^k(\L)|U + \sum_{j=1}^{n} f^j|U \cdot i^k\big(D^k(W_j)\big)|U \\ &= \big(i^k(\L) + \sum_{j=1}^{n} f^j \cdot i^k\big(D^k(W_j)\big)\big)|U \end{aligned}$$

is an element of the module $J^k(W)_U$. q.e.d.

THEOREM 4.2. *If $(\mathcal{T}_p)_{p\in M}$ is splitting of the exact jet-module sequence of order $k$, then there exists exactly one homomorphism of linear field modules*

$$T\colon J^k(\mathcal{W}) \to \L^k_s(\mathcal{X}(M, \mathcal{C}), \mathcal{W})$$

*such that*:

(i) $\ker T_p = \mathcal{T}_p$, $p \in M$,

(ii) $T \circ i^k = \mathrm{id}$.

*Moreover, $T \circ j^k$ is a complete differential of order $k$ in the module $\mathcal{W}$.*

Proof. Consider the projection $P_p$ and the projection $R_p$: $J^k_p(\mathcal{W}) \to \mathcal{T}_p$ defined by the direct sum $J^k_p(\mathcal{W}) = \ker r^{k,(k-1)}_p \oplus \mathcal{T}_p$, $p \in M$. Since $P_p(j^k_p(W)) \in \ker r^{k,(k-1)}_p = \operatorname{im} i^k_p$ for $W \in \mathcal{W}$, there exists exactly one element $s_p \in \L^k_s((M, \mathcal{C})'_p, \Phi_{\mathcal{W}}(p))$ associated with $W \in \mathcal{W}$ such that $P_p(j^k_p(W)) = i^k_p(s_p)$. Hence for $W \in \mathcal{W}$

$$T_p(j^k_p(W)) = T_p\big(R_p(j^k_p(W)) + P_p(j^k_p(W))\big) = T_p\big(P_p(j^k_p(W))\big) = T_p(i^k_p(s_p)) = s_p.$$

This proves the uniqueness of the mapping $T_p$ and gives the method of computing it. Now it must be proved that

$$T(S) = (M \ni p \mapsto T_p(S_p)) \in \L^k_s(\mathcal{X}(M, \mathcal{C}), \mathcal{W})$$

for any field $S \in J^k(\mathcal{W})$. As in the foregoing theorem, $S$ will be given in the form $S|U = i^k(\L)|U + \big(\sum_{j=1}^{n} f^j j^k(W_j)\big)|U$. Then

$$T(S)|U = T\big(i^k(\L) + \sum_{j=1}^{n} f^j j^k(W_j)\big)|U = \L|U + \sum_{j=1}^{n} \big(f^j \cdot T(j^k(W_j))\big)|U.$$

From the exactness of the jet-module sequence of order $k$ follows the existence of fields $\L_j$, $j = 1, \ldots, n$, from the spaces $\L^k_s(\mathcal{X}(M, \mathcal{C}), \mathcal{W})$ such that $P(j^k(W_j)) = i^k(\L_j)$. Hence

$$T(S)|U = \big(\L + \sum_{j=1}^{n} f^j \L_j\big)|U.$$

It is easy to check that $T\circ j^k$ is an $R$-linear mapping. Finally, if $f\in\mathscr{C}$, $f-f(p)\in I_p^{(k)}(M,\mathscr{C})$ and $W\in\mathscr{W}$, then from (3.4.6) we have

$$\begin{aligned} T_p\circ j_p^k(f\cdot W) &= T_p\circ j_p^k\big((f-f(p))\,W\big)+T_p\circ j_p^k(f(p)\,W)\\ &= T_p\Big(i_p^k\Big(d_p^{(k)}\big((f-f(p))\,W\big)\Big)\Big)+f(p)\,T_p\circ j_p^k(W)\\ &= d_p^{(k)}(f-f(p))\otimes W(p)+f(p)\,T_p\circ j_p^k(W). \qquad \text{q.e.d.}\end{aligned}$$

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