ANNALES
POLONICI MATHEMATICI
XII (1962)

On the limits of moduli of analytic functions

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In affectionate memory
of Professor Mieczysław Biernacki

1. Introduction. Suppose that $f(z)$ is regular and bounded in a half-strip S: $a \leqslant x \leqslant \beta$, $y \geqslant 0$, and that $a < a < \beta$. Then it follows from a classical theorem of Montel that if

$$f(a+iy) \to l, \quad \text{as} \quad y \to +\infty,$$

then

$$f(x+iy) \to l, \quad \text{as} \quad y \to +\infty,$$

uniformly for $a+\delta \leqslant x \leqslant \beta-\delta$, whenever $\delta > 0$.

Hardy, Ingham and Pólya [2] considered the corresponding problem for $|f(z)|$. They noted that the analogue of Montel's theorem is false in general since for instance $f(z) = e^{\sinh z}$ is bounded in any S, but $|f(x+iy)|$ approaches a limit as $y \to +\infty$ if and only if $x = 0$.

However they showed that in certain circumstances the analogue of Montel's theorem holds if $|f(x+iy)|$ approaches a limit as $y \to \infty$ for two distinct values of x. More precisely their main results are contained in

THEOREM A. *Suppose that $f(z)$ is bounded in S, that $a < a < b < \beta$, and that*

$$|f(a+iy)| \to A, \tag{1}$$

$$|f(b+iy)| \to B \tag{2}$$

as $y \to +\infty$. Then

$$|f(x+iy)| \to A^{(b-x)/(b-a)} B^{(x-a)/(b-a)} \tag{3}$$

as $y \to +\infty$, uniformly for $a+\delta \leqslant x \leqslant \beta-\delta$, provided that either

$$f(z) \neq 0 \quad \text{in} \quad S \tag{4}$$

or

$$b-a \leqslant \tfrac{1}{2}(\beta-a). \tag{5}$$

They also showed by examples that the condition (5) is best possible in the above result, in the sense that if a, b are any numbers such that

$a < a < b < \beta$ and $b-a > \frac{1}{2}(\beta - a)$ then there exists a function $f(z)$ satisfying (1) and (2) but such that $|f(x+iy)|$ does not approach a limit as $y \to +\infty$ for any x in $a < x < \beta$ other than $x = a$ or $x = b$.

More recently Cartwright [1] showed that (5) can be relaxed somewhat if something is known about the precise bounds for $|f(z)|$ in S and their relation to A and B. She proved for instance

THEOREM B. *Suppose that* (1) *and* (2) *hold with* $A = B = 1$ *and that* $|f(z)| < M$ *in* S, *where*

$$M < \min\left\{\frac{\beta - a}{a+\beta-2a}, \frac{\beta-a}{2b-a-\beta}\right\}.$$

Then (3) *holds.*

THEOREM C. *If* $f(z)$ *satisfies* (1) *and* $|f(z)| < A$ *in* S, *then* (3) *holds with* $A = B$.

2. Statement of results. It is the aim of this paper to prove a more precise form of Theorem B.

We suppose that $f(z)$ is bounded in S, continuous in the closure of S (1), that

(6) $$a < a < b < \beta, \quad \text{and} \quad (b-a) > \tfrac{1}{2}(\beta - a),$$

and further that (1) and (2) hold and that

(7) $$|f(a+iy)| \leqslant M_1, \quad |f(b+iy)| \leqslant M_2, \quad 0 \leqslant y < +\infty$$

and ask under what circumstances these conditions imply (3). Following Hardy, Ingham and Pólya, we note that w. l. g. we may assume that $A = B = 1$, since the general case reduces to this by means of the substitution

$$F(z) = A^{-(b-z)/(b-a)}B^{-(z-a)/(b-a)}f(z).$$

Also (7) is equivalent to

$$|F(a+iy)| \leqslant M_1', \quad |F(b+iy)| \leqslant M_2',$$

where

$$M_1' = M_1 A^{-(b-a)/(b-a)}B^{-(a-a)/(b-a)},$$
$$M_2' = M_2 A^{(\beta-b)/(b-a)}B^{-(\beta-a)/(b-a)}.$$

Further we may suppose that $M_1 > 1$, and $M_2 > 1$ if (1) and (2) hold with $A = B = 1$. For if $M_1 \leqslant 1$, it follows from Theorem C, that (3) holds uniformly for $a+\delta \leqslant x \leqslant b-\delta$, and hence from Theorem A that (3) holds also for $a+\delta \leqslant x \leqslant \beta-\delta$, since we may now apply (5), with $a, (a+b)/2$ instead of a, b. Thus Theorem A certainly holds in this case.

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(1) This condition is not essential but makes it possible to write down (7) more simply.

Finally we assume w. l. g. that $a = -\pi/4$, $b = \pi/4$, since this may be achieved by a linear transformation in the z-plane. Then our results are the following

THEOREM I. *Suppose that $f(z)$ satisfies the hypotheses of Theorem* A *with $A = B = 1$ and $a = -\pi/4$, $b = \pi/4$, and further that* (6) *and* (7) *hold, where $M_1 > 1$, $M_2 > 1$. Then* (3) *holds provided that*

$$\frac{M_1+M_2}{1+M_1M_2} > \frac{\sin(\beta-\alpha)}{\cos(\alpha+\beta)} . \tag{8}$$

THEOREM II. *If α, β, M_1, M_2 are any numbers satisfying the other conditions of Theorem* I *but not* (8), *then there exists $f(z)$ satisfying the hypotheses but not the conclusion of Theorem* I. *More precisely $f(z)$ has infinitely many zeros near a certain ray $x = x_0$, $y > 0$, where $|x_0| < \pi/4$.*

3. Proof of Theorem II. We note that $\tan z$ is regular in the strip $|x| \leqslant \pi/4$, and that

$$|\tan z| = 1, \quad \text{for} \quad x = \mp\pi/4 .$$

Further

$$\tan z \to \mp i, \quad \text{as} \quad y \to \mp\infty ,$$

uniformly in x. For $-\pi/4 < x_0 < \pi/4$, set

$$\psi(z, x_0) = \frac{\tan z - \tan x_0}{1-\tan z \tan x_0} . \tag{9}$$

Then

$$|\psi(z, x_0)| = 1, \quad \text{for} \quad x = \mp\pi/4 ,$$

and

$$\begin{aligned} &\psi(z, x_0) \to Z_0 = \frac{-(i+\tan x_0)}{1+i\tan x_0}, &&\text{as} \quad y \to -\infty , \\ &\psi(z, x_0) \to \bar{Z}_0 , &&\text{as} \quad y \to +\infty , \end{aligned} \tag{10}$$

uniformly in x and in a range, $-\pi/4+\delta \leqslant x_0 \leqslant \pi/4-\delta$.

Let K_n be a sequence of positive numbers such that

$$1 < K_n \leqslant 2 , \quad n = 1, 2, \ldots \quad \text{and } K_n \to 1 , \quad \text{as} \quad n \to +\infty , \tag{11}$$

and y_n a rapidly increasing sequence of positive numbers and set

$$f(z) = \prod_{n=1}^{\infty} \left\{ \psi\left[\frac{z-iy_n}{K_n}, x_0\right] \Big/ Z_0 \right\} . \tag{12}$$

It follows from (10) that given x_0, so that $-\pi/4 < x_0 < \pi/4$, and an integer $p \geqslant 2$ we may choose Y_n positive, and such that

$$|1-\psi(x+iy, x_0)/Z_0| < p^{-n}, \tag{13}$$

provided $y < -Y_n$. Thus the product (12) converges on every compact set provided that

$$y_n > 3Y_n, \quad n = 1 \text{ to } \infty.$$

We now assume further that

$$(14) \qquad y_{n+1} - y_n > 4Y_{n+1}, \quad n = 1 \text{ to } \infty.$$

Suppose that y is real and that m is the least integer such that $y_{m+1} - 2Y_{m+1} \geqslant y$. Thus if $m > 0$, we have

$$y_m - 2Y_m < y \leqslant y_{m+1} - 2Y_{m+1}.$$

Then we deduce from (11), (13) and (14) that if $n > m$, and $z = x + iy$, then

$$\frac{y - y_n}{K_n} \leqslant -Y_n,$$

and so

$$1 - p^{-n} < \left|\psi\left(\frac{z - iy_n}{K_n}, x_0\right)\right| \leqslant 1 + p^{-n}, \quad n \geqslant m+1.$$

Also if $n < m$,

$$\frac{y - y_n}{K_n} \geqslant Y_m,$$

and by taking complex conjugates in (13) we deduce that

$$1 - p^{-m} < \left|\psi\left(\frac{z - iy_n}{K_n}, x_0\right)\right| \leqslant 1 + p^{-m}, \quad n = 1 \text{ to } m-1.$$

We deduce that independently of x we have the inequality

$$(15) \qquad 1 - \sigma_m < \left|\prod_{\substack{n=1 \\ n \neq m}}^{\infty} \left[\psi\left(\frac{z - iy_n}{K_n}, x_0\right)\Big/ Z_0\right]\right| < 1 + \sigma_m,$$

where

$$\sigma_m = (1 + p^{-m})^m \prod_{n=m+1}^{\infty} (1 + p^{-n}) - 1,$$

so that

$$\sigma_m \to 0, \quad \text{as} \quad m \to \infty.$$

Further if $x = \mp \pi/4$, we have for large m

$$1 \leqslant \left|\psi\left(\frac{z - iy_m}{K_m}, x_0\right)\right| \leqslant \left|\psi\left(\frac{\pi}{4K_m}, x_0\right)\right| \to 1, \quad \text{as} \quad m \to +\infty.$$

Thus we deduce that regardless of the choice of the constants K_n, or the y_n subject to (11) and (14) we have that

$$(16) \qquad |f(x + iy)| \to 1, \quad \text{as} \quad y \to +\infty,$$

for $x = \mp \pi/4$.

Consider next $f(a+iy)$, where a is a fixed number in the range $-\pi/2-x_0 < a < -\pi/4$. Suppose again that m is defined as before. Then for $z = a+iy$

$$\left|\psi\left(\frac{z-iy_m}{K_m}, x_0\right)\right| \leqslant \max\left[1, \left|\psi\left(\frac{a}{K_m}, x_0\right)\right|\right] < |\psi(a, x_0)| .$$

Also

$$\prod_{\substack{n=1\\ n\neq m}}^{\infty}\left|\psi\left(\frac{z-iy}{K_n}, x_0\right)\right| \leqslant 1+\sigma_m .$$

Thus our function $f(z)$ satisfies for $z = a+iy$

$$|f(x+iy)| \leqslant (1+\sigma_m)\max\{1, \psi(a/K_m, x_0)\} . \tag{17}$$

This inequality holds also for $|a| \leqslant \pi/4$, so that $f(z)$ is uniformly bounded in any strip $-\pi/2-x_0+\delta \leqslant x \leqslant \pi/2-x_0-\delta$. For $|a| > \pi/4$ in this range we first choose the integer p which occurs in σ_m so large that

$$(1+\sigma_m)\max\{1, |\psi(a/2, x_0)|\} < |\psi(a, x_0)|$$

for all m. Since $\sigma_m \to 1$ as $m \to +\infty$, it follows that we can then choose K_m to satisfy the conditions (11) for any fixed a in the range $-\pi/2-x_0 < a < -\pi/4$, so that (17) yields

$$|f(a+iy)| \leqslant |\psi(a, x_0)| , \quad \text{all real } y. \tag{18}$$

Simultaneously we can ensure that for a fixed real β in the range $\pi/4 < \beta < \pi/2-x_0$, we have

$$|f(\beta+iy)| \leqslant |\psi(\beta, x_0)| , \quad \text{all real } y. \tag{19}$$

Thus we have found a function $f(z)$, satisfying the hypotheses of Theorem I, except possibly (8) with $M_1 = |\psi(a, x_0)|$, $M_2 = |\psi(\beta, x_0)|$. The conclusions of Theorem I are not satisfied, since $f(z)$ has infinitely many zeros near the line $x = x_0$. To conclude the proof of Theorem II we note that given M_1, M_2 we can find x_0 in the range $-\pi/4 < x_0 < \pi/4$ as required, provided that

$$\psi(\beta, x_0) = \frac{\tan\beta - \tan x_0}{1-\tan\beta\tan x_0} = M_2 , \quad \psi(a, x_0) = \frac{\tan a - \tan x_0}{1-\tan a\tan x_0} = -M_1 ,$$

$$\tan x_0 = \frac{\tan\beta - M_2}{1-\tan\beta M_2} = \frac{\tan a + M_1}{1+\tan a M_1},$$

i.e. provided that

$$\frac{M_1+M_2}{M_1M_2+1} = \frac{\tan\beta-\tan a}{1-\tan a\tan\beta} = \frac{\sin(\beta-a)}{\cos(a+\beta)} .$$

This proves Theorem II in case equality holds in (8). The general case follows also since we may replace M_1, M_2 by larger numbers and the L.H.S. of (8) is a decreasing function of each of M_1, and M_2, for $M_1 > 1$, and $M_2 > 1$.

When M_1 and $\beta - a$ are given the value of M_2 is largest when $a + \beta = 0$, so that the inner strip is symmetrical w.r.t. the outer one. If $M_1 = M_2 = M$, we obtain in this critical case

$$\frac{2M}{1+M^2} = \sin(\beta - a)\,, \qquad \text{so that} \qquad M = \tan\frac{(\beta - a)}{2}.$$

Thus if $\beta - a < 2(b-a)$, the conclusion of Theorem I does not hold unless

$$M < \tan\frac{(\beta - a)}{2}\,,$$

in any case and if $a + \beta \neq 0$ this condition can be sharpened.

4. Proof of Theorem I. We now assume that the hypotheses of Theorem I are satisfied, and show that in this case $f(z)$ can have at most a finite number of zeros in the half strip $|x| \leqslant \pi/4$, $y > 0$. The desired conclusion then follows.

Consider for $z = x + iy$

$$\varphi(z) = f(z)\overline{f(-\pi/2 - \bar{z})} = f(x+iy)\overline{f(-\pi/2 - x + iy)}\,.$$

Then our hypotheses imply that $\varphi(z)$ is regular and bounded in the half-strip, $a \leqslant x \leqslant -a - \pi/2$, $y \geqslant 0$, and

$$\varphi(x+iy) \to 1\,, \quad \text{as} \quad y \to +\infty \tag{20}$$

for $x = -\pi/4$. Thus by Montel's Theorem (20) holds uniformly in the range $a + \delta \leqslant x \leqslant -a - \pi/2 - \delta$, when δ is any small positive number.

Also we have for sufficiently large positive y by the Phragmén-Lindelöf principle and our hypotheses,

$$\begin{aligned} |f(x+iy)| &\leqslant M_1(1+\delta)\,, & a \leqslant x \leqslant -\pi/4\,, \\ |f(x+iy)| &\leqslant 1 + \delta\,, & -\pi/4 \leqslant x \leqslant \pi/4\,, \\ |f(x+iy)| &\leqslant M_2(1+\delta)\,, & \pi/4 \leqslant x \leqslant \beta\,. \end{aligned} \tag{21}$$

We deduce that for $y > y_0(\delta)$, we have the inequality (21) and also (20) yields

$$|f(x+iy)| \geqslant \frac{1}{M_1(1+\delta)^2}\,, \quad -\frac{\pi}{4} \leqslant x \leqslant -a - \frac{\pi}{2} - \delta\,, \tag{22}$$

and similarly

$$|f(x+iy) \geqslant \frac{1}{M_2(1+\delta)^2}\,, \quad \frac{\pi}{2} - \beta + \delta \leqslant x \leqslant \frac{\pi}{4}. \tag{23}$$

We now assume, after a shift in the origin if necessary, that $f(z)$ is regular and satisfies the inequalities (21), (22) and (23) in the half-strip $-\pi/4 \leqslant x \leqslant \pi/4$, $y > -1/\delta$. We shall see that if (8) holds and δ is sufficiently small this implies that $f(z)$ has no zeros in $|x| \leqslant \pi/4$, $y \geqslant 0$.

Suppose in fact that there is such a zero z_0. We may assume w.l.g. that $z_0 = x_0$ is real since otherwise we can apply the argument to $f(z-iy_0)$ instead of $f(z)$. Consider

$$F(z) = f(z)/\psi(z, x_0)\,,$$

where $\psi(z, x_0)$ is given by (9). Then on the rays $|x| = \pi/4$, $y \geqslant -1/\delta$, $|\psi(z, x_0)| = 1$. Also by (22) and (23) we must have

$$-a-\pi/2-\delta \leqslant x_0 \leqslant \pi/2-\beta+\delta\,,$$

so that by (10) $|\psi(z, x_0)| > 1-\varepsilon$, for $|y| \geqslant 1/\delta$, where ε is small if δ is small. In view of (21) we deduce that

$$|F(z)| \leqslant \frac{1+\delta}{1-\varepsilon}$$

on the boundary of our half strip and so inside.

Thus

$$|f(z)| \leqslant \frac{1+\delta}{1-\varepsilon}\,|\psi(z, x_0)|$$

in our half strip. Setting in turn $z = -a-\pi/2-\delta$, $\pi/2-\beta+\delta$, we obtain from (22) and (23)

$$\left|\psi\left(-a-\frac{\pi}{2}-\delta\,,\; x_0\right)\right| \geqslant \frac{1-\varepsilon}{(1+\delta)^3 M_1}\,, \qquad \left|\psi\left(\frac{\pi}{2}-\beta+\delta\,,\; x_0\right)\right| \geqslant \frac{1-\varepsilon}{(1+\delta)^3 M_2}$$

and so

$$|\psi(a+\delta\,,\, x_0)| = \frac{1}{|\psi(-a-\pi/2-\delta\,,\; x_0)|} \leqslant \frac{M_1(1+\delta)^3}{(1-\varepsilon)}\,,$$

$$|\psi(\beta-\delta\,,\, x_0)| \leqslant \frac{M_2(1+\delta)^3}{1-\varepsilon}\,.$$

In view of (9) this in turn yields

$$\frac{(1-\varepsilon)}{(1+\delta)^3}\cdot\frac{M_1+M_2}{1+M_1M_2} \leqslant \frac{\psi(\beta-\delta\,, x_0)-\psi(a+\delta\,, x_0)}{1-\psi(a+\delta\,, x_0)\psi(\beta-\delta\,, x_0)} = \frac{\sin(\beta-a-2\delta)}{\cos(a+\beta)}\,.$$

This contradicts (8) if δ and hence ε is sufficiently small. We deduce that with the hypotheses of Theorem II $f(z)$ has no zeros in $|x| \leqslant \pi/4$, $y \geqslant y_0$, if y_0 is sufficiently large. By applying the Phragmén-Lindelöf principle in this half-strip to $f(z)$ and $f(z)^{-1}$, we deduce from (1) and (2) that (3) holds uniformly for $|x| \leqslant \pi/4$. Using Theorem A we deduce that (3) holds for $a+\delta \leqslant x < -\pi/4$ and similarly for $\pi/4 \leqslant x \leqslant \beta-\delta$. This completes the proof of Theorem I.

I am indebted to Dr. M. L. Cartwright for acquainting me with the problem, showing me her M. S. [1] and for helpful advice and discussion.

References

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Reçu par la Rédaction le 12. 10. 1961