ANNALES
POLONICI MATHEMATICI
XLIII (1983)

Some vector subbundles of a cotangent bundle of order 2

by Jacek Gancarzewicz (Kraków)

Abstract. Let M be a differentiable manifold. We denote by $T_r^* M$ its cotangent boundle of order r. $T_1^* M = T^* M$ is just the cotangent bundle of a manifold M. Let β: $T_2^* M \to T^* M$ be the canonical vector bundle epimorphism. Then $K = \ker \beta$ is a vector subbundle of $T_2^* M$. The main theorem of this note is the following

THEOREM. *There is one-to-one correspondence between the set of all vector subbundles L of $T_2^* M$ such that $T_2^* M = K \oplus L$ and the set of all linear connections without torsion on M.*

1. Introduction. Let M be a manifold of dimension n. Differentiability means always differentiability of class C^∞. By $T_x M$, $T_x^* M$, TM, $T^* M$, $\mathscr{X}(M)$, $\mathscr{X}^*(M)$ and $C^\infty(M)$ we denote, respectively, the tangent space at x, its dual space, the tangent bundle, the cotangent bundle, the module of all differentiable vector fields, the module of all covector fields and the ring of all differentiable functions on M.

For a point x of M we write

$$(T_r^* M)_x = \{j_x^r f \colon f \in C^\infty(M),\ f(x) = 0\},$$

where $j_x^r f$ denotes the r-jet of f at x. $(T_r^* M)_x$ is a vector space with the natural vector space structure defined by the formula

$$a j_x^r f + a' j_x^r f' = j_x^r (af + a'f'),$$

where $a, a' \in R$, $f, f' \in C^\infty(M)$, $f(x) = f'(x) = 0$. Using the standard methods we can define a differentiable structure on

$$T_r^* M = \bigcup_{x \in M} (T_r^* M)_x$$

such that

$$\pi \colon\ T_r^* M \to M, \qquad \pi(j_x^r f) = x$$

will be a vector bundle over M. This vector bundle will be called the *cotangent bundle of order r.* $T_1^* M$ is isomorphic in a natural way to T^*M. Namely, each element $j_x^1 f$ of $T_1^* M$ defines a linear mapping

$$\widehat{j_x^1 f}\colon T_x M \ni X \to (Xf)(x) = \frac{d}{dt}(f\circ\gamma)|_{t=0} \in R,$$

where $\dot{\gamma}(0) = X$. Now, the mapping

$$T_1^* M \ni j_x^1 f \to \widehat{j_x^1 f} \in T^*M$$

defines the natural vector bundle isomorphism between $T_1^* M$ and T^*M. In the sequel we will always identify T_1^*M with T^*M using the above isomorphism.

Let (U, φ) be a chart on M and $\varphi = (x^1, \ldots, x^n)$. We define

$$\tilde{\varphi}\colon \pi^{-1}(U) \to R^n \times R^n \times \Omega_2 \times \ldots \times \Omega_r, \qquad \tilde{\varphi} = (x^i, x_i, x_{i_1 i_2}, \ldots, x_{i_1 \ldots i_r}),$$

where

$$\Omega_s = \{(a_{i_1\ldots i_s}) \in R^{n^s}\colon a_{i_1\ldots i_s} \text{ is symmetric}\}$$

and for $A = j_x^r f \in \pi^{-1}(U) \subset T_r^* M$

$$x^i(A) = x^i(x), \qquad x_{i_1\ldots i_s}(A) = \frac{\partial^s (f\circ\varphi^{-1})}{\partial x^{i_1}\ldots\partial x^{i_s}}(\varphi(x)), \qquad s = 1, \ldots, r.$$

$(\pi^{-1}(U), \tilde{\varphi})$ is now a chart on T_r^*M – it is called the *induced chart.*

In this note we will consider only cotangent bundles of order 1 and 2. We remark that in the case of cotangent bundles of order 2 we have the following "transformation formula". If (U, φ) and (U', φ') are two charts on M and if we denote

$$(\varphi'\circ\varphi^{-1})(x^1,\ldots,x^n) = (x^{1'}(x^1,\ldots,x^n),\ldots,x^{n'}(x^1,\ldots,x^n)),$$
$$(\varphi\circ\varphi'^{-1})(x^{1'},\ldots,x^{n'}) = (x^1(x^{1'},\ldots,x^{n'}),\ldots,x^n(x^{1'},\ldots,x^{n'})),$$

then $(\tilde{\varphi}'\circ\tilde{\varphi}^{-1})(x^i, x_i, x_{ij}) = (x^{i'}, x_{i'}, x_{i'j'})$, where

$$x^{i'} = x^{i'}(x^1,\ldots,x^n), \qquad x_{i'} = x_i \frac{\partial x^i}{\partial x^{i'}},$$

$$x_{i'j'} = x_{ij}\frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^j}{\partial x^{j'}} + x_i \frac{\partial^2 x^i}{\partial x^{i'}\partial x^{j'}}. \tag{1.1}$$

2. The main theorem. Let

$$\beta\colon T_2^* M \ni j_x^2 f \to j_x^1 f \in T^* M.$$

β is the canonical vector boundle epimorphism. $K = \ker\beta$ is a vector subbundle of T_2^*M, since if we denote by β_x the restriction of β to the fibre $(T_2^* M)_x$, then the function $x \to \dim\ker\beta_x$ is constant ($\dim\ker\beta_x = \binom{n+1}{2}$). If

(U, φ) is a chart on M, we can consider the induced charts on $T_2^* M$ and $T^* M$. The epimorphism β is represented in these induced charts by the mapping

$$\beta\colon (x^i, x_i, x_{ij}) \to (x^i, x_i).$$

Hence, the vector subbundle K is defined in the induced chart (x^i, x_i, x_{ij}) by the equations

$$x_i = 0, \quad i = 1, \dots, n.$$

In this note we prove the following theorem.

THEOREM. *Let $\beta\colon T_2^* M \to T^* M$ be the canonical vector boundle epimorphism defined as above and let $K = \ker \beta$. There is one-to-one correspondence between the set of all vector subbundles L of $T_2^* M$ such that $T_2^* M = K \oplus L$ and the set of all linear connections without torsion on M.*

Let us remark that a vector subbundle L of $T_2^* M$ complementary to K is isomorphic to the cotangent bundle $T^* M$.

We will prove this theorem in the next section. Firstly, for a given vector subbundle L of $T_2^* M$ complementary to K we will construct a linear connection without torsion on M. Secondly, we will prove that a connection Γ without torsion on M will determine a vector subbundle L of $T_2^* M$ complementary to K. The proof will be carried in a few steps.

3. Proof of the main theorem. We fix a vector subbundle L of $T_2^* M$ such that $K \oplus L = T_2^* M$. For a point x of M, L_x denotes a fibre of L over x. L_x is a vector space of dimension n, where $n = \dim M$.

We choose a chart (U, φ) on M. Now, the vector bundles $T_2^* M|U = \pi^{-1}(U)$ and $T^* M|U$ are trivial. The bundle $L|U$ is also trivial because L is isomorphic to $T^* M$. Thus we can find n sections of $T_2^* M|U$

$$\varkappa^1, \dots, \varkappa^n\colon U \to T_2^* M$$

such that, for every point x of U, $\varkappa^1(x), \dots, \varkappa^n(x)$ is a base of L_x.

Conversely, if we have n sections $\varkappa^1, \dots, \varkappa^n$ defined on U such that $\varkappa^1(x), \dots, \varkappa^n(x)$ are linearly indenpendent for all x, then writing

$$L = \bigcup_x L_x,$$

where L_x is the subspace of $(T_2^* M)_x$ spanned by $\varkappa^1(x), \dots, \varkappa^n(x)$, we see that L is a vector subbundle of $T_2^* M|U$ of fibre dimension n. We will say that L is spanned by sections $\varkappa^1, \dots, \varkappa^n$. We prove the following lemma:

LEMMA 1. *Let*

$$\varkappa^1, \dots, \varkappa^n\colon U \to T_2^* M$$

be sections of $T_2^* M|U$ and let

$$\varkappa^i(x) = \big(x, x_j^{(i)}(x), x_{rs}^{(i)}(x)\big) \tag{3.1}$$

be the equation of $\varkappa^i$ *in the induced chart* $(\pi^{-1}(U), \tilde{\varphi})$. (*Of course,* $x_{rs}^{(i)} = x_{sr}^{(i)}$.) *Then* $\varkappa^1, \ldots, \varkappa^n$ *span a vector subbundle* L *of* $T_2^* M|U$ *such that*

$$L \oplus K|U = T_2^* M|U$$

if and only if for all x

$$\det [x_j^{(i)}(x)]_{i,j=1,\ldots,n} \neq 0.$$

Proof. Let $e^1, \ldots, e^n$ be the canonical base of R^n and $\{E^{(r,s)}\}_{r,s=1,\ldots,n}$, $E^{(r,s)} = E^{(s,r)}$, be a canonical base of Ω. Now, the sections $\lambda^i, \lambda^{rs}, i, r, s = 1, \ldots, n$, form a base of $(T_2^* M)_x$ for all $x \in U$, where λ^i and λ^{rs} are given in the induced chart $(\pi^{-1}(U), \tilde{\varphi})$ by the formulas

$$\lambda^i(x) = (x, e^i, 0), \quad i = 1, \ldots, n, \quad \lambda^{rs}(x) = (x, 0, E^{rs}), \quad r, s = 1, \ldots, n.$$

Let us remark that λ^{rs} is a section of K for $r, s = 1, \ldots, n$.

The given sections $\varkappa^1, \ldots, \varkappa^n$ span a vector subbundle L of $T_2^* M|U$ such that $L \oplus K|U = T_2^* M|U$ if and only if the sections

$$\varkappa^i, \lambda^{rs}, \quad i, r, s = 1, \ldots, n, \tag{3.2}$$

form a base of $(T_2^* M)_x$ for all $x \in U$. The matrix which transforms the system $\{\lambda^i, \lambda^{rs}\}$ into (3.2) has the following form:

$$\left[\begin{array}{c:c} x_j^{(i)}(x) & x_{rs}^{(i)}(x) \\ \hdashline 0 & I \end{array}\right].$$

(I denotes the identity matrix of dimension $\binom{n+1}{2}$.) Thus system (3.2) forms a base of $(T_2^* M)_x$ for all $x \in U$ if and only if the above matrix is non-singular for all x. The last condition is equivalent to the following one:

$$\det [x_j^{(i)}(x)] \neq 0 \quad \text{for all } x.$$

This finishes the proof of the lemma.

We fix a vector subbundle L of $T_2^* M$ complementary to K and we fix sections $\varkappa^i\colon U \to T_2^* M$, $i = 1, \ldots, n$, which span $L|U$. We suppose that $\varkappa^i$ is given by (3.1). According to Lemma 1, for each point x of M the elements

$$\omega_i(x) = \left(x_i^{(1)}(x), \ldots, x_i^{(n)}(x)\right) \in R^n,$$

$i = 1, \ldots, n$, form a base of R^n. Thus, for each pair (r, s), the vector

$$\omega_{rs}(x) = \left(x_{rs}^{(1)}(x), \ldots, x_{rs}^{(n)}(x)\right)$$

in R^n is a linear combination of $\omega_1, \ldots, \omega_n$. Hence we can find functions Γ_{rs}^i such that

$$\omega_{rs}(x) = \Gamma_{rs}^i(x)\,\omega_i(x). \tag{3.3}$$

This is equivalent to

$$x_{rs}^{(j)}(x) = \Gamma_{rs}^{i}(x)\,x_i^{(j)}(x), \quad j, r, s = 1,\dots,n. \tag{3.3'}$$

Since $\omega_{rs} = \omega_{sr}$, we have

$$\Gamma_{rs}^{i} = \Gamma_{sr}^{i}. \tag{3.4}$$

According to our definition, the functions Γ_{rs}^{i}, $i, r, s = 1,\dots,n$, defined on U depend on L, on the sections $\varkappa^1,\dots,\varkappa^n$ and also on the chart (U, φ). The first to prove is

LEMMA 2. *The functions Γ_{rs}^{i} are independent of the choice of the sections $\varkappa^1,\dots,\varkappa^n$.*

Proof. Let $\varkappa^1,\dots,\varkappa^n$ and $\bar{\varkappa}^1,\dots,\bar{\varkappa}^n$ be two systems of sections of $T_2^* M$ which span the same vector subbundle L of $T_2^* M|U$ such that $L \oplus K|U = T_2^* M|U$. We suppose that these sections are given by the equations

$$\varkappa^i(x) = \big(x, x_j^{(i)}(x), x_{rs}^{(i)}(x)\big), \qquad \bar{\varkappa}^i(x) = \big(x, \bar{x}_j^{(i)}(x), \bar{x}_{rs}^{(i)}(x)\big),$$

and let ω_i, $\bar{\omega}_i$, ω_{rs}, $\bar{\omega}_{rs}$, Γ_{rs}^{i}, $\bar{\Gamma}_{rs}^{i}$ be vectors and functions defined respectively for $\varkappa^1,\dots,\varkappa^n$ and $\bar{\varkappa}^1,\dots,\bar{\varkappa}^n$. According to (3.3′) we have

$$x_{rs}^{(j)} = \Gamma_{rs}^{i}\,x_i^{(j)}, \qquad \bar{x}_{rs}^{(j)} = \bar{\Gamma}_{rs}^{i}\,\bar{x}_i^{(j)}.$$

Since $\varkappa^1(x),\dots,\varkappa^n(x)$ and $\bar{\varkappa}^1(x),\dots,\bar{\varkappa}^n(x)$ are two bases of L_x there are functions $A_j^i(x)$, $i, j = 1,\dots,n$, such that

$$\bar{\varkappa}^i = A_j^i \varkappa^j \tag{3.5}$$

and $A(x) = [A_j^i(x)]_{i,j}$ is a non-singular matrix for all $x \in U$. Condition (3.5) is equivalent to

$$\bar{x}_p^{(i)} = A_j^i\,x_{(p)}^{(j)}, \qquad \bar{x}_{rs}^{(i)} = A_j^i\,x_{rs}^{(j)}. \tag{3.5'}$$

Now we have two formulas

$$\bar{x}_{rs}^{(j)} = \bar{\Gamma}_{rs}^{i}\,\bar{x}_i^{(j)} = \bar{\Gamma}_{rs}^{i}\,x_i^{(p)} A_p^j, \qquad \bar{x}_{rs}^{(j)} = x_{rs}^{(p)} A_p^j = \Gamma_{rs}^{i}\,x_i^{(p)} A_p^j.$$

Hence $\Gamma_{rs}^{i} = \bar{\Gamma}_{rs}^{i}$.

LEMMA 3. *The functions Γ_{jk}^{i} are Christofel's symbols of some linear connection without torsion on M.*

Proof. Let $\varkappa^1,\dots,\varkappa^n$ be sections which span on U a vector subbundle L complementary to K. For two charts (U, φ) and (U', φ') on M the equations of $\varkappa^i$ in the induced charts are given respectively by the formulas

$$\varkappa^i(x) = \big(x, x_j^{(i)}(x), x_{rs}^{(i)}(x)\big), \qquad '\varkappa^i(x) = \big(x, x_{j'}^{(i)}(x), x_{r's'}^{(i)}(x)\big).$$

Let Γ_{rs}^{i} and $\Gamma_{r's'}^{i'}$ be the systems functions defined by means of (U, φ) and (U', φ'), respectively. Thus, by (3.3′), we have

$$x_{r's'}^{(i)} = \Gamma_{r's'}^{j'}\,x_{j'}^{(i)}, \qquad x_{rs}^{(i)} = \Gamma_{rs}^{j}\,x_j^{(i)}. \tag{3.6}$$

On account of (3.4) it is sufficient to prove[1] that

$$\Gamma^{i'}_{r's'} = \Gamma^{i}_{rs}\frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial x^{r}}{\partial x^{r'}}\frac{\partial x^{s}}{\partial x^{s'}}+\frac{\partial x^{i'}}{\partial x^{i}}\frac{\partial^2 x^{i}}{\partial x^{r'}\partial x^{s'}}. \tag{3.7}$$

According to (1.4) we have

$$'x^{(i)}_{j'} = x^{(i)}_{j}\frac{\partial x^{j}}{\partial x^{j'}},\qquad 'x^{(i)}_{r's'} = x^{(i)}_{rs}\frac{\partial x^{r}}{\partial x^{r'}}\frac{\partial x^{s}}{\partial x^{s'}}+x^{(i)}_{r}\frac{\partial^2 x^{r}}{\partial x^{r'}\partial x^{s'}},$$

and hence, by (3.6),

$$'x^{(i)}_{r's'} = \Gamma^{j'}_{r's'}\, x'x^{(i)}_{j'} = \Gamma^{j'}_{r's'}\frac{\partial x^{j}}{\partial x^{j'}}x^{(i)}_{j},$$

$$'x^{(i)}_{r's'} = x^{(i)}_{rs}\frac{\partial x^{r}}{\partial x^{r'}}\frac{\partial x^{s}}{\partial x^{s'}}+x^{(i)}_{r}\frac{\partial^2 x^{r}}{\partial x^{r'}\partial x^{s'}} = \left(\Gamma^{j}_{rs}\frac{\partial x^{r}}{\partial x^{r'}}\frac{\partial x^{s}}{\partial x^{s'}}+\frac{\partial^2 x^{r}}{\partial x^{r'}\partial x^{s'}}\right)x^{(i)}_{j}.$$

From the above two formulas it follows that

$$\Gamma^{j'}_{r's'}\frac{\partial x^{i}}{\partial x^{j'}} = \Gamma^{i}_{rs}\frac{\partial x^{r}}{\partial x^{r'}}\frac{\partial x^{s}}{\partial x^{s'}}+\frac{\partial^2 x^{i}}{\partial x^{r'}\partial x^{s'}}$$

and this formula is equivalent to (3.7). The proof of Lemma 3 is now complete.

Our three lemmas imply the following

PROPOSITION. *If L is a vector subbundle of $T_2^* M$ such that $L \oplus K = T_2^* M$, then L defines a linear connection Γ without torsion on M. The Christofel's symbols of Γ are given by* (3.3′).

Let Γ be any linear connection without torsion on M. To finish the proof of the main theorem we must give a method of reconstruction of a vector subbundle L of $T_2^* M$.

We choose a chart (U, φ) on M. Since $T^*M|U$ is a trivial bundle, we can choose 1-forms $\omega^1,\ldots,\omega^n$ on U such that $\omega^1(x),\ldots,\omega^n(x)$ is a base of $T_x^* M$ for every $x \in U$. Let

$$\omega^i(x) \in x^{(i)}_{j}(x)\, dx^j.$$

Now, $\det[x^{(i)}_{j}(x)]_{i,j} \neq 0$. We define the sections

$$\varkappa^i\colon U \to T_2^* M,\qquad i = 1,\ldots,n,$$

---

[1] S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, vol. I, New York–London 1963.

in the induced chart $(\pi^{-1}(U), \tilde{\varphi})$ by the formula

$$\varkappa^i(x) = \big(x, x_j^{(i)}(x), x_{rs}^{(i)}(x)\big), \qquad x_{rs}^{(i)}(x) = \Gamma_{rs}^j(x)\, x_j^{(i)}(x),$$

where Γ_{rs}^j denote Christofel's symbols of Γ in (U, φ). By Lemma 1, the sections $\varkappa^1, \ldots, \varkappa^n$ span a vector subbundle L of $T_2^* M|U$ such that $L \oplus K|U = T_2^* M|U$. We now prove further two lemmas.

LEMMA 4. *L is independent of ω^i, $i = 1, \ldots, n$.*

Proof. Let $\bar{\omega}^i = \bar{x}_j^{(i)} dx^j$ be another system of 1-forms on U such that $\bar{\omega}^1(x), \ldots, \bar{\omega}^n(x)$ is a base of $T_x^* M$ for all x in U, and let $\bar{\varkappa}^1, \ldots, \bar{\varkappa}^n$ be the sections of $T_2^* M|U$ defined by $\bar{\omega}^1, \ldots, \bar{\omega}^n$. There is a matrix $A(x) = [A_j^i(x)]_{i,j}$ such that

$$\bar{\omega}^i = A_j^i \omega^j,$$

that is,

$$\bar{x}_p^{(i)} = A_j^i x_p^{(j)}.$$

Now,

$$\bar{x}_{rs}^{(i)} = \Gamma_{rs}^j \bar{x}_j^{(i)} = \Gamma_{rs}^j x_j^{(p)} A_p^i = A_p^i x_{rs}^{(p)};$$

and hence

$$\bar{\varkappa}^{(i)} = A_j^i \varkappa^{(j)}.$$

This means that $\bar{\varkappa}^{(1)}, \ldots, \bar{\varkappa}^{(n)}$ span the same vector subbundle as of the sections $\varkappa^1, \ldots, \varkappa^n$.

LEMMA 5. *L is independent of the chart (U, φ).*

Proof. Let (U, φ) and (U', φ') be two charts on M. We choose 1-forms $\omega^1, \ldots, \omega^n$ on $U \cap U'$ such that $\omega^1(x), \ldots, \omega^n(x)$ is a base of $T_x^* M$ for all $x \in U \cap U'$. If

$$\omega^i = x_j^{(i)} dx^j = x_{j'}^{(i)} dx^{j'},$$

then

$$x_{j'}^{(i)} = x_j^{(i)} \frac{\partial x^j}{\partial x^{j'}}.$$

We define sections $\varkappa^1, \ldots, \varkappa^n$ and $'\varkappa^1, \ldots, '\varkappa^n$ using the induced charts $(\pi^{-1}(U), \tilde{\varphi})$ and $(\pi^{-1}(U'), \tilde{\varphi}')$, respectively. Thus

$$\varkappa^i(x) = \big(x, x_j^{(i)}(x), x_{rs}^{(i)}(x)\big), \qquad '\varkappa^i(x) = \big(x, x_{j'}^{(i)}(x), x_{r's'}^{(i)}(x)\big),$$

where

$$x_{rs}^{(i)} = \Gamma_{rs}^j x_j^{(i)}, \qquad x_{r's'}^{(i)} = \Gamma_{r's'}^{j'} x_{j'}^{(i)}$$

and Γ^j_{rs}, $\Gamma^{j'}_{r's'}$ are Christoffel's symbols of Γ with respect to (U, φ) and (U', φ'). Now,

$$'x^{(i)}_{r's'} = \Gamma^{j'}_{r's'}\,'x^{(i)}_{j'} = \left\{\Gamma^j_{rs}\frac{\partial x^{j'}}{\partial x^j}\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^s}{\partial x^{s'}} + \frac{\partial x^{j'}}{\partial x^j}\frac{\partial^2 x^j}{\partial x^{r'}\partial x^{s'}}\right\}\frac{\partial x^p}{\partial x^{j'}}x^{(i)}_p$$

$$= \Gamma^j_{rs}x^{(i)}_j\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^s}{\partial x^{s'}} + x^{(i)}_j\frac{\partial^2 x^j}{\partial x^{r'}\partial x^{s'}} = x^{(i)}_{rs}\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^s}{\partial x^{s'}} + x^{(i)}_j\frac{\partial^2 x^j}{\partial x^{r'}\partial x^{s'}}.$$

By (1.4) it follows that $'\varkappa^i = \varkappa^i$. This finishes the proof of Lemma 5.

The above two lemmas finish the proof of the main theorem, because using a covering of M by charts we can define a global vector subbundle L of $T_2^* M$. Of course, $L \oplus K = T_2^* M$ (we employ here Lemma 1) and if we construct for L a linear connection as in Proposition, we obtain the connection Γ.

4. Remark. In an analogous way we can formulate a theorem which gives an interpretation of any linear connection (with torsion) on M. To this end, it is sufficient to replace the cotangent bundle $T_2^* M$ of order 2 by a semi-holonomic cotangent bundle $\tilde{T}_2^* M$ of order 2. The bundle $\tilde{T}_2^* M$ is defined as follows.

Let $\operatorname{atl}(M)$ be an atlas on a given manifold M, $n = \dim M$. In the disjoint union

$$\mathcal{T} = \bigcup_{(U,\varphi)\in\operatorname{atl}(M)} U \times \{\varphi\} R^n \times R^{n^2}$$

we define an equivalence relation "$\sim$" by the formula

$$(x, \varphi, x_i, x_{rs}) \sim (x', \varphi', x_{i'}, x_{r's'})$$

if and only if $x = x'$ and

$$x_{i'} = x_i \frac{\partial x^i}{\partial x^{i'}}(x), \qquad x_{r's'} = x_{rs}\frac{\partial x^r}{\partial x^{r'}}\frac{\partial x^s}{\partial x^{s'}} + x_i\frac{\partial^2 x^i}{\partial x^{r'}\partial x^{s'}},$$

where $(\varphi \circ \varphi'^{-1})(x^{1'}, \ldots, x^{n'}) = \left(x^i(x^{1'}, \ldots, x^{n'})\right)_{i=1,\ldots,n}$ (cf. formula (1.1)). Now

$$\tilde{T}_2^* M = \mathcal{T}/\sim, \qquad \pi\colon \tilde{T}_2^* M \to M, \qquad \pi\langle x, \varphi, x_i, x_{rs}\rangle = x,$$

where $\langle x, \varphi, x_i, x_{rs}\rangle$ denotes the equivalence class in $\tilde{T}_2^* M$ of $(x, \varphi, x_i, x_{rs})$, is a vector bundle with natural linear structures on fibres. Let

$$\tilde{\beta}\colon \tilde{T}_2^* M \ni \langle x, \varphi, x_i, x_{rs}\rangle \to x_i dx^i \in T^* M.$$

$\tilde{\beta}$ is well defined and it is a vector bundle epimorphism. $T_2^* M$ can be considered as a vector subbundle of $\tilde{T}_2^* M$. $T_2^* M$ is given by the equations

$$x_{rs} = x_{sr}, \qquad r, s = 1, \ldots, n.$$

Then $\tilde{\beta}|T_2^* M = \beta$. Let $\tilde{K} = \ker \tilde{\beta}$. $\bar{K}$ is a vector subbundle of $\tilde{T}_2^* M$. Using the same arguments as in the proof of our main theorem we can state

THEOREM 2. *There is one-to-one correspondence between the set of all vector subbundles L of $\tilde{T}_2^* M$ such that $L \oplus K = \tilde{T}_2^* M$ and the set of all linear connections on M.*

INSTITUTE OF MATHEMATICS,
JAGIELLONIAN UNIVERSITY, KRAKOW

Reçu par la Rédaction le 13.1.1979