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# On the random version of Ważewski theorem

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**Abstract.** In this paper the proof of the generalization of the retract theorem of Ważewski on the case of $SP$-solution of random differential equation is presented.

**Introduction.** The Ważewski theorem [1] is one of the most important theorems of the qualitative theory of differential equations. The thesis of it says, that under some assumptions the differential equation has the solution totally included in the given open set $D$. Now, this theorem has a lot of generalizations and to omit the exact formulations author defines the Ważewski condition as the thesis of the Ważewski theorem. To obtain more general results the author does not assume the uniqueness of solution. To prove the random Ważewski theorem the author used the Nowak method of measurable selectors [2]. In this paper, the author obtained only the result for $SP$-solution. The result formulated for the case of autonomic equation can be in natural way extended to the case of non-autonomic equation.

## 1. Formulation of the theorem

Definition 1. The function $\bar{f}$: $\boldsymbol{R}^d \to \boldsymbol{R}^d$ satisfies in the open set $D$ the Ważewski condition if there exists such $x_0 \in D$ for which every solution of the problem

$$x'(t) = \bar{f}(x), \tag{1}$$

$$x(0) = x_0 \tag{2}$$

defined on $\boldsymbol{R}_+$ satisfies the property

$$\forall t > 0 \;\; x(t) \in D. \tag{3}$$

Let us consider the differential equation

$$\xi' = f(\xi), \tag{4}$$

where $(\Omega, \Sigma, P)$ is the given complete probabilistic space and $f$: $\boldsymbol{R}^d \times \Omega \to \boldsymbol{R}^d$ is some random function. Let $\xi_0$ be a $d$-dimensional random variable on the space $\Omega$.

DEFINITION 2. The function $\xi\colon \boldsymbol{R}_+ \times \Omega \to \boldsymbol{R}^d$ is the *SP-solution* of (4) with the initial condition

$$\xi(0) = \xi_0 \tag{5}$$

if $\xi(t, \cdot)$ is measurable for every $t \geqslant 0$ and with the probability 1 the function $\xi(\cdot, \omega)\colon \boldsymbol{R}_+ \to \boldsymbol{R}^d$ is the classical solution of the Cauchy problem

$$x'(t) = f(x, \omega), \tag{1'}$$

$$x(0) = \xi_0(\omega). \tag{2'}$$

THEOREM. *Let $f\colon \boldsymbol{R}^d \times \Omega \to \boldsymbol{R}^d$ be such that the function $f(\cdot, \omega)$ is continuous and satisfies the Ważewski condition and there exist $a_\omega$, $b_\omega \geqslant 0$ such that $|f(x, \omega)| \leqslant a_\omega |x| + b_\omega$ with the probability 1. Under these assumptions, there exists a random variable $\xi_0\colon \Omega \to \boldsymbol{R}^d$ such that $P(\xi_0 \in D) = 1$ and $P(\forall t\ \xi(t) \in D) = 1$, where $\xi$ is an SP-solution of* (4), (5).

Remark. The assumption of linear growth of $f$ guarantees the existence of the solution defined on whole half-line.

**2. Auxiliary lemmas.** Let us consider the map $\tilde{f}\colon \Omega_0 \to C(\boldsymbol{R}^d)$ defined as follows:

$$\Omega_0 = \{\omega \in \Omega\colon f(\cdot, \omega) \text{ is uniformly continuous on } \boldsymbol{R}^d\}, \tag{6}$$

$$\tilde{f}(\omega) = f(\cdot, \omega). \tag{7}$$

Clearly, $P(\Omega_0) = 1$. Let us consider the space $C(\boldsymbol{R}^d)$ with the topology of almost uniform convergence.

LEMMA 1. *The function $\tilde{f}$ is measurable.*

Proof. To prove the lemma it is sufficient to show that for every compact set $K \subset \boldsymbol{R}^d$, every $f_0 \in C(\boldsymbol{R}^d)$ and every $\varepsilon > 0$ the set $B(\varepsilon, K, f_0) = \{\omega\colon \|f(\cdot, \omega) - f_0\|_K < \varepsilon\}$ is measurable. From the properties of continuous functions,

$$\begin{aligned} B(\varepsilon, K, f_0) &= \bigcup_{n=1}^{\infty} \left\{\omega\colon \|f(\cdot, \omega) - f_0\|_K \leqslant \frac{n}{n+1}\varepsilon\right\} \\ &= \bigcup_{n=1}^{\infty} \bigcap_{x \in K \cap \boldsymbol{Q}^d} \left\{\omega\colon |f(x, \omega) - f_0(x)| \leqslant \frac{n}{n+1}\varepsilon\right\}. \ \square \end{aligned}$$

Let now $F$ be an arbitrary closed subset of $\boldsymbol{R}^d$ and let $B_M(F, t)$ be the set of all pairs $(x_0, \omega) \in \boldsymbol{R}^d \times \Omega$ for which there exists the solution of (1′), (2) satisfying the conditions

$$\forall s \in [0, t]\ |x(s)| \leqslant M, \tag{8}$$

$$x(t) \in F. \tag{9}$$

LEMMA 2. *The set $B_M(F, t)$ is measurable.*

Proof. We shall prove that the set of all pairs $(x_0, f) \in \boldsymbol{R}^d \times C(\boldsymbol{R}^d)$ for which there exists the solution of Cauchy problem satisfying (8), (9), is closed.

Let $\overset{n}{x}_0 \to x_0$ and let $f_n \to f$ almost uniformly. Let $x_n$ be a function such that

$$(10) \qquad x_n(t') = \overset{n}{x}_0 + \int_0^{t'} f(x_n(s))ds \qquad \text{for } 0 \leqslant t' \leqslant t$$

and satisfying (8), (9). From the properties of continuous functions follows that

$$(11) \qquad |x_n(t')| \leqslant \sup_n |\overset{n}{x}_0| + t \sup_{\substack{|x| \leqslant M \\ n \in N}} |f_n(x)| < \infty$$

and

$$(12) \qquad |x_n'(t)| \leqslant \sup_{\substack{|x| \leqslant M \\ n \in N}} |f_n(x)|.$$

Since $f_n \to f$ uniformly on the set $\{x: |x| \leqslant M\}$, it follows that the sequence $x_n$ contains the subsequence $x_{\alpha_n}$ uniformly convergent to the function $x$. But for every $n \in N$

$$\begin{aligned}
&\Big|x(t') - x_0 - \int_0^{t'} f(x(s))ds\Big| \\
&\leqslant |x(t') - x_{\alpha_n}(t')| + |x_0 - x_{\alpha_n}(0)| + \int_0^{t'} \big|f_{\alpha_n}(x_{\alpha_n}(s)) - f(x(s))\big|ds \\
&\leqslant |x(t') - x_{\alpha_n}(t')| + |x_0 - x_{\alpha_n}(0)| + \int_0^{t'} \big|f_{\alpha_n}(x_{\alpha_n}(s)) - f(x_{\alpha_n}(s))\big|ds \\
&\qquad + \int_0^{t'} \big|f(x_{\alpha_n}(s)) - f(x(s))\big|ds = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.
\end{aligned}$$

The components I and II are clearly convergent to zero. III $\leqslant t \,\|f_{\alpha_n} - f\|_{\{|x| \leqslant M\}} \to 0$. The convergence of IV to zero follows from the Lebesgue majorized convergence theorem and the continuity of $f$. Hence for every $t' \in [0, t]$

$$x(t') = x_0 + \int_0^{t} f(x(s))ds.$$

The observation that $x$ is the solution of Cauchy problem for the pair $(x_0, f)$ satisfying (8), (9) and using Lemma 1 complete the proof. □

**3. Proof of the theorem.** Let $\Phi(\omega)$ denote the set of all $x_0$ for which every solution of the problem (1'), (2) satisfies (3). Let $\Omega_1 = \{\omega: \Phi(\omega) \neq \emptyset\}$. From this assumption it follows that $P(\Omega_1) = 1$. We shall prove that $\Phi: \Omega_1 \to \mathscr{P}(\boldsymbol{R}^d)$ has a measurable selector. Since the space $\boldsymbol{R}^d$ is Polish, from the Sainte-Beuve theorem [3] follows that it is sufficient to prove that graph $\Phi$ is measurable.

Let $\{K_n\}$ be a sequence of closed sets such that $K_n \subset \operatorname{int} K_{n+1}$ and $D = \bigcup_{n=1}^{\infty} K_n$. Let $A(m, n)$ be the set of all pairs $(x_0, \omega)$ for which every solution of the problem (1′), (2) satisfies the condition

$$\forall t \in (0, m) \ x(t) \in K_n. \tag{13}$$

Clearly, graph $\Phi = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} A(m, n)$ and it is sufficient to show that for all positive integers $m, n$ the set $A(m, n)$ is measurable. Let us denote by $A'(n, t)$ the set of all pairs $(x_0, \omega)$ for which each solution of the problem (1′), (2) satisfies the condition

$$x(t) \in K_n. \tag{14}$$

Let $\{F_{nm}\}$ be a family of closed subsets of $\boldsymbol{R}^d$ such that $\boldsymbol{R}^d \setminus K_n = \bigcup_{m=0}^{\infty} F_{nm}$. It is obvious that

$$A'(n, t) = \bigcap_{M=1}^{\infty} \bigcap_{n=0}^{\infty} [(\boldsymbol{R}^d \times \Omega) \setminus B_M(F_{nm}, t)]$$

and as a consequence of Lemma 2 follows that $A'(n, t)$ is measurable. Since every solution of differential equation is continuous and $K_n$ are closed, the formula

$$A(m, n) = \bigcap_{\substack{t \in \boldsymbol{Q} \\ t < m}} A'(n, t) \tag{15}$$

is true and in consequence $\Phi$ has a measurable selector. Let $\xi_0$ be such selector. Now, it is sufficient to prove that (4), (5) has a $SP$-solution. Let $\psi\colon \Omega \to \mathscr{P}(C(\boldsymbol{R}_+, \boldsymbol{R}^d))$ be a multivalued map defined as follows: $\psi(\omega)$ is the set of all $x\colon \boldsymbol{R}_+ \to \boldsymbol{R}^d$ such that $x$ satisfies (1′), (2′). From the assumption follows that $\psi(\omega)$ is nonempty with probability 1. Thus, it is sufficient to prove that graph $\psi$ is measurable. Let $\hat{\psi}\colon \Omega \times C(\boldsymbol{R}_+, \boldsymbol{R}^d) \to \boldsymbol{R}_+ \cup \{\infty\}$ be defined by the formula

$$\hat{\psi}(\omega, x) = \sup_{t \geqslant 0} \left| x(t) - \xi_0(\omega) - \int_0^t f(x(s), \omega) ds \right|. \tag{16}$$

Clearly, $\hat{\psi}$ is measurable and graph $\psi = \hat{\psi}^{-1}(\{0\})$. Since $C(\boldsymbol{R}_+, \boldsymbol{R}^d)$ is a Polish space, it follows from the Sainte-Beuve Theorem [3] that $\psi$ has a measurable selector. Let $\psi\colon \Omega \to C(\boldsymbol{R}_+, \boldsymbol{R}^d)$ be this selector. Define the $SP$-solution by the formula

$$\xi(t, \omega) = \psi(\omega)(t). \quad \square \tag{17}$$

**4. Remarks.** These methods cannot be generalized on the case of $L^p$- and $W^p$-solutions. Since the set $\{\xi\colon P(\xi \in D) = 1\}$ is nowhere dense in the

$L^p$-topology, the author supposes that it is impossible to obtain the analogous result. It may be interesting to give a different formulation of the Ważewski theorem which would be true for $L^p$- and $W^p$-solutions of random differential equations and also for stochastic differential equations with Ito integral.

**References**


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