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Extreme points and support points of subordination families with p-valent majorants

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Abstract. Let S_p^*, K_p and C_p denote the p-valent starlike, convex and close-to-convex analytic functions in the unit disc Δ normalized so as to have p zeros at the origin. We study the classes defined by subordination to S_p^*, K_p and C_p. We determine the support points of these classes and the extreme points of their closed convex hulls. This information is used to solve extremal problems.

Introduction. Let $\mathbf{A}$ denote the set of functions analytic in the open unit disk $\Delta = \{z: |z| < 1,\ z \in \mathbf{C}\}$. Then $\mathbf{A}$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of Δ. A function f in $\mathbf{A}$ is said to be *subordinate* to a function F in $\mathbf{A}$ (written $f \prec F$), if there is a function φ in B_0 such that $f = F \circ \varphi$, where $B_0 = \{\varphi: \varphi \in \mathbf{A},\ \varphi(0) = 0,\ |\varphi(z)| < 1\}$. For $F \in \mathbf{A}$ we let $s(F)$ denote the set $\{f: f \prec F\}$ and note that $s(F)$ is a compact subset of $\mathbf{A}$.

Let $\mathbf{F}$ be a compact subset of $\mathbf{A}$. A function f in $\mathbf{F}$ is a support point of $\mathbf{F}$ if there is a continuous linear functional J on $\mathbf{A}$ such that $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g): g \in \mathbf{F}\}$ and $\operatorname{Re} J$ is non-constant on $\mathbf{F}$. We let $s(\mathbf{F}) = \{f: f \prec F \text{ for some } F \in \mathbf{F}\}$ and note that $s(\mathbf{F})$ is compact. We [17] let $H\mathbf{F}$, $EH\mathbf{F}$ and $\operatorname{supp} \mathbf{F}$ denote respectively the closed convex hull of $\mathbf{F}$, the set of extreme points of the closed convex hull of $\mathbf{F}$, and the set of support points of $\mathbf{F}$. We let $\operatorname{supp}\{\mathbf{F}, J\}$ denote the support points of $\mathbf{F}$ associated with a specific continuous linear functional J.

Let S denote the set of functions in $\mathbf{A}$ that are univalent and satisfy $f(0) = 0$ and $f'(0) = 1$. Let N denote the natural numbers and $S^p = \{f^p: f \in S,\ p \in N\}$. Let S_p^* denote the family of p-valent starlike functions with power series expansion $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ $(z \in \Delta)$. It is known that $f \in S_p^*$ if and only if $f = g^p$ where $g \in S$ and $g(\Delta)$ is starlike with respect to the origin [19]. A function f is said to be in the *family of p-valent convex functions* which we denote by K_p if and only if $zf'(z)/p \in S_p^*$. It is known that $K_p \subset S_p^*$ [8]. It is known ([11], p. 341, [16]) that the class C_p of p-valent close-to-convex functions consists of those functions $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$ $(z \in \Delta)$ for which there exist $g \in S_p^*$ and an α, $0 \leqslant \alpha < 2\pi$ such that

$$\operatorname{Re} e^{i\alpha} \frac{zf'(z)}{g(z)} > 0 \quad \text{for } 0 < |z| < 1.$$

Clearly $S_p^* \subset C_p$. In [11], p. 346, it was essentially proved that $C_p \subset S^p$. The sets HS_p^*, EHS_p^*, HK_p, EHK_p, HC_p and EHC_p were determined in [11].

In this paper we first determine the sets $Hs(S_p^*)$, $EHs(S_p^*)$, $Hs(K_p)$, $EHs(K_p)$, $Hs(C_p)$ and $EHs(C_p)$. We use this information to significantly improve results found in [11] dealing with coefficient and integral mean problems. We also determine the sets $\operatorname{supp} s(S_p^*)$ and $\operatorname{supp} s(K_p)$. For $p = 1$ these results were proved in [12]. In the case of $s(K_p)$ our results for $p \geqslant 2$ provide a striking contrast to the result found in [11], p. 531, when $p = 1$. We also prove that $\operatorname{supp} s(C_p) \subset EHs(C_p)$. This result for $p = 1$ was proved in [2]. All our arguments provide a different proof for the case $p = 1$ than offered previously.

1. Closed convex hulls and extreme points. It is known [11] that

$$HS_p^* = \left\{ \int_{\partial\Delta} \frac{z^p}{(1-xz)^{2p}} d\mu(x):\ \mu \in \Lambda_1 \right\},$$

where Λ_1 denotes the set of probability measures on $\partial\Delta$ and that

$$EHS_p^* = \left\{ \frac{z^p}{(1-xz)^{2p}}:\ |x| = 1 \right\}.$$

The next lemma is a technical tool needed for determining $Hs(S_p^*)$. Let Λ_2 be the set of probability measures $\partial\Delta \times \partial\Delta$.

LEMMA 1. *For any natural number* γ *let*

$$G_\gamma = \left\{ \int_{\partial\Delta \times \partial\Delta} \frac{xz^\gamma}{(1-yz)^{2\gamma}} d\mu(x, y):\ \mu \in \Lambda_2 \right\}.$$

If $f \in G_\alpha$ *and* $g \in G_\beta$, *then* $fg \in G_{\alpha+\beta}$.

Proof. The proof of this lemma follows directly from Theorem 1 ([4], p. 415) and the techniques used to prove Theorem 6 ([3], p. 97).

THEOREM 2. *Let* $s(S_p^*) = \{F \circ \varphi:\ F \in S_p^*,\ \varphi \in B_0\}$. *Then*

$$Hs(S_p^*) = \left\{ \int_{\partial\Delta \times \partial\Delta} \frac{xz^p}{(1-yz)^{2p}} d\mu(x, y):\ \mu \in \Lambda_2 \right\} \tag{1}$$

and

$$EHs(S_p^*) = \left\{ \frac{xz^p}{(1-yz)^{2p}}: |x| = |y| = 1 \right\}. \tag{2}$$

Proof. Let $f \in s(S_p^*)$. It follows that there exists $\mu_1 \in \Lambda_2$ such that

$$f(z) = \left[\int_{\partial\Delta \times \partial\Delta} \frac{xz}{(1-yz)^2} d\mu_1(x, y) \right]^p$$

([12], p. 458; [19]). Now $p - 1$ applications of Lemma 1 imply that

$$f(z) = \int_{\partial\Delta \times \partial\Delta} \frac{xz^p}{(1-yz)^{2p}} d\mu(x, y) \tag{3}$$

for some $\mu \in \Lambda_2$. Hence (3) implies

$$s(S_p^*) \subset \mathbf{F} = \left\{ \int_{\partial\Delta \times \partial\Delta} \frac{xz^p}{(1-yz)^{2p}} d\mu(x, y): \ \mu \in \Lambda_2 \right\}$$

and since the latter set is closed and convex also $Hs(S_p^*) \subset \mathbf{F}$. The kernel functions $xz^p/(1-yz)^{2p}$ belong to $s(S_p^*)$ and consequently $Hs(S_p^*) \supset \mathbf{F}$. This proves (1). To prove (2) it is enough to show that each function of the form $xz^p/(1-yz)^{2p}$ is in $EHs(S_p^*)$. Let $f(z) = xz^p/(1-yz)^{2p}$. Then $f(z) = xz^p + 2pxyz^{p+1} + \ldots$ Let $\alpha = 1/x$, $\beta = 1/xy$, and define a continuous linear functional J by

$$J(g) = \alpha \frac{g^{(p)}(0)}{p!} + \beta \frac{g^{(p+1)}(0)}{(p+1)!} \quad \text{for } g \in \mathbf{A}.$$

It is easy to verify that f uniquely maximizes $\operatorname{Re} J$ over

$$\left\{ \frac{uz^p}{(1-vz)^{2p}}: |u| = |v| = 1 \right\}.$$

It follows that $f \in EHs(S_p^*)$ and (2) holds. This completes the proof.

It is known [11] that

$$HK_p = \left\{ \int_{\partial\Delta} \left[\int_0^z \frac{p\tau^{p-1}}{(1-x\tau)^{2p}} d\tau \right] d\mu(x): \ \mu \in \Lambda_1 \right\},$$

where Λ_1 is the set of probability measures on $\partial\Delta$ and that

$$EHK_p = \left\{ \int_0^z \frac{p\tau^{p-1}}{(1-x\tau)^{2p}} d\tau: \ |x| = 1 \right\}.$$

We next determine the sets $Hs(K_p)$ and $EHs(K_p)$

THEOREM 3. *Let* $s(K_p) = \{F \circ \varphi \colon F \in K_p,\ \varphi \in B_0\}$. *Then*

$$Hs(K_p) = \left\{ \int_{\partial\Delta \times \partial\Delta} \left[\int_0^z \frac{px\tau^{p-1}}{(1-y\tau)^{2p}} d\tau \right] d\mu(x, y) \colon \mu \in \Lambda_2 \right\} \tag{4}$$

and

$$EHs(K_p) = \left\{ \int_0^z \frac{px\tau^{p-1}}{(1-y\tau)^{2p}} d\tau \colon |x| = |y| = 1 \right\}. \tag{5}$$

Proof. Let $f \in EH_s(K_p)$. Then there is a function $F \in EHK_p$ such that $f \prec F$ ([17], p. 366). Hence for some $\varphi \in B_0$

$$f(z) = \int_0^{\varphi(z)} \frac{p\tau^{p-1}}{(1-x\tau)^{2p}} d\tau. \tag{6}$$

It follows that

$$f'(z) = \frac{p(\varphi(z))^{p-1}}{(1-x\varphi(z))^{2(p-1)}} \cdot \frac{\varphi'(z)}{(1-x\varphi(z))^2}. \tag{7}$$

It is easy to prove that there exists $\nu \in \Lambda_1$ such that

$$\frac{\varphi'(z)}{(1-x\varphi(z))^2} = \int_{\partial\Delta} \frac{\bar{x}y}{(1-yz)^2} d\nu(y). \tag{8}$$

It follows from (8) and Theorem 2 that

$$f'(z) = \int_{\partial\Delta \times \partial\Delta} \frac{puz^{p-1}}{(1-vz)^{2(p-1)}} d\mu(u, v) \cdot \int_{\partial\Delta} \frac{\bar{x}y}{(1-yz)^2} d\nu(y). \tag{9}$$

We conclude that

$$f'(z) = \int_{\partial\Delta \times \partial\Delta \times \partial\Delta} \frac{puz^{p-1}}{(1-vz)^{2(p-1)}} \cdot \frac{\bar{x}y}{(1-yz)^2} d\mu(u, v)\, d\nu(y). \tag{10}$$

We shall show that the expression on the right of (10) can be written as

$$\int_{\partial\Delta \times \partial\Delta} \frac{pxz^{p-1}}{(1-wz)^{2p}} d\tau(x, w) \quad \text{for some } \tau \in \Lambda_2.$$

Clearly it is enough to prove it for the integrand. It follows from Theorem 1 ([4], p. 415) that

$$\frac{1}{(1-vz)^{2(p-1)}} \frac{1}{(1-yz)^2} = \int_{\partial\Delta} \frac{1}{(1-wz)^{2p}} d\mu_1(w) \tag{11}$$

for some $\mu_1 \in \Lambda_1$. Hence

$$\frac{puz^{p-1}\, xy}{(1-vz)^{2(p-1)}(1-yz)^2} = \int_{\partial\Delta} \frac{pu\bar{x}yz^{p-1}}{(1-wz)^{2p}} d\mu_1(w). \tag{12}$$

Let $d\tau(x, w) = d\mu_1(w)d\mu_2(x)$ where μ_2 is a unit point mass at $u\bar{x}y$. Then (10) and (12) imply that

$$f'(z) = \int_{\partial\Delta\times\partial\Delta} \frac{pxz^{p-1}}{(1-wz)^{2p}}d\tau(x, w). \tag{13}$$

It follows that f has the representation given by the right-hand side of equality (4). This proves the inclusion

$$EHs(K_p) \subset \left\{ \int_{\partial\Delta\times\partial\Delta} \left[\int_0^z \frac{px\tau^{p-1}}{(1-y\tau)^{2p}}d\tau\right] d\mu(x, y):\ \mu\in\Lambda_2 \right\}. \tag{14}$$

Hence, we also have $Hs(K_p)$ contained in the set of functions on the right-hand side of equality (4). It is easy to verify that each function

$$f(z) = \int_0^z \frac{px\tau^{p-1}}{(1-y\tau)^{2p}}d\tau, \qquad x, y\in\partial\Delta$$

is in $s(K_p)$. It follows that (4) holds. Since $Hs(K_p)$ is homeomorphic to $Hs(S_p^*)$ through the homeomorphism L: $Hs(K_p)\to Hs(S_p^*)$ defined by $L(g) = zS'(z)/p$ we conclude from (2) that (5) holds and this completes the proof.

The third and last subordination family of multivalent functions we present is $s(C_p)$. For $p = 1$, $Hs(C_p)$ and $EHs(C_p)$ were determined in [12]. In [11] it was proved that

$$HC_p = \left\{ \int_{\partial\Delta\times\partial\Delta} \left[\int_0^z \frac{p\tau^{p-1}(1-y\tau)}{(1-x\tau)^{2p+1}}d\tau\right] d\mu(x, y):\ \mu\in\Lambda_2 \right\}$$

and

$$EHC_p = \left\{ \int_0^z \frac{p\tau^{p-1}(1-y\tau)}{(1-x\tau)^{2p+1}}d\tau:\ |x| = |y| = 1,\ x\neq y \right\}.$$

THEOREM 4. *Let Λ_3 be the set of probability measures on $\partial\Delta\times\partial\Delta\times\partial\Delta$. Then*

$$Hs(C_p) = \left\{ \int_{(\partial\Delta)^3} \left[\int_0^z \frac{px\tau^{p-1}(1-u\tau)}{(1-v\tau)^{2p+1}}d\tau\right] d\mu(x, u, v):\ \mu\in\Lambda_3 \right\} \tag{15}$$

and

$$EH_s(C_p) = \left\{ \int_0^z \frac{px\tau^{p-1}(1-u\tau)}{(1-v\tau)^{2p+1}}d\tau:\ u\neq v,\ |u| = |v| = |x| = 1 \right\}. \tag{16}$$

Proof. Let $f\in EHs(C_p)$. Then there is a function $F\in EHC_p$ such that $f\prec F$ ([17], p. 366). Hence for some $\varphi\in B_0$

$$f(z) = \int_0^{\varphi(z)} \frac{p\tau^{p-1}(1-y\tau)}{(1-x\tau)^{2p+1}}d\tau \qquad (x\neq y,\ |x| = |y| = 1). \tag{17}$$

It follows that

$$f'(z) = p\left(\frac{\varphi(z)}{(1-x\varphi(z))^2}\right)^{p-1}\frac{1-y\varphi(z)}{1-x\varphi(z)}\frac{\varphi'(z)}{(1-x\varphi(z))^2}. \tag{18}$$

It is easy to see that there exists $\tau\in\Lambda_1$ such that

$$\frac{1-y\varphi(z)}{1-x\varphi(z)} = \int_{\partial\Delta}\frac{1-\bar{x}ywz}{1-wz}\,d\tau(w). \tag{19}$$

It follows from (8), (18), (19) and Theorem 2 that

$$f'(z) = \int_{(\partial\Delta)^4}\frac{puz^{p-1}}{(1-vz)^{2(p-1)}}\frac{1-\bar{x}ywz}{1-wz}\frac{\bar{x}y}{(1-yz)^2}\,d\mu(u,v)\,d\tau(w)\,dy(y), \tag{20}$$

where $\mu\in\Lambda_2$ and $v, \tau\in\Lambda_1$. We will show that the integrand in (20) has the form

$$\int_{(\partial\Delta)^3}\frac{pxz^{p-1}(1-uz)}{(1-vz)^{2p+1}}\,d\mu_1(x,u,v), \tag{21}$$

where $\mu_1\in\Lambda_3$. Theorem 1 ([4], p. 415) implies

$$\frac{1}{(1-vz)^{2(p-1)}}\frac{1}{1-wz}\frac{1}{(1-yz)^2} = \int_{\partial\Delta}\frac{1}{(1-xz)^{2p+1}}\,d\tau_1(x), \tag{22}$$

where $\tau_1\in\Lambda_1$. It follows from (22) that

$$\frac{puz^{p-1}(1-\bar{x}ywz)\bar{x}y}{(1-vz)^{2(p-1)}(1-wz)(1-yz)^2} = \int_{\partial\Delta}\frac{pu\bar{x}yz^{p-1}(1-\bar{x}ywz)}{(1-vz)^{2p+1}}\,d\tau_1(v). \tag{23}$$

Now define $d\mu_1(x,u,v) = d\tau_3(x)\,d\tau_2(u)\,d\tau_1(v)$, where τ_3 is a unit point mass at $u\bar{x}y$ and τ_2 is a unit point mass at $\bar{x}yw$. It follows from (23) that

$$\frac{puz^{p-1}(1-\bar{x}ywz)\bar{x}y}{(1-vz)^{2(p-1)}(1-wz)(1-yz)^2} = \int_{(\partial\Delta)^3}\frac{pxz^{p-1}(1-uz)}{(1-vz)^{2p+1}}\,d\mu_1(x,u,v). \tag{24}$$

We conclude from (20) and (24) that

$$f'(z) = \int_{(\partial\Delta)^3}\frac{pxz^{p-1}(1-uz)}{(1-vz)^{2p+1}}\,d\mu(x,u,v) \tag{25}$$

for some $\mu\in\Lambda_3$. This proves the inclusion

$$EH_s(C_p) \subset \left\{\int_{(\partial\Delta)^3}\left[\int_0^z\frac{px\tau^{p-1}(1-u\tau)}{(1-v\tau)^{2p+1}}\,d\tau\right]d\mu(x,u,v):\ \mu\in\Lambda_3\right\}. \tag{26}$$

Hence, we also have $Hs(C_p)$ contained in the set of functions on the right-hand side of equality (15). It is easy to verify that each

$$f(z) = \int_0^z \frac{px\tau^{p-1}(1-u\tau)}{(1-v\tau)^{2p+1}}\,d\tau, \qquad x, u, v \in \partial\Delta$$

is in $s(C_p)$. It follows that (15) holds.

Note that if $u_0 = v_0$ then

$$g(z) = \int_0^z \frac{px_0\tau^{p-1}}{(1-v_0\tau)^{2p}}\,d\tau \in s(K_p)$$

and so $g \in s(S_p^*)$. Since Theorem 2 implies $g \notin EHs(S_p^*)$ we conclude that $g \notin EHs(C_p)$. Now suppose $x_0, u_0, v_0 \in \partial\Delta$ and $u_0 \neq v_0$ and assume that

$$\int_0^z \frac{px_0\tau^{p-1}(1-u_0\tau)}{(1-v_0\tau)^{2p+1}}\,d\tau = \int_{(\partial\Delta)^3} \left[\int_0^z \frac{px\tau^{p-1}(1-u\tau)}{(1-v\tau)^{2p+1}}\,d\tau\right] d\mu(x, u, v). \tag{27}$$

This is equivalent to

$$\frac{px_0\tau^{p-1}(1-u_0z)}{(1-v_0z)^{2p+1}} = \int_{(\partial\Delta)^3} \frac{pxz^{p-1}(1-uz)}{(1-vz)^{2p+1}}\,d\mu(x, u, v). \tag{28}$$

Hence

$$\frac{x_0(1-u_0z)}{(1-v_0z)^{2p+1}} = \int_{(\partial\Delta)^3} \frac{x(1-uz)}{(1-vz)^{2p+1}}\,d\mu(x, u, v). \tag{29}$$

It follows from (24) that $x_0 = \int_{(\partial\Delta)^3} x\,d\mu(x, u, v)$. Define a measure $\lambda \in \Lambda$, in the following way: for each measurable subset A of $\partial\Delta$, $\lambda(A) = \mu(A \times \partial\Delta \times \partial\Delta)$. Then $x_0 = \int_{\partial\Delta} x\,d\lambda(x)$ and we conclude that λ is a unit point mass at x_0, therefore (29) implies

$$\frac{1-u_0z}{(1-v_0z)^{2p+1}} = \int_{\{x_0\}\times(\partial\Delta)^2} \frac{1-uz}{(1-vz)^{2p+1}}\,d\mu(x_0, u, v). \tag{30}$$

A standard argument ([14], p. 58), the assumption $u_0 \neq v_0$ and (30) shows that μ has to be a unit point mass at (x_0, u_0, v_0). It follows that (16) holds and this completes the proof.

2. Applications. As one would expect knowledge of the extreme points of the sets $Hs(S_p^*)$, $Hs(K_p)$ and $Hs(C_p)$ helps to generalize coefficient and integral mean results known for the classes S_p^*, K_p, C_p.

Let us recall the concepts of majorization and quasi-subordination.

DEFINITION 5. We say that f is *majorized by g in Δ* if

$$f, g \in \mathbf{A} \quad \text{and} \quad |f(z)| \leqslant |g(z)| \quad (|z| < 1).$$

It is easy to see that f is majorized by g if and only if there is a function $w \in B$ such that $f = wg$, where

$$B = \{\varphi\colon\ \varphi \in A,\ |\varphi(z)| < 1\}.$$

DEFINITION 6. We say that f is *quasi-subordinate to* g [20] if and only if there is a function $w \in B$ and a function $\varphi \in B_0$ such that $f(z) = w(z)g(\varphi(z))$, $z \in \Delta$.

THEOREM 7. *Let* $f(z) = \sum_{n=p}^{\infty} a_n z^n$ *be quasi-subordinate to* $g \in HK_p$, *and suppose*

$$\int_0^z \frac{p\tau^{p-1}}{(1-\tau)^{2p}}\,d\tau = \sum_{n=p}^{\infty} A_n z^n.$$

Then $|a_n| \leqslant A_n$ *for* $n = p, p+1, \ldots$

Proof. If is sufficient to consider functions of the form $f = wG$, where $w \in B$ and $G \in EHs(K_p)$. We see from (5) and the remark made prior to the proof of Theorem 3 that $EHs(K_p) = \{ch\colon\ h \in EHK_p,\ |c| = 1\}$. The result now follows directly from Theorem 3.1 in [11], p. 348.

Remark. This generalizes Theorems 3.1 and 3.3 in [11].

THEOREM 8. *Let* $f(z) = \sum_{n=p}^{\infty} a_n z^n$ *be quasi-subordinate to* $g \in HC_p$, *and suppose*

$$\frac{z^p}{(1-z)^{2p}} = \sum_{n=p}^{\infty} A_n z^n.$$

Then $|a_n| \leqslant A_n$ *for* $n = p, p+1, \ldots$

Proof. This result follows from (16), the remark made prior to the proof of Theorem 4 and Theorem 3.2 in [11], p. 349.

Remark. This result generalizes Theorems 3.2 and 3.4 in [11].

THEOREM 9. *Let*

$$f \in Hs(K_p), \qquad F(z) = \int_0^z \frac{p\tau^{p-1}}{(1-\tau)^{2p}}\,d\tau, \qquad \lambda \geqslant 1,\ n = 0, 1, 2, \ldots$$

Then

$$\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^{\lambda}\,d\theta \leqslant \int_0^{2\pi} |F^{(n)}(re^{i\theta})|^{\lambda}\,d\theta \quad \textit{for } r < 1. \tag{31}$$

Proof. Since $\lambda \geqslant 1$ it is enough to consider $f \in EHs(K_p)$ [17]. As remarked in the proof of Theorem 7, $EHs(K_p) = \{ch\colon\ h \in EHK_p,\ |c| = 1\}$. Hence (31) follows directly from Theorem 4.1 ([11], p. 353).

THEOREM 10. *Let*

$$f \in Hs(S_p^*), \quad F(z) = \frac{z^p}{(1-z)^{2p}}, \quad \lambda \geqslant 1, \; n = 0, 1, 2, \ldots$$

Then

$$\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^\lambda \, d\theta \leqslant \int_0^{2\pi} |F^{(n)}(re^{i\theta})|^\lambda \, d\theta \quad \textit{for } r < 1. \tag{32}$$

Proof. Since $\lambda \geqslant 1$ (32) follows form Theorem 4.2 ([11], p. 354) and the fact that $EHs(S_p^*) = \{ch\colon h \in EHS_p^*, |c| = 1\}$.

THEOREM 11. *Let*

$$f \in Hs(C_p), \quad F(z) = \frac{z^p}{(1-z)^{2p}}, \quad \lambda \geqslant 1, \; n = 0 \textit{ or } 1.$$

Then

$$\int_0^{2\pi} |f^{(n)}(re^{i\theta})|^\lambda \, d\theta \leqslant \int_0^{2\pi} |F^{(n)}(re^{i\theta})|^\lambda \, d\theta \quad \textit{for } r < 1. \tag{33}$$

Proof. In case $n = 0$ the result follows from Theorem 4.3 ([11], p. 354) by much the same argument as Theorems 9, 10. For $n = 1$ one has to appeal to Theorem 4.4 in [11], p. 355.

Remark. Theorems 9, 10 and 11 generalize Theorems 4.1–4.4 in [11]. Theorem 10 also holds for $n = 0$ and $0 < \lambda < 1$.

3. Support points. We first consider $s(S_p^*)$. Let J be a continuous linear functional on **A**. It is known [21] that there exists a sequence $\{b_n\}$ of complex numbers so that $\overline{\lim\limits_{n\to\infty}} \sqrt[n]{|b_n|} < 1$ and $J(f) = \sum\limits_{n=0}^{\infty} a_n b_n$, where $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ is analytic in Δ. Clearly if for all $k \geqslant p$, $b_k = 0$ then J is zero on $s(S_p^*)$. Hence, if for all $k \geqslant p$, $b_k = 0$, then $\operatorname{Re} J = 0$ on $s(S_p^*)$ and J does not generate any support points of $s(S_p^*)$. The converse is also true, i.e., if $\operatorname{Re} J$ is constant on $s(S_p^*)$ then $b_k = 0$ for $k \geqslant p$. Indeed, if $\operatorname{Re} J$ is constant on $s(S_p^*)$ then $\operatorname{Re} J$ is constant on $Hs(S_p^*)$ and so is constant on HS_p^*. We shall prove that this latter fact implies $b_k = 0$ for $k > p$.

LEMMA 12. *Let*

$$\frac{z^p}{(1-z)^{2p}} = \sum_{k=0}^{\infty} A_{p+k} z^{p+k}.$$

Then there is an $\varepsilon > 0$ *such that if* $|x_0| < \varepsilon$ *then* $z^p + A_{p+k} x_0 z^{p+k} \in HS_p^*$ *for* $k = 1, 2, \ldots$

Proof. For a fixed $k \in N$ and $|x_0| < 1$, the function $1+2x_0 z^k$ belongs to the class of normalized functions of positive real part in Δ. The uniqueness of the Herglotz representation implies the existence of a probability measure μ on $\partial\Delta$ such that $x_0 = \int_{\partial\Delta} x^k d\mu(x)$ and $0 = \int_{\partial\Delta} x^j d\mu(x)$ for $j \in N$, $j \neq k$. It follows that

$$z^p + A_{p+k} x_0 z^{p+k} = \int_{\partial\Delta} \frac{z^p}{(1-xz)^{2p}} d\mu(x). \tag{34}$$

It follows from Theorem 2.1 in [11], p. 343, and (34) that $z^p + A_{p+k} x_0 z^{p+k} \in HS_p^*$ and this completes the proof.

Now the assumption $\operatorname{Re} J$ constant on the set HS_p^* implies that $\operatorname{Re} J(z^p + A_{p+k} x_0 z^{p+k})$ is constant for all sufficiently small $|x_0|$ and $k = 1, 2, \ldots$ It follows that $J(z^{p+k}) = 0$ for $k = 1, 2, \ldots$ since $A_{p+k} \neq 0$ for all k. Notice now that $xz^p \in s(S_p^*)$ for all $|x| < \frac{1}{4}$ by the $\frac{1}{4}$ covering Theorem, [15], p. 3. We conclude that $\operatorname{Re} J(xz^p)$ is constant for all $|x| < \frac{1}{4}$. Hence $J(z^p) = 0$.

Therefore a continuous linear functional J on **A** given by a sequence $\{b_n\}$ generates a support point of $s(S_p^*)$ if and only if there is $k \geqslant p$ such that $b_k \neq 0$. The situation is analogous in $s(K_p)$ and $s(C_p)$. We will see later that functionals whose only non-zero coefficient is b_p generate a great number of support points in these classes, while support points generated by all the other functionals must be extreme points of the corresponding closed convex hulls. We need the following algebraic lemma to proceed with our investigation.

LEMMA 13. *Let $p \in N$ and $a_1, a_2, \ldots, a_{p-1}$ be complex numbers such that $a_i \neq 0$, $i = 1, 2, \ldots, p-1$, and $a_i \neq a_j$, $i \neq j$, $i, j \in (1, 2, \ldots, p-1)$. Then there exist numbers $x_1, x_2, \ldots, x_{p-1}$ such that*

$$\sum_{i=1}^{p-1} a_i^k x_i = \begin{cases} 1 & \text{for } k = 1, \\ 0 & \text{for } k = 2, \ldots, p-1. \end{cases} \tag{35}$$

Proof. Consider the system

$$a_1 x_1 + a_2 x_2 + \ldots + a_{p-1} x_{p-1} = 1,$$
$$a_1^2 x_1 + a_2^2 x_2 + \ldots + a_{p-1}^2 x_{p-1} = 0,$$
$$\ldots$$
$$a_1^{p-1} x_1 + a_2^{p-1} x_2 + \ldots + a_{p-1}^{p-1} x_{p-1} = 0.$$

The determinant of this system is easily seen to be $\prod_{i=1}^{p-1} a_i \prod_{\substack{i,j=1 \\ i<j}}^{p-1} (a_i - a_j) \neq 0$.

Hence the existence of the $x_1, x_2, \ldots, x_{p-1}$ such that (35) holds follows directly.

LEMMA 14. *Let* $a_i, x_i \in C$, $i = 1, 2, \ldots, p-1$, *and let* $g \in S^p$. *Then there exists a complex number* $w_0 \neq 0$ *such that* $wx_i(z+a_i z^m)^p \prec g$ *whenever* $|w| \leqslant |w_0|$, $i = 1, 2, \ldots, p-1$, $m = 2, 3, \ldots$

Proof. Since $g \in S^p$ there is a function $f \in S$ such that $g = f^p$. The $\frac{1}{4}$ covering Theorem ([15], p. 3) implies that there exists $w_0 \in C$ such that if $|w| \leqslant |w_0|$ then $(wx_i)^{1/p}(z+a_i z^m) \prec f(z)$, $i = 1, 2, \ldots, p-1$, $m = 2, 3, \ldots$ Hence $wx_i(z+a_i z^m)^p = [f(\varphi(z))]^p = g(\varphi(z))$ for some $\varphi \in B_0$ and this completes the proof.

THEOREM 15. *Let* $\mathbf{F}$ *be a family of functions such that* $z^p \in \mathbf{F} \subset S^p$, *and let* J *be a continuous linear functional on* $\mathbf{A}$. *If there exist a function* $g \in \mathbf{F}$ *such that* $\operatorname{Re} J$ *is constant on* $s(g)$, *then* $\operatorname{Re} J$ *is constant on* $s(\mathbf{F})$.

Proof. Suppose g satisfies the assumptions of the theorem. Since $g \in S^p$ it follows from the $\frac{1}{4}$ covering Theorem ([15], p. 3) that for all $|x|$ sufficiently small and $m = 1, 2, \ldots$ we have $xz^{pm} \in s(g)$. Hence $\operatorname{Re} J$ constant on $s(g)$ implies that $J(z^{pm}) = 0$ for $m = 1, 2, \ldots$ Let $a_1, a_2, \ldots, a_{p-1}$ be numbers satisfying the assumptions of Lemma 13 and let $x_1, x_2, \ldots, x_{p-1}$ be a corresponding set of numbers whose existence is assured by Lemma 13. By Lemma 14, $wx_i(z+a_i z^m)^p \in s(g)$ for all complex w with sufficiently small absolute value and $i = 1, 2, \ldots, p-1$, $m = 2, 3, \ldots$ Hence $\operatorname{Re} J(wx_i(z+a_i z^m)^p)$ is constant for all sufficiently small w and so $J(x_i(z+a_i z^m)^p) = 0$, $i = 1, 2, \ldots, p-1$, $m = 2, 3, \ldots$ Therefore

$$J\Big[\sum_{i=1}^{p-1} x_i(z+a_i z^m)^p\Big] = 0, \qquad m = 2, 3, \ldots \tag{36}$$

A short computation using (35) shows that

$$\sum_{i=1}^{p-1} x_i(z+a_i z^m)^p = \Big[\sum_{i=1}^{p-1} x_i\Big] z^p + pz^{p+m-1} + \Big[\sum_{i=1}^{p-1} a_i^p x_i\Big] z^{pm}. \tag{37}$$

Since $J(z^{pm}) = 0$ for $m = 1, 2, \ldots$ we conclude from (36) and (37) that $J(z^{p+m-1}) = 0$ for $m = 2, 3, \ldots$ or equivalently $J(z^{p+n}) = 0$ for $n = 1, 2, \ldots$ If $f \in s(\mathbf{F})$ then since $\mathbf{F} \subset S^p$ we have $f(z) = \sum_{n=p}^{\infty} a_n z^n$. Hence J is zero on $s(\mathbf{F})$ and this completes the proof.

THEOREM 16. $\operatorname{supp}(S_p^*) = EHS_p^*$; $\operatorname{supp}(K_p) = EHK_p$.

Proof. This follows easily by standard techniques form Theorem 2.1 in [11], p. 343.

THEOREM 17.

$$\operatorname{supp} Hs(S_p^*) = \left\{ \int_{\partial \Delta} \frac{B(y)z^p}{(1-yz)^{2p}} d\mu(y) \colon \ \mu \in \Lambda_1,\ B \in \operatorname{supp} B_0 \right\}.$$

Proof. This can be proved by the method used in [18] to prove Theorem 2.2.

Remark. We recall that $\operatorname{supp} B_0$ consists of all finite Blaschke products which vanish at the origin [13], p. 526.

THEOREM 18. $\operatorname{supp} s(S_p^*) = \{f(xz): f \in S_p^*, |x| = 1\}$. *Also real parts of continuous linear functionals on* $\mathbf{A}$ *not of the form*

$$J(g) = \sum_{n=0}^{p} b_n \frac{g^{(n)}(0)}{n!}$$

are maximized over $s(S_p^*)$ *only by functions in* $EHs(S_p^*)$.

Proof. We first show that $EHs(S_p^*) \subset \operatorname{supp} s(S_p^*)$. Let

$$f(z) = \frac{uz^p}{(1-yz)^{2p}}, \quad |u| = |y| = 1$$

and define a continuous linear functional J by

$$J(g) = \frac{1}{u}\frac{g^{(p)}(0)}{p!} + \frac{1}{uy}\frac{g^{(p+1)}(0)}{(p+1)!} \quad \text{for } g \in \mathbf{A}.$$

It is easy to verify that $\operatorname{Re} J$ is not constant on $s(S_p^*)$ and $\operatorname{Re} J(g) \leqslant 1+2p$ for all $g \in Hs(S_p^*)$ with equality only for $g = f$. This proves the inclusion. It is also easy to see that for each $f \in S_p^*$ and each $x \in \partial\Delta$ the function $f(xz)$ supports over $s(S_p^*)$ a functional of the form

$$J(g) = c\frac{g^{(p)}(0)}{p!}, \quad c \in \partial\Delta,$$

[13], p. 545.

Now suppose $f \in \operatorname{supp} s(S_p^*)$, then $f = F \circ \varphi$, where $F \in S_p^*$ and by familiar arguments ([2], p. 89) and Theorem 15 we conclude that $\varphi \in \operatorname{supp} B_0$. We first prove that $\varphi(z) = uz$ for some $u \in \partial\Delta$. Consider the case $F \in EHS_p^*$. Then there is $x \in \partial\Delta$ such that

$$f(z) = \left[\frac{\varphi(z)}{1-x\varphi(z)}\right]^p \frac{1}{(1-x\varphi(z))^p}. \tag{38}$$

It follows from Lemma 4 in [7], p. 82, that

$$\frac{\varphi(z)}{1-x\varphi(z)} = \sum_{k=1}^{n} \lambda_k \frac{\bar{x}x_k z}{1-x_k z}, \tag{39}$$

where $\sum_{k=1}^{n} \lambda_k = 1$, $\lambda_k \geqslant 0$, $x_k \in \partial\Delta$, $k = 1, 2, \ldots, n$. Since $p \in N$ it is easy to see that (39) implies

$$(40)\qquad \left[\frac{\varphi(z)}{1-x\varphi(z)}\right]^p = \sum_{k_1,\dots,k_p=1}^{n} (\lambda_{k_1}\dots\lambda_{k_p})\frac{(\bar{x})^p x_{k_1}\dots x_{k_p} z^p}{(1-x_{k_1}z)\dots(1-x_{k_p}z)}.$$

We deduce from (38) and (40) that

$$(41)\qquad f(z) = \sum_{k_1,\dots,k_p=1}^{n} (\lambda_{k_1}\dots\lambda_{k_p})\frac{(\bar{x})^p x_{k_1}\dots x_{k_p} z^p}{(1-x_{k_1}z)\dots(1-x_{k_p}z)}\frac{1}{(1-x\varphi(z))^p}.$$

Since $\sum_{k_1,k_2,\dots,k_p=1}^{n} \lambda_{k_1}\dots\lambda_{k_p} = \sum_{k_1=1}^{n}\lambda_{k_1}\dots\sum_{k_p=1}^{n}\lambda_{k_p} = 1$, it follows from Theorem 1 ([4], p. 415), the Herglotz representation, (41) and Theorem 2 that f is a finite convex combination of functions in $Hs(S_p^*)$. Since $f\in\operatorname{supp} Hs(S_p^*)$, each of these functions must also be a support point of $Hs(S_p^*)$. Theorem 17 implies that

$$(42)\qquad \frac{(\bar{x})^p x_{k_1}\dots x_{k_p} z^p}{(1-x_{k_1}z)\dots(1-x_{k_p}z)}\frac{1}{(1-x\varphi(z))^p} = \int_{\partial\Delta}\frac{\overline{B(y)}z^p}{(1-yz)^{2p}}d\mu(y)$$

for some finite Blaschke product B and $\mu\in\Lambda_1$. Hence

$$(43)\qquad \frac{(\bar{x})^p x_{k_1}\dots x_{k_p}}{(1-x_{k_1}z)\dots(1-x_{k_p}z)}\left(\sum_{j=1}^{n}\lambda_j\frac{1}{1-x_j z}\right)^p = \int_{\partial\Delta}\frac{\overline{B(y)}}{(1-yz)^{2p}}d\mu(y).$$

It follows from (43) that $(\bar{x})^p x_{k_1}\dots x_{k_p} = \int_{\partial\Delta}\overline{B(y)}\,d\mu(y)$. Put $x_0 = (\bar{x})^p x_{k_1}\dots x_{k_p}$, and consider the function

$$h(z) = \int_{\partial\Delta}\frac{1+\overline{B(y)}z}{1-\overline{B(y)}z}d\mu(y).$$

Since $\int_{\partial\Delta}\overline{B(y)}\,d\mu(y) = x_0$ and $B(y)\in\partial\Delta$ for $y\in\partial\Delta$, $h(z)$ is a normalized function of positive real part satisfying $h'(0) = 2x_0$. This implies that

$$h(z) = \frac{1+x_0 z}{1-x_0 z}.$$

Hence

$$\int_{\partial\Delta}\frac{1+\overline{B(v)}z}{1-\overline{B(y)}z}d\mu(y) = \frac{1+x_0 z}{1-x_0 z}.$$

If we multiply this equation by $1-x_0 z$, we may write

$$1+x_0 z = \int_{\Gamma}(1+\overline{B(y)}z)\,d\mu(y) + \int_{\partial\Delta\setminus\Gamma}\frac{1+\overline{B(y)}z}{1-\overline{B(y)}z}(1-x_0 z)\,d\mu(y),$$

where $\Gamma = \{y\colon y \in \partial\Delta, \overline{B(y)} = x_0\}$. If we put $z = rx_0$ $(0 < r < 1)$ in (44) and let $r \to 1$, then the Lebesgue dominated convergence theorem implies that the second integral in (44) tends to zero. Consequently

$$1 + x_0 \bar{x}_0 = \int_\Gamma (1 + \overline{B(y)}\, x_0)\, d\mu(y)$$

and the definition of Γ implies that $1 = \int_\Gamma d\mu(\Gamma)$. Since B is a finite Blaschke product, $\Gamma = \{y_1, y_2, \ldots, y_m\}$ for some $y_k \in \partial\Delta$, $k = 1, 2, \ldots, m$. Therefore (43) implies

$$(45) \qquad \frac{x_0}{(1 - x_{k_1} z)\ldots(1 - x_{k_p} z)} \Big(\sum_{j=1}^{n} \lambda_j \frac{1}{1 - x_j z} \Big)^p = \sum_{k=1}^{m} \alpha_k \frac{x_0}{(1 - y_k z)^{2p}},$$

where $\sum_{k=1}^{m} \alpha_k = 1$, $\alpha_k > 0$, $k = 1, 2, \ldots, m$. A comparison of the poles in (45) implies that $n = 1$ and hence $\varphi(z) = uz$ for some $u \in \partial\Delta$. This proves that if $f = F \circ \varphi$ is a support point of $s(S_p^*)$ and $F \in EHS_p^*$, then $\varphi(z) = uz$ for $u \in \partial\Delta$.

Now assume that $F \notin EH(S_p^*)$. Therefore, by Theorem 16, $F \notin \operatorname{supp}(S_p^*)$. Define a continuous linear functional L on $\mathbf{A}$ by $L(g) = J(g \circ \varphi)$ for $g \in \mathbf{A}$, where J is the functional which f supports on $s(S_p^*)$. Since $\operatorname{Re} L(F) = \max_{g \in S_p^*} \operatorname{Re} L(f)$, yet $F \notin \operatorname{supp}(S_p^*)$, it follows that $\operatorname{Re} L(g)$ is constant on S_p^*. In particular

$$\operatorname{Re} J\Big(\frac{\varphi(z)^p}{(1 - x\varphi(z))^{2p}}\Big) = \operatorname{Re} J(f) \quad \text{for some } x \in \partial\Delta.$$

Hence

$$\Big| \frac{\varphi(z)}{(1 - x\varphi(z))^2} \Big|^p \in \operatorname{supp} s(S_p^*).$$

But as we proved earlier, this implies $\varphi(z) = uz$ for some $u \in \partial\Delta$. Therefore, if $f = F \circ \varphi \in \operatorname{supp} s(S_p^*)$, then $\varphi(z) = uz$, $u \in \partial\Delta$.

Now let J be a continuous linear functional whose real part is not constant on $s(S_p^*)$, and $\operatorname{Re} J(f) = \max_{h \in s(S_p^*)} \operatorname{Re} J(h)$. If $\operatorname{Re} J(g(uz))$ is not constant on S_p^* $(g \in S_p^*)$, then $F \in \operatorname{supp}(S_p^*)$ and $f(z) = yz^p/(1 - xz)^{2p}$ for some $x, y \in \partial\Delta$. If on the other hand $\operatorname{Re} J(g(uz))$ is constant on S_p^* then in particular $\operatorname{Re} J(g(uz))$ is constant for all $g \in EHS_p^*$. Hence $\operatorname{Re} J((uz)^p/(1 - xuz)^{2p})$ is constant for all $x \in \partial\Delta$. Since $A(x) = J((uz)^p/(1 - xuz)^{2p})$ is analytic in $\bar{\Delta}$ it follows that $J(z^n) = 0$ for $n \geqslant p + 1$. This implies that J has the form

$$J(g) = \sum_{n=0}^{p} b_n \frac{g^{(n)}(0)}{n!} \quad \text{for } g \in \mathbf{A}$$

and it is clear that then all functions $h(uz)$, $h \in S_p^*$ support J over $s(S_p^*)$ ([13], p. 545). In particular we proved that if J is not of the form

$$J(g) = \sum_{n=0}^{p} b_n \frac{g^{(n)}(0)}{n!},$$

then $\operatorname{Re} J(g(uz))$ is not constant on S_p^* and so $f \in EHs(S_p^*)$. This completes the proof.

Remark. The case $p = 1$ of the previous theorem is in [13], p. 544.

It is known that $\operatorname{supp} s(K_1) = \{h \circ \varphi \colon h \in K_1,\ \varphi \in \operatorname{supp} B_0\}$ ([13], p. 541). The situation for K_p is quite different in the case $p \geqslant 2$.

THEOREM 19. *Let $p \geqslant 2$. Then $\operatorname{supp} s(K_p) = \{f(xz)\colon f \in K_p,\ x \in \partial\Delta\}$. Also the real parts of continuous linear functionals on* **A** *not of the form*

$$J(g) = \sum_{n=0}^{p} b_n \frac{g^{(n)}(0)}{n!}, \qquad g \in \mathbf{A}$$

are maximized over $s(K_p)$ only by functions in $EHs(K_p)$.

Proof. We first show that $EHs(K_p) \subset \operatorname{supp} s(K_p)$. Note that

$$L(f(z)) = p \int_0^z \frac{f(\tau)}{\tau} d\tau$$

defines a linear homeomorphism between $Hs(S_p^*)$ and $Hs(K_p)$. Hence, since we have previously proved that $EHs(S_p^*) \subset \operatorname{supp} Hs(S_p^*)$ we conclude that $EHs(K_p) \subset \operatorname{supp} Hs(K_p)$. But $EHs(K_p) \subset s(K_p)$ and so we have $EHs(K_p) \subset \operatorname{supp} s(K_p)$.

Now suppose $f \in \operatorname{supp} s(K_p)$. Then $f = F \circ \varphi$, $F \in K_p$, $\varphi \in B_0$, since $K_p \subset S^p$ ([11], p. 346) we conclude from familiar arguments ([2], p. 89) and Theorem 15 that $\varphi \in \operatorname{supp} B_0$. We will show that $\varphi(z) = uz$ for some $u \in \partial\Delta$. Assume first that $f \in EHK_p$. Then we have

$$f(z) = \int_0^{\varphi(z)} \frac{p\tau^{p-1}}{(1-x\tau)^{2p}} d\tau \qquad (|x| = 1). \tag{46}$$

We deduce from (46)

$$f'(z) = p\left[\frac{\varphi(z)}{1-x\varphi(z)}\right]^{p-1} \frac{\varphi'(z)}{(1-x\varphi(z))^2} \frac{1}{(1-x\varphi(z))^{p-1}}. \tag{47}$$

We conclude from (39) that

$$\frac{\varphi'(z)}{(1-x\varphi(z))^2} = \bar{x} \sum_{k=1}^{n} \lambda_k \frac{x_k}{(1-x_k z)^2}, \tag{48}$$

where $\sum_{k=1}^{n} \lambda_k = 1$, $\lambda_k \geqslant 0$, $x_k \in \partial\Delta$, $k = 1, 2, \ldots, n$. Furthermore

$$(49)\quad \left(\frac{\varphi(z)}{1-x\varphi(z)}\right)^{p-1} = \sum_{k_1,\dots,k_{p-1}=1}^{n} \lambda_{k_1}\dots\lambda_{k_{p-1}} \frac{(\bar{x})^{p-1} x_{k_1}\dots x_{k_{p-1}} z^{p-1}}{(1-x_{k_1}z)\dots(1-x_{k_{p-1}}z)}.$$

It follows from (47), (48) and (49) that

$$(50)\quad f'(z) = p \sum_{\substack{k_1,\dots,k_{p-1}=1\\ k=1}} \lambda_{k_1}\dots\lambda_{k_{p-1}}\lambda_k \frac{(\bar{x})^p x_{k_1}\dots x_{k_{p-1}} x_k z^{p-1}}{(1-x_{k_1}z)\dots(1-x_{k_{p-1}}z)(1-x_k z)^2}.$$

Theorem 1 ([4], p. 415) implies that f' is a finite convex of functions of the form

$$\int_{\partial\Delta\times\partial\Delta} \frac{pxz^{p-1}}{(1-yz)^{2p}} d\mu(x, y), \qquad \mu\in\Lambda_2.$$

Since $f\in\operatorname{supp} Hs(K_p)$, linear considerations imply that

$$f'(z)\in\operatorname{supp}\left\{\int_{\partial\Delta\times\partial\Delta} \frac{pxz^{p-1}}{(1-yz)^{2p}} d\mu(x, y)\colon\ \mu\in\Lambda_2\right\}.$$

An argument analogous to that used in the proof of Theorem 2.2 in [18] shows that the support points of this family have the form

$$\int_{\partial\Delta} \frac{\overline{B(y)}\, z^{p-1}}{(1-yz)^{2p}} dv(y),$$

where B is a finite Blaschke product and $v\in\Lambda_1$. Therefore for each choice of $k_1, k_2, \dots, k_{p-1}$, $k\in(1, 2, \dots, n)$ there is a finite Blaschke product B and a measure $v\in\Lambda_1$ such that

$$(51)\quad \frac{(\bar{x})^p x_{k_1}\dots x_{k_{p-1}} x_k z^{p-1}}{(1-x_{k_1}z)\dots(1-x_{k_{p-1}}z)(1-x_k z)^2} \left(\sum_{j=1}^{n} \lambda_j \frac{1}{1-x_j z}\right)^{p-1} = \int_{\partial\Delta} \frac{\overline{B(y)}\, z^{p-1}}{(1-yz)^{2p}} d\mu(y).$$

Equation (51) can be treated identically to (43) in the proof of Theorem 18. Consequently, v is supported on a finite set $\Gamma = \{y_1, y_2, \dots, y_m\}$, $y_j\in\partial\Delta$, $i = 1, 2, \dots, m$, and we deduce from (51)

$$(52)\quad \frac{x_0}{(1-x_{k_1}z)\dots(1-x_{k_{p-1}}z)} \frac{1}{(1-x_k z)^2} \left(\sum_{j=1}^{n} \lambda_j \frac{1}{1-x_j z}\right)^{p-1} = \sum_{i=1}^{m} \alpha_i \frac{x_0}{(1-y_i z)^{2p}},$$

where $x_0 = (\bar{x})^p x_{k_1}\dots x_{k_{p-1}}$ $\sum_{i=1}^{m} \alpha_i = 1$, $\alpha_i > 0$, $i = 1, 2, \dots, m$. Notice that since $p \geqslant 2$ the fact $\left(\sum_{j=1}^{n} \lambda_j \cdot 1/(1-x_j z)\right)^{p-1}$ is not degenerate, and hence a comparison of poles in (52) implies that $n = 1$. Hence $\varphi(z) = uz$ for some $u \in \partial\Delta$. This proves that if $f = F \circ \varphi \in \operatorname{supp} s(K_p)$ and $F \in EHK_p$ then $\varphi(z) = uz$, $u \in \partial\Delta$. When $f \in F \circ \varphi \in \operatorname{supp} s(K_p)$ but $F \notin EHK_p$ we argue as in the proof of Theorem 18 to conclude that $\varphi(z) = uz$, $u \in \partial\Delta$. Since also each function of the form $f(xz)$, where $f \in K_p$ and $x \in \partial\Delta$, belongs to $\operatorname{supp} s(K_p)$ we have completed the proof of the first assertion of the theorem. The second assertion can be proven in the same way as the second assertion of Theorem 18. This completes the proof.

We now make some observations about the construction of continuous linear functionals whose solution sets consist of chosen apriori extreme points. Suppose $f \in EHs(S_p^*)$. We have seen in the proof of Theorem 18 that there exists a continuous linear functional J such that f is the only function in $Hs(S_p^*)$ for which the maximum of $\operatorname{Re} J$ is attained.

Consider now an arbitrary set E of extreme points of $Hs(S_p^*)$. Let

$$E = \left\{\frac{x_k z^p}{(1-y_k z)^{2p}} \colon k = 1, 2, \ldots, n,\ x_k, y_k \in \partial\Delta\right\}.$$

We would like to construct a continuous linear functional J on $\mathbf{A}$ whose real part is maximized over $s(S_p^*)$ only by functions from E. Notice first that if this happens then $y_i \neq y_j$, $i \neq j$, $i, j \in (1, 2, \ldots, n)$. Indeed, this follows from considering $G(y) = J\left(z^p/(1-yz)^{2p}\right)$. Since G is analytic in $\bar{\Delta}$, Lemma 6 in [13], p. 539, implies that if $M = \max_{y \in \partial\Delta} |G(y)|$ then either $|G(v)| = M$ for all $y \in \partial\Delta$ or $|G(y)| = M$ for finitely many values of y. Clearly, $\operatorname{Re} J\left(xz^p/(1-yz)^{2p}\right) = \operatorname{Re}\{xG(y)\}$. Hence $\max_{x,y \in \partial\Delta} \operatorname{Re} J\left(xz^p/(1-yz)^{2p}\right) = M$. Also x is uniquely determined by $G(y)$. Hence if $y_i = y_j$ for $i \neq j$ then $x_i = x_j$. We now show how to construct a functional J with the desired property.

THEOREM 20. *Suppose* $|x_k| = |y_k| = 1$, $k = 1, 2, \ldots, n$, $y_i \neq y_j$ *for* $i \neq j$ *and* $f_k(z) = x_k z^p/(1-y_k z)^{2p}$. *There is a continuous linear functional* J *on* $\mathbf{A}$ *such that*

$$\operatorname{Re} J(f_k) = \max_{g \in s(S_p^*)} \operatorname{Re} J(g) \tag{53}$$

and

$$\operatorname{Re} J(g) < \max_{h \in s(S_p^*)} \operatorname{Re} J(h) \quad \textit{for } g \neq f_k \ (k = 1, 2, \ldots, n).$$

Proof. Assume first that x_k's are distinct, by virtue of Theorem 1 in [6] there is a function G analytic in $\bar{\Delta}$ so that $G(y_k) = \bar{x}_k$ for $k = 1, \ldots, n$ and

$|G(y)| < 1$ for $y \in \bar{\Delta}$ distinct from y_k, $k = 1, 2, \ldots, n$. Suppose

$$G(z) = \sum_{n=0}^{\infty} b_n z^n \quad \text{and} \quad \frac{z^p}{(1-z)^{2p}} = z^p + \sum_{k=1}^{\infty} A_{p+k} z^{p+k}.$$

We note that $A_{p+k} \neq 0$ for all k. Set $A_p = 1$. Let $c_n = b_n/A_{p+n}$, $n = 0, 1, \ldots$, and note that $\overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|} < 1$ since $\overline{\lim_{n\to\infty}} \sqrt[n]{A_{p+n}} = 1$ and $\overline{\lim_{n\to\infty}} \sqrt[n]{|b_n|} < 1$. Define $F(z) = \sum_{n=0}^{\infty} c_n z^n$. Then F is analytic in $\bar{\Delta}$ and it generates a continuous linear functional J. We have

$$J\left(\frac{xz^p}{(1-yz)^{2p}}\right) = x \sum_{n=0}^{\infty} c_n A_{p+n} y^n = x \sum_{n=0}^{\infty} b_n y^n = xG(y). \tag{54}$$

It follows from (54) that

$$\max_{x,y \in \partial\Delta} \operatorname{Re} J\left(\frac{xz^p}{(1-yz)^{2p}}\right) = \max_{x,y \in \partial\Delta} \operatorname{Re}\{xG(y)\} = 1. \tag{55}$$

Equality occurs in (55) if and only if $|G(y)| = 1$ and $G(y) = \bar{x}$. Clearly the only functions from $s(S_p^*)$ for which the real part of J is maximal are f_k $(k = 1, 2, \ldots, n)$.

Suppose now that the x_k's are not all distinct. Theorem 1 in [6] implies the existence of a function F analytic in Δ such that $F(y_k) = y_k$, $k = 1, 2, \ldots, n$ and $|F(y)| < 1$ for all other $y \in \Delta$. It follows from the interpolation theorem in [5] that there is a finite Blaschke product B such that $B(y_k) = x_k$, $k = 1, 2, \ldots, n$. Define $G(z) = B(F(z))$, $z \in \Delta$. Clearly G is analytic in Δ and if we define functional J as we did before we will have $J(xz^p/(1-yz)^{2p}) = xG(y)$. Since $|G(y)| = 1$ only for $y = y_k$, $k = 1, 2, \ldots, n$ and $G(y_k) = B(y_k) = x_k$, $k = 1, 2, \ldots, n$, J has the desired property. This completes the proof.

Remark. It is easy to prove an analogous theorem for finite subsets of $EHs(k_p)$.

In the process of determining $\operatorname{supp} s(S_p^*)$ we proved that if $f = F \circ \varphi \in \operatorname{supp} s(S_p^*)$, $F \in S_p^*$, $\varphi \in B_0$ and f maximizes over $s(S_p^*)$ the real part of a continuous linear functional J not of the form

$$J(g) = \sum_{k=0}^{p} b_k \frac{g^{(k)}(0)}{k!}, \tag{J}$$

then $F \in \operatorname{supp}(S_p^*)$. We next prove a generalization of this result which will be useful in our examination of $\operatorname{supp} s(C_p)$.

THEOREM 21. *Let* $\mathbf{F}$ *be a compact subset of* $\mathbf{A}$ *such that* $S_p^* \subset \mathbf{F} \subset S^p$ *and let* J *be a continuous linear functional on* $\mathbf{A}$ *not of the form* (J). *Then*

$$\operatorname{supp}\{\mathbf{F}, J\} \subset \{F \circ \varphi \colon F \in \operatorname{supp} \mathbf{F},\ \varphi \in \operatorname{supp} B_0\}. \tag{56}$$

Proof. Theorems 15 and 18 give the result in case $\mathbf{F} = S_p^*$. Assume now that $S_p^* \not\subset \mathbf{F}$. Let J satisfy the assumption of the theorem, and let $f = F \circ \varphi$ be a support point of $s(\mathbf{F})$ associated with J. Theorem 14 implies $\varphi \in \operatorname{supp} B_p$ since $z^p \in \mathbf{F}$. Define a continuous linear functional L on $\mathbf{A}$ by $L(g) = J(g \circ \varphi)$. If $\operatorname{Re} L$ is constant on $\mathbf{F}$, then since $z^p \in S_p^* \quad \mathbf{F}$ it follows that $\varphi^p \in \operatorname{supp} s(\mathbf{F})$. Hence since $\varphi^p \in s(S_p^*)$ and by Theorem 15 $\operatorname{Re} J$ is non-constant on $s(S_p^*)$ we conclude that $\varphi^p \in \operatorname{supp} s(S_p^*)$. This is a contradiction of Theorem 18 since $\varphi^p \notin EHs(S_p^*)$. It follows that $F \in \operatorname{supp} \mathbf{F}$ and so (56) holds.

Remark. This theorem generalizes Theorem 1 in [2] to the p-valent case.

THEOREM 22. $\operatorname{supp} C_p \subset EHC_p$.

Proof. This result may be proved using the methods that were used to prove the case $p = 1$ in [9]. It is only necessary to note that

$$\int_0^z \frac{p\tau^{p-1}}{(1-x\tau)^{2p}} d\tau \notin \operatorname{supp} S_p^* \qquad \text{for any } |x| = 1;$$

this follows from Theorem 2.1 in [11] and Theorem 16 in this paper. It is also necessary to use the fact that $C_p \subset S^p$.

We shall show that an analogous result holds for $s(C_p)$ when we consider only non-trivial functionals. Clearly functionals of the form

$$(\mathrm{J}') \qquad J(g) = \sum_{n=0}^{p} b_n \frac{g^{(n)}(0)}{n!}$$

generate support points $f(uz)$, $f \in C_p$, $u \in \partial\Delta$ ([13], p. 545).

THEOREM 23. *Let J be a continuous linear functional on* $\mathbf{A}$ *not of the form* (J′). *Then*

$$(57) \qquad \operatorname{supp} \{s(C_p); J\} \subset EHC_p.$$

Proof. Let J satisfy the assumptions of the theorem. Suppose $f \in \operatorname{supp} \{s(C_p); J\}$. Theorem 21 implies that $f = F \circ \varphi$, where $F \in \operatorname{supp} C_p$ and $\varphi \in \operatorname{supp} B_0$. By Theorem 22 we know that $F \in EHC_p$. Hence

$$(58) \qquad f(z) = \int_0^{\varphi(z)} \frac{p\tau^{p-1}(1-y\tau)}{(1-x\tau)^{2p+1}} d\tau, \qquad x, y \in \partial\Delta,\ x \neq y.$$

Let

$$\mathbf{F} = \left\{ \int_{(\partial\Delta)^3} \frac{puz^{p-1}(1-yz)}{(1-xz)^{2p+1}} d\mu(u, y, x) \colon \mu \in \Lambda_3 \right\}.$$

Since $f \in \operatorname{supp} Hs(C_p)$ we conclude from (15) that $f' \in \mathbf{F}$. It can easily be proved that $f' \in \operatorname{supp}\{\mathbf{F}, L\}$, where L is not of the form

$$L(g) = \sum_{n=0}^{p-1} d_n \frac{g^{(n)}(0)}{n!}.$$

It follows from (58) that

$$(59) \qquad f'(z) = p\left[\frac{\varphi(z)}{1-x\varphi(z)}\right]^{p-1} \frac{1-y\varphi(z)}{1-x\varphi(z)} \frac{\varphi'(z)}{(1-x\varphi(z))^2} \frac{1}{(1-x\varphi(z))^{p-1}}.$$

It follows from (48), (49) and

$$(60) \qquad \frac{1-y\varphi(z)}{1-x\varphi(z)} = \sum_{k=1}^{n} \lambda_k \frac{1-\bar{x}yx_k z}{1-x_k z}$$

that

$$(61) \qquad f'(z) = pz^{p-1} \sum_{\substack{k_1,\dots,k_{p-1}=1\\ m=1,k=1\\ j_1,\dots,j_{p-1}=1}}^{n} \lambda_{k_1}\dots\lambda_{k_{p-1}}\lambda_m\lambda_{j_1}\dots\lambda_{j_{p-1}} \times$$

$$\times\left(\frac{(\bar{x})^p x_{k_1}\dots x_{k_{p-1}} x_m}{(1-x_{k_1}z)\dots(1-x_{k_{p-1}}z)(1-x_m z)^2} \frac{1}{1-x_k z} \frac{1-\bar{x}yx_k z}{1-x_{j_1} z} \dots \frac{1}{1-x_{j_{p-1}} z}\right).$$

Clearly,

$$(62) \qquad \sum_{\substack{k_1,\dots,k_{p-1}\\ k=1,m=1\\ j_1,\dots,j_{p-1}=1}}^{n} \lambda_{k_1}\dots\lambda_{k_{p-1}}\lambda_m\lambda_k\lambda_{j_1}\dots\lambda_{j_{p-1}} = 1.$$

Theorem 1 ([4], p. 415), (61) and (62) imply that $f'(z)$ is a finite convex combination of functions from $\mathbf{F}$. Since $f'(z) \in \operatorname{supp}\{\mathbf{F}, L\}$ it follows that for every choice of $k_1, \dots, k_{p-1}, m, k, j_1, \dots, j_{p-1}$ in $\{1, 2, \dots, n\}$ the function

$$(63) \qquad h(z) = \frac{(\bar{x})^p x_{k_1}\dots x_{k_{p-1}} x_m z^{p-1}}{(1-x_{k_1}z)\dots(1-x_{k_{p-1}}z)(1-x_m z)^2} \frac{1}{1-x_k z} \frac{1-\bar{x}yx_k z}{1-x_{j_1} z} \dots \frac{1}{1-x_{j_{p-1}} z}$$

is in $\operatorname{supp}\{\mathbf{F}, L\}$. Let $a = (\bar{x})^p x_{k_1}\dots x_{k_{p-1}} x_m$. Then $|a| = 1$ and $h(z) = az^{p-1} + \dots$ Since $h \in \mathbf{F}$ we have

$$(64) \qquad a = \int_{(\Gamma,1)^3} u\, d\mu(u, y, x) \quad \text{for some } \mu \in \Lambda_3.$$

By arguing as we did near the end of the proof of Theorem 4 we conclude that

$$h(z) = \int_{\{a\}\times\partial\Delta\times\partial\Delta} \frac{upz^{p-1}(1-yz)}{(1-xz)^{2p+1}}\, d\mu(u, y, x). \tag{65}$$

Now for each measurable subset C of $\partial\Delta\times\partial\Delta$ define a probability measure $v \in \Lambda_2$ by $v(C) = \mu(\{a\}\times C)$. We can rewrite (65) as

$$h(z) = \int_{\partial\Delta\times\partial\Delta} \frac{apz^{p-1}(1-yz)}{(1-xz)^{2p+1}}\, dv(y, x) \tag{66}$$

for $v \in \Lambda_2$. Define the class

$$\mathbf{F}_1 = \left\{a \int_{\partial\Delta\times\partial\Delta} pz^{p-1} \frac{(1-yz)}{(1-xz)^{2p+1}}\, dv(y, x):\ v \in \Lambda_2\right\}.$$

By arguments similar to those used in [9] it can be proved that the support points of $\mathbf{F}_1$ consist of finite convex combinations of functions of the form $apz^{p-1}(1-yz)/(1-xz)^{2p+1}$, $y \neq x$. We have $h \in \operatorname{supp}\{\mathbf{F}, L\}$ and $h \in \mathbf{F}_1$, where $\mathbf{F}_1 \subset \mathbf{F}$. We wish to conclude that $h \in \operatorname{supp}\{\mathbf{F}_1, L\}$. It is only necessary to prove that $\operatorname{Re} L$ is non-constant on $\mathbf{F}_1$. Suppose $\operatorname{Re} L$ is constant on $\mathbf{F}_1$. Then

$$\operatorname{Re} L\left(\frac{z^{p-1}(1-yz)}{(1-xz)^{2p+1}}\right)$$

is constant for all $x, y \in \partial\Delta$. Note that for each fixed y

$$A(x) = L\left(\frac{z^{p-1}(1-yz)}{(1-xz)^{2p+1}}\right)$$

defines an analytic function on $\bar{\Delta}$. Since $\operatorname{Re} A(x)$ is constant for $x \in \bar{\Delta}$ we conclude $A(x) = A(0) = L(z^{p-1} - yz^p)$. Since $\operatorname{Re} L(z^{p-1} - yz^p)$ is constant for $y \in \bar{\Delta}$ we conclude $L(z^n) = 0$ for $n = p, p+1, p+2, \ldots$ This contradicts the form of L and so we conclude $\operatorname{Re} L$ is non-constant on $\mathbf{F}_1$. Hence $h \in \operatorname{supp}\{\mathbf{F}, L\}$. It follows that

$$h(z) = az^{p-1} \sum_{i=1}^{q} t_i \frac{1-y_i z}{(1-x_i z)^{2p+1}}, \quad \sum_{i=1}^{q} t_i = 1, \tag{67}$$

$$t_i \geqslant 0,\ x_i, y_i \in \partial\Delta,\ x_i \neq y_i,\ i = 1, 2, \ldots, q,$$

$$\frac{az^{p-1}}{(1-x_{k_1} z)\ldots(1-x_{k_{p-1}} z)}\frac{1}{(1-x_m z)^2}\frac{1-\bar{x}yx_k z}{1-x_k z}\frac{1}{1-x_{j_i} z}\cdots\frac{1}{1-x_{j_{p-1}} z} = az^{p-1} \sum_{i=1}^{q} t_i \frac{1-y_i z}{(1-x_i z)^{2p+1}}. \tag{68}$$

Since all the poles of the right-hand side of (68) are of order $2p+1$ and the sum of the orders of the poles on the left-hand side is $2p+1$ we conclude that $x_{k_1} = \ldots = x_{k_{p-1}} = x_m = x_{j_1} = \ldots = x_{j_{p-1}}$ for all choices of $k_1, \ldots, k_{p-1}, m, j, \ldots, j_{p-1} \in (1, 2, \ldots, n)$ and that $q = 1$. In particular $n = 1$ and therefore $\varphi(z) = uz$ for some $u \in \partial\Delta$. Hence

$$(69) \qquad f'(z) = \left(\int_0^{uz} \frac{p\tau^{p-1}(1-y\tau)}{(1-x\tau)^{2p+1}} d\tau \right)' = \frac{puz^{p-1}(1-yz)}{(1-xz)^{2p+1}}, \qquad x \neq y.$$

It follows that $f \in EHs(C_p)$ and (57) is proved.

Remark. In [2] it was proved that (57) holds for $p = 1$.

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