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Uniqueness of optimal trajectories for non-linear control systems

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Abstract. A uniqueness theorem for time-optimal trajectories of non-linear control systems is proved. In the theorem certain regularity conditions for the control system and smoothness for the surfaces of accessible sets are assumed. In applications the smoothness assumption is often satisfied for not too large (positive) time. Two variants of the uniqueness theorem are given.

The non-uniqueness of time-optimal trajectories for control systems occurs frequently. Nevertheless a general sufficient condition can be given. Under a weak assumption (for instance Lipschitz-continuity in space variables of a control system) the time-optimal solutions must be contained in the boundary of an emission zone [1] (the notion of emission zone, to be defined later, is closely connected with accessibility of sets). Therefore it is enough to consider the uniqueness of trajectories on the boundary of the emission zone. The uniqueness is proved under certain regularity conditions imposed upon a control system and a certain smoothness condition for the boundary of the emission zone. Two variants of the regularity condition for a control system are given. For the formulation of the first variant we introduce a suitable distance in the space of convex sets. In formulating the second variant a kind of Lipschitz-continuity of first order derivatives in the space of convex bodies is used. The smoothness condition for the boundary of the emission zone is satisfied for sufficiently small positive time, provided the control system satisfies certain regularity conditions, but it is not necessarily satisfied for large time even for regular control systems.

Notations and definitions. $|x| = (x_1^2+\ldots+x_n^2)^{\frac{1}{2}}$, $x\in R^n$,

$$q(a, B) = \inf_{x\in B}|a-x|, \qquad a\in R^n, B \subset R^n,$$

$$q(A, B) = \sup_{x\in A} q(x, B), \qquad A \subset R^n.$$

Hausdorff's distance is defined by the following well-known formula:

$$r(A, B) = \max\big(q(A, B), q(B, A)\big).$$

After T. Ważewski, we call any convex compact and non-empty subset of R^n an *orientor*. We say that an orientor function $F(t, x)$ is *of class L* at a point (s, y) if and only if there exist such positive numbers d, e and a neighbourhood N of the point (s, y) that for any point (t, x) of N and any vector w: $|w| = d$ there exists such an orientor A that

$$r\big(F(t, x+lw), (1-l)F(t, x)+lA\big) \leqslant el^2 \quad \text{for } |l| \leqslant 1. \tag{1}$$

Remark 1. For our purpose the independence of the constant d on a point from N is essential.

Let ∂A denote the boundary of an orientor A. We write $W(a, A) = \{w \in R^n\colon |w| = 1, w(x-a) \leqslant 0 \text{ for } x \in A\}$. For $a \in \partial A$ the set $W(a, A)$ is non-empty and compact. An orientor is called *p-convex*, where p is a positive number, if and only if for each point $a \in \partial A$ and each vector $w \in W(a, A)$ we have

$$(x-a)w+p|x-a|^2 \leqslant 0 \quad \text{for } x \in A. \tag{2}$$

Let w be a unit vector and A an orientor. Let us define $N(w, A) = \{x\colon w \in W(x, A)\}$. Let A, B be orientors. We define

$$s(A, B) = \sup\big\{r\big(N(w, A), N(w, B)\big)\colon |w| = 1\big\}. \tag{3}$$

It is easy to see that $s(A, B)$ is a (metric) distance and $s(A, B) \geqslant r(A, B)$. For strictly convex orientors we have

$$s(A, B) = \sup\{|a-b|\colon a \in \partial A, b \in \partial B, W(a, A) \cap W(b, B) \neq \emptyset\}, \tag{4}$$

where $\emptyset$ denotes the empty set. We shall show that for a family Q of orientors contained in a given ball S and its subfamily P of p-convex orientors we have the inequality

$$s(A, B)^2 \leqslant kr(A, B) \quad \text{for } A \in P, B \in Q \text{ (1)}, \tag{5}$$

where k is a positive constant. Let $a \in \partial A$, $b \in \partial B$ be such points that $s(A, B) = |a-b|$ and $W(a, A) \cap W(b, B) \neq \emptyset$. Let $w \in W(a, A) \cap W(b, B)$. If $(b-a)\, w+\frac{1}{2}p|b-a|^2 \leqslant 0$, then $(a-b)w \geqslant \frac{1}{2}p|b-a|^2$ and we get $(a-x)w \geqslant \frac{1}{2}p|b-a|^2$ for $x \in B$. Hence $|a-x| \geqslant \frac{1}{2}p|b-a|^2$ for $x \in B$ and therefore $r(A, B) \geqslant q(a, B) \geqslant \frac{1}{2}p|b-a|^2 = \frac{1}{2}ps(A, B)^2$. If $(b-a)w+\frac{1}{2}p|b-a|^2 \geqslant 0$, then for $h(x) = (x-a)w+p|x-a|^2$ we have the inequality $h(b) \geqslant \frac{1}{2}p|b-a|^2$. Because of the p-convexity of set A, $h(x) \leqslant 0$ for $x \in A$. The function $h(x)$ satisfies the inequality $|h(x)-h(y)| \leqslant k|x-y|$ for $x \in S$, $y \in S$, where the positive constant k depends on S and p only. We

(1) The possibility of extracting this property from the previous proof was suggested by S. Łojasiewicz.

have $k|x-b| \geqslant |h(x)-h(b)| \geqslant \frac{1}{2}p|b-a|^2$ for $x \in A$, and therefore $r(A, B) \geqslant q(A, b) \geqslant \frac{1}{2}pk^{-1}|b-a|^2 = \frac{1}{2}pk^{-1}s(A, B)^2$ and inequality (5) follows.

We recall the following property of Hausdorff's distance $r\big(B, (1-l)B+lA\big) = lr(A, B)$ for $0 \leqslant l \leqslant 1$, where A, B are orientors. It is easy to see that $s(A, B)$ satisfies the same formula

$$s\big(B, (1-l)B+lA\big) = ls(B, A) \quad \text{for } 0 \leqslant l \leqslant 1, \tag{6}$$

where A, B are orientors.

We write the control system in the form introduced by T. Ważewski [3] (the orientor form). A vector function $x(t)$ defined on an open interval J is said to be a *trajectory* of the control system

$$x' \in F(t, x), \tag{7}$$

(of the orientor field $F(t, x)$) if it is absolutely continuous on every compact subintegral of J and satisfies the condition $x'(t) \in F\big(t, x(t)\big)$ for almost every $t \in J$.

Let D be a given subset of R^{n+1}. The union of all trajectories of the control system (7) having at least one common point with D is called the *emission zone* of D with respect to (7) (to the orientor field $F(t, x)$). The intersection of the emission zone with the hyperplane $t = s$ (projected on R^n) is called an *accessible set* for $t = s$.

Control systems (orientor fields) can also be considered on differential manifolds (in particular, on differential submanifolds). The generalization is straightforward. We shall consider, as in [2], the orientor field induced by a field given on a subdomain of R^{n+1}, on the boundary of the emission zone. If the orientors are strictly convex and the boundary of the emission zone is differentiable manifold, the induced orientor field on the boundary reduces to a vector field, i.e. we get on the boundary an ordinary differential equation. The uniqueness of solutions of this equation implies in our case the uniqueness of time-optimal trajectories for the control system.

ASSUMPTION A. *Let the emission zone of a closed set contained in a half-space $t < q$, with respect to an orientor field $F(t, x)$ be represented in a neighbourhood of a point (q, v) by the inequality*

$$g(t, x) \leqslant 0, \tag{8}$$

where the function $g(t, x)$ is of class C^1 and its derivatives $g_{x_1}, \ldots, g_{x_n}$ are Lipschitz-continuous in x and do not vanish simultaneously at the point (q, v).

THEOREM 1. *Suppose Assumption* A. *Let the orientor field $F(t, x)$ be continuous (in Hausdorff's topology and satisfy the condition*

$$s\big(F(t, x), F(t, y)\big) \leqslant k|x-y|, \tag{9}$$

where k is a constant and $s(A, B)$ is defined by (3), *and let the orientors $F(t, x)$ be p-convex, $p > 0$, on a neighbourhood of the point (q, v).*

Under these assumptions the solutions of the initial problems for the ordinary differential equation induced by the orientor field $F(t, x)$ on the boundary of the emission zone are unique in a neighbourhood of the point (q, v).

Proof. We shall show that the right-hand side of the induced ordinary differential equation is Lipschitz-continuous in x and the uniqueness property will follow.

Let $(s, y), (s, z)$ be any points of the boundary of the emission zone in a sufficiently small neighbourhood of (q, v). We have $g(s, y) = g(s, z) = 0$. Consider the functions

$$h(u) = g_t(s, y) + g_{x_1}(s, y)u_1 + \ldots + g_{x_n}(s, y)u_n,$$

$$j(u) = g_t(s, z) + g_{x_1}(s, z)u_1 + \ldots + g_{x_n}(s, z)u_n.$$

It can be seen that the set $Y = F(s, y)$ (set $Z = F(s, z)$) is contained in the half-space $h(u) \leqslant 0$ (the half-space $j(u) \leqslant 0$) and the intersection of the set Y (set Z) with the hyperplane $h(u) = 0$ (hyperplane $j(u) = 0$) consists of a single point a (point b). The induced differential equation $x' = f(t, x)$ satisfies the equalities $f(s, y) = a$, $f(s, z) = b$. Let e be such a number that Z is contained in the half-space $h(u) + e \leqslant 0$ and the intersection of Z with the hyperplane $h(u) + e$ is non-empty. The intersection consists of a single point c. It follows from (9), (4), that $|c - a| \leqslant k|z - y|$. If $c = b$ the proof is complete. Suppose $b \neq c$. Let $w \in W(b, Z)$, $\tilde{w} \in W(c, Z)$. In virtue of (2) we have $p|c - b|^2 \leqslant -(c - b)w$, $p|b - c|^2 \leqslant -(b - c)\tilde{w}$. We get $2p|c - b|^2 \leqslant (b - c)(w - \tilde{w})$ and it follows that $2p|c - b|^2 \leqslant |b - c||w - \tilde{w}|$. Hence $2p|c - b| \leqslant |w - \tilde{w}|$ for $w \in W(b, Z)$, $w \in W(c, Z)$. Consider the vectors $d = (g_{x_1}(s, z), \ldots, g_{x_n}(s, z))$, $\tilde{d} = (g_{x_1}(s, y), \ldots, g_{x_n}(s, y))$. It is easy to see that $|d|^{-1}d \in W(b, Z)$, $|\tilde{d}|^{-1}\tilde{d} \in W(c, Z)$. Therefore $2p|b - c| \leqslant \big||d|^{-1}d - |\tilde{d}|^{-1}\tilde{d}\big| \leqslant k_1|y - z|$, where k_1 is a suitable constant, in virtue of Assumption A. We obtain $|a - b| \leqslant |a - c| + |c - b| \leqslant (k + \frac{1}{2}p^{-1}k_1)|z - y|$. We have proved the Lipschitz-continuity of the induced equation. The proof of Theorem 1 is thus complete.

Remark 2. The metric $s(A, B)$ in condition (9) cannot be replaced by Hausdorff's distance $r(A, B)$. An example of a Lipschitz-continuous (in Hausdorff's metric) orientor field with the induced equation without the uniqueness property is given in [2].

THEOREM 2. *Assume Assumption* A. *Let the orientor field $F(t, x)$ be continuous on a neighbourhood of point (q, v) and of class L at (q, v) and let the orientors $F(t, x)$ be p-convex, $p > 0$.*

Under these assumptions the solutions of the initial problems for the *differential equation induced by $F(t, x)$ on the boundary of the emission zone are unique in a neighbourhood of point (q, v).*

Proof. We shall show that our assumptions imply (9) and Theorem 2 will follow from Theorem 1.

Let $(t, x), (t, y)$ be points from a sufficiently small neighbourhood of (q, v). We assume $x \neq y$; for $x = y$ (9) is trivial. We apply (1) for $w = (y-x)d|y-x|^{-1}$, $l = |y-x|d^{-1}$. We get $r\big(F(t, y), (1-l)F(t, x)+lA\big) \leqslant el^2$. In virtue of (5) we have

$$s\big(F(t, y), (1-l)F(t, x)+lA\big) \leqslant k^{\frac{1}{2}}e^{\frac{1}{2}}l. \tag{10}$$

For (t, x) from a sufficiently small neighbourhood of (q, v), $F(t, x)$ are contained in a common ball because of the continuity of $F(t, x)$. In virtue of (1) the sets A are contained in a common ball for arbitrary w; $|w| = d$ and (t, x) from a neighbourhood of (q, v); therefore $s\big(F(t, x), A\big) \leqslant m$, where m is a constant independent of (t, x) or w. In virtue of (6) $s\big(F(t, x), (1-l)F(t, x)+lA\big) \leqslant lm$. Hence in virtue of (10)

$$s\big(F(t, x), F(t, y)\big) \leqslant (m+k^{\frac{1}{2}}e^{\frac{1}{2}})d^{-1}|y-x|.$$

Remark 3. Theorem 2 can be reformulated for the control system in the form $x' = f(t, x, u)$, $u \in U$.

References

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