ANNALES
POLONICI MATHEMATICI
XX (1968)

On certain functional equations for quasiconformal mappings

by J. Ławrynowicz (Łódź)

Introduction. The Cauchy-Riemann equations for analytic functions and the Beltrami equations for pseudo-analytic functions do not give information on the invertibility of these functions if there are no additional assumptions. Therefore, if we investigate various properties of conformal or quasiconformal mappings, for instance extremal properties, we usually have to apply more complicated equations, such as the parametric ones, which quarantee the invertibility of a solution, and characterize a priori the image of the domain considered if the initial or boundary conditions are known.

In this paper the author obtains a system of two non-parametric functional equations for quasiconformal mappings of the unit disc onto itself, applying the Shah Tao-shing method (cf. [7] and [5]). Unfortunately, one of those equations is obtained in a subclass, and the problem of its density in the whole class considered remains open. Nevertheless, the author conjectures that the system of equations constructed always has a unique solution for a dense subclass of the class of all complex dilatations (i.e. functions measurable and bounded by constants <1), under suitable boundary conditions, and that this solution represents a quasiconformal mapping of the unit disc onto itself.

Analogous results are obtained for quasiconformal mappings of an annulus onto another annulus, and an analogous conjecture is posed.

Theorems 1 and 2 of this paper have been proved in [7] and [5]. Theorems 3 and 6 concern the first of the equations obtained (in dense subclasses) in the cases of quasiconformal mappings in the unit disc and in an annulus, respectively. Theorems 4 and 7 concern the second of the equations obtained in the cases of quasiconformal mappings in the unit disc and in an annulus, respectively. Those equations are presented in a more symmetric form in Theorems 5 and 8.

The theorems proved in this paper were presented to the Conference on Analytic Functions in Łódź on the 5^{th} of September, 1966 (see [6]).

On this place I should like to express my sincere thanks to Professors Ch. Pommerenke from Göttingen (now in London) and J. Krzyż from Lublin for their helpful remarks. During the preparation of this paper the author held a Scholarship of the Polish Academy of Sciences at the Imperial College, London, under the guidance of Professor W. K. Hayman.

§ 1. Parametric equations. Let

$$K = \{z\colon\ |z| < 1\}, \quad \bar{K} = \{z\colon\ |z| \leqslant 1\},$$
$$K_r = \{z\colon\ r < |z| < 1\}, \quad \bar{K}_r = \{z\colon\ r \leqslant |z| \leqslant 1\}.$$

Let "$g(z,t) \rightrightarrows g(z)$ as $t \to t_0$" for an open set D mean the so-called almost uniform convergence in D (i.e. the uniform convergence on compact subsets of D) and the convergence of $\operatorname{re}\{(1/z)g(z,t)\}$ on its closure.

In this paper we need the following lemmas and theorems (see [5]):

LEMMA 1. *Let $\sigma = \mu(z,t)$ be a function defined in $\bar{K} \times \{t\colon\ 0 < t \leqslant T\}$, belonging to C^1 and bounded by $Q^*(t) < 1$ in absolute value in $\bar{K}$ for any t and such that*

$$(1) \qquad (1/t)\mu(z,t) \rightrightarrows \varphi(z) \quad \text{for} \quad t \to 0+,$$

$$(2) \qquad (1/t)\,|\partial\mu(z,t)/\partial z| \leqslant k(z) \quad \text{for} \quad 0 < t \leqslant T,$$

where φ and k are bounded. Let $Q(t) = \{1+Q^*(t)\}/\{1-Q^*(t)\}$. Then, for the $Q(t)$-quasiconformal mapping $w = f(z,t)$ of $\bar{K}$ onto itself, generated by the complex dilatation μ so that $f(0,t) = 0$ and $f(1,t) = 1$, the following formula holds in K:

$$(3) \qquad (1/t)\{f(z,t)-z\} \rightrightarrows (1/\pi)z(1-z)\iint\limits_{|\zeta|\leqslant 1}\{\varphi(\zeta)/\zeta(1-\zeta)(z-\zeta)+$$
$$+\overline{\varphi(\zeta)}/\bar{\zeta}(1-\bar{\zeta})(1-z\bar{\zeta})\}\,d\xi\,d\eta \quad \text{for } t\to 0+ \quad (\zeta = \xi+i\eta).$$

THEOREM 1. *Let $\sigma = \mu(z,t)$ be a function defined in $\bar{K} \times \{t\colon\ 0 \leqslant t \leqslant T\}$, bounded by $Q^*(t) < 1$ in absolute value and such that the partial derivatives $\partial\mu/\partial z$, $\partial\mu/\partial\bar{z}$, $\partial\mu/\partial t$, $\partial^2\mu/\partial z\partial t$ exist and $\partial\mu/\partial z$, $\partial\mu/\partial\bar{z}$ fulfil a global Hölder condition with a certain exponent δ $(0 < \delta \leqslant 1)$. Then the $Q(t)$-quasiconformal* (1) *mapping $w = f(z,t)$ of $\bar{K}$ onto itself, generated by the complex dilatation μ so that $f(0,t) = 0$ and $f(1,t) = 1$, satisfies in K the equation*

$$(4) \qquad \partial w/\partial t = (1/\pi)w(1-w)\iint\limits_{|\zeta|\leqslant 1}\{\varphi(\zeta,t)/\zeta(1-\zeta)(w-\zeta)+$$
$$+\overline{\varphi(\zeta,t)}/\bar{\zeta}(1-\bar{\zeta})(1-w\bar{\zeta})\}\,d\xi\,d\eta \quad (\zeta = \xi+i\eta),$$

where the function φ is defined by the formula

$$(5) \qquad \varphi(w,t)$$
$$= \frac{1}{1-|\mu(f^{-1}(w,t),t)|^2}\cdot\frac{\partial}{\partial t}\mu(f^{-1}(w,t),t)\exp\left(-2i\arg\frac{\partial}{\partial w}f^{-1}(w,t)\right).$$

(1) The connection between $Q(t)$ and $Q^*(t)$ is the same as in Lemma 1.

LEMMA 2. *Let $\sigma = \mu(z, t)$ be a function defined in $\bar{K}_r \times \{t:\ 0 < t \leqslant T\}$, belonging to C^1 and bounded by $Q^*(t) < 1$ in absolute value in $\bar{K}_r$ for any t and fulfilling conditions* (1) *and* (2), *where φ and k are bounded. Then, for the $Q(t)$-quasiconformal* (¹) *mapping $w = f(z, t)$ of $\bar{K}_r$ onto $\bar{K}_{R(t)}$, generated by the complex dilatation μ so that $f(1, t) = 1$, where $\varrho = R(t)$ is determined uniquely, the following formula holds in K_r* (²):

$$(6)\qquad (1/t)\{f(z, t) - z\} \rightrightarrows (1/2\pi) \iint\limits_{r \leqslant |\zeta| \leqslant 1} z \sum_{\nu=-\infty}^{+\infty} \left\{ \frac{\varphi(\xi)}{\zeta^2}\left(\frac{z + r^{2\nu}\zeta}{z - r^{2\nu}\zeta} - \frac{1 + r^{2\nu}\zeta}{1 - r^{2\nu}\zeta}\right) - \frac{\overline{\varphi(\zeta)}}{\bar{\zeta}^2}\left(\frac{1 + r^{2\nu}z\bar{\zeta}}{1 - r^{2\nu}z\bar{\zeta}} - \frac{1 + r^{2\nu}\bar{\zeta}}{1 - r^{2\nu}\bar{\zeta}}\right)\right\} d\xi\, d\eta \quad \textit{for} \quad t \to 0+ \qquad (\zeta = \xi + i\eta).$$

Moreover,

$$(7)\qquad (1/t)\{R(t) - r\} \to (1/2\pi) \iint\limits_{r \leqslant |\zeta| \leqslant 1} r\{\varphi(\zeta)/\zeta^2 + \overline{\varphi(\zeta)}/\bar{\zeta}^2\}\, d\xi\, d\eta \quad \textit{for} \quad t \to 0+ .$$

THEOREM 2. *Let $\sigma = \mu(z, t)$ be a function defined in $\bar{K}_r \times \{t:\ 0 \leqslant t \leqslant T\}$, bounded by $Q^*(t) < 1$ in absolute value and such that the partial derivatives $\partial\mu/\partial z$, $\partial\mu/\partial\bar{z}$, $\partial\mu/\partial t$, $\partial^2\mu/\partial z\partial t$ exist and $\partial\mu/\partial z$, $\partial\mu/\partial\bar{z}$ fulfil a global Hölder condition with a certain exponent δ $(0 < \delta \leqslant 1)$. Then the $Q(t)$-quasiconformal* (¹) *mapping $w = f(z, t)$ of $\bar{K}_r$ onto $\bar{K}_{R(t)}$, generated by the complex dilatation μ so that $f(1, t) = 1$, where $\varrho = R(t)$ is determined uniquely, satisfies in K_r the equation* (²)

$$(8)\qquad \partial w/\partial t = (1/2\pi) \iint\limits_{R(t) \leqslant |\zeta| \leqslant 1} w \sum_{\nu=-\infty}^{+\infty} \left\{ \frac{\varphi(\zeta, t)}{\zeta^2}\left(\frac{w + R^{2\nu}(t)\zeta}{w - R^{2\nu}(t)\zeta} - \frac{1 + R^{2\nu}(t)\zeta}{1 - R^{2\nu}(t)\zeta}\right) - \frac{\overline{\varphi(\zeta, t)}}{\bar{\zeta}^2}\left(\frac{1 + R^{2\nu}(t)w\bar{\zeta}}{1 - R^{2\nu}(t)w\bar{\zeta}} - \frac{1 + R^{2\nu}(t)\bar{\zeta}}{1 - R^{2\nu}(t)\bar{\zeta}}\right)\right\} d\xi\, d\eta$$

$$(R^{2\nu}(t) = \{R(t)\}^{2\nu},\ \zeta = \xi + i\eta),$$

where φ is defined by (5). *Moreover, $R \in C^1$ in $\{t:\ 0 \leqslant t \leqslant T\}$, and*

$$(9)\qquad R'(t) = (1/2\pi) \iint\limits_{R(t) \leqslant |\zeta| \leqslant 1} R(t)\{\varphi(\zeta, t)/\zeta^2 + \overline{\varphi(\zeta, t)}/\bar{\zeta}^2\}\, d\xi\, d\eta .$$

§ 2. The first functional equation in the unit disc. We confine ourselves to quasiconformal mappings whose complex dilatations belong to C^2. It is known (see [2] and [5]) that the class of these mappings is dense in the whole class of quasiconformal mappings, in the case of both the unit disc and an annulus. We prove the following

---

(²) For the sake of simplicity we apply the notation $\sum_{\nu=-\infty}^{+\infty} a_\nu$ instead of $a_0 + \sum_{\nu=1}^{+\infty} (a_\nu + a_{-\nu})$ provided the last series converges.

THEOREM 3. *Let $\sigma = \mu(z)$ be a function of the class C^2 defined in $\bar{K}$ and bounded in it by $Q^* < 1$ in absolute value. Let $Q = (1+Q^*)/(1-Q^*)$. Then the Q-quasiconformal mapping $w = f(z)$ of $\bar{K}$ onto itself, generated by the complex dilatation μ so that $f(0) = 0$ and $f(1) = 1$, satisfies in K the equation*

$$\partial w/\partial\vartheta = w(\partial w/\partial\vartheta)_{z=1} + \Phi_1(w)\,, \tag{10}$$

where

$$\Phi_1(w) = \frac{w(1-w)}{\pi}\iint\limits_{|\zeta|\leqslant 1}\left\{\frac{\varphi_1(\zeta)}{\zeta(1-\zeta)(w-\zeta)} + \frac{\overline{\varphi_1(\zeta)}}{\bar{\zeta}(1-\bar{\zeta})(1-w\bar{\zeta})}\right\}d\xi\,d\eta\,, \tag{11}$$

$$\vartheta = \arg z\,, \qquad \xi = \mathrm{re}\,\zeta\,, \qquad \eta = \mathrm{im}\,\zeta\,, \tag{12}$$

$$\varphi_1(w) = \frac{1}{1-\left|\mu\left(f^{-1}(w)\right)\right|^2}\left\{\left(\frac{\partial}{\partial\vartheta}\mu(z)\right)_{z=f^{-1}(w)} - 2i\left(f^{-1}(w)\right)\right\}\times \times\exp\left(-2i\arg\frac{\partial}{\partial w}f^{-1}(w)\right). \tag{13}$$

Proof. We apply here Lemma 1 and the proof runs analogously to that of Theorem 1. For more clearness it is divided into four steps.

Step A. Construction of a suitable function satisfying the assumptions of Lemma 1. In order to verify that $w = f(z)$ satisfies equation (10) we construct a suitable function satisfying the assumptions of Lemma 1; we denote this function by G. By a suitable function we understand any function of the variable w that maps $\bar{K}$ onto itself, the points 0 and 1 being fixed, depends on one real parameter $\Delta\vartheta$ $(0 < \Delta\vartheta \leqslant \Delta\vartheta^*)$, and fulfils in K the condition

$$(1/\Delta\vartheta)\{G(w, \Delta\vartheta) - w\} \rightrightarrows \partial w/\partial\vartheta + B(w) \quad \text{for} \quad \Delta\vartheta \to 0+\,, \tag{14}$$

where $w = f(z)$ and B is a function chosen in such a way that G fulfils also the remaining conditions of Lemma 1.

The proof of Theorem 1 suggests that we should consider the function

$$G(w, \Delta\vartheta) = f\left(e^{i\Delta\vartheta}f^{-1}(w)\right)/f\left(e^{i\Delta\vartheta}\right). \tag{15}$$

Clearly, G maps $\bar{K}$ onto itself and preserves the points 0 and 1 for any real $\Delta\vartheta$. Moreover,

$$\lim_{\Delta\vartheta\to 0+}\frac{G(w, \Delta\vartheta) - w}{\Delta\vartheta}$$
$$= \lim_{\Delta\vartheta\to 0+}\frac{f\left(e^{i\Delta\vartheta}f^{-1}(w)\right) - f\left(f^{-1}(w)\right)}{\Delta\vartheta f\left(e^{i\Delta\vartheta}\right)} + \lim_{\Delta\vartheta\to 0+}\frac{f\left(f^{-1}(w)\right)f(e^0) - wf\left(e^{i\Delta\vartheta}\right)}{\Delta\vartheta f\left(e^{i\Delta\vartheta}\right)}$$
$$= \frac{\partial}{\partial\vartheta}f(z) - f(z)\left(\frac{\partial}{\partial\vartheta}f(z)\right)_{z=1},$$

where $\vartheta = \arg z$, $z = f^{-1}(w)$. Hence

$$B(w) = w(\partial w/\partial\vartheta)_{z=1} \,. \tag{16}$$

It remains to verify whether the function G, defined by (15) in $\bar{K} \times \{\Delta\vartheta\colon\ 0 < \Delta\vartheta \leqslant \Delta\vartheta^*\}$, satisfies the remaining conditions of Lemma 1, and, of course, to evaluate in our case the function φ appearing in this lemma. In order to avoid ambiguity we shall write in our case φ_1. Thus, we introduce in $\bar{K} \times \{\Delta\vartheta\colon\ 0 < \Delta\vartheta \leqslant \partial\vartheta^*\}$ the following notation:

$$\mu_1(w, \Delta\vartheta) = \frac{\partial}{\partial\bar{w}} G(w, \Delta\vartheta) \Big/ \frac{\partial}{\partial w} G(w, \Delta\vartheta) \,, \tag{17}$$

$$\varphi_1(w) = \lim_{\Delta\vartheta\to 0+} \{(1/\Delta\vartheta)\mu_1(w, \Delta\vartheta)\} \,. \tag{18}$$

The derivatives in (17) exist everywhere in $\bar{K}$, because, in view of Theorem 7.3 of [8], the assumption that $\mu \in C^2$ implies that $f \in C^2$, and thus, by (15), the function G must also be of class C^2 for any real $\Delta\vartheta$.

For reasons of arrangement we start with expressing the functions μ_1 and φ_1 in terms of f and μ.

Step B. Evaluation of the functions μ_1 and φ_1. Note first the identities

$$\partial\bar{w}/\partial z = \overline{\partial w/\partial\bar{z}} \,, \qquad \partial\bar{w}/\partial\bar{z} = \overline{\partial w/\partial z} \tag{19}$$

which can easily be verified. Since in our case the functions $w - f$ and $\bar{w} - \bar{f}$, considered as functions of the variables $z, \bar{z}, w, \bar{w}$, satisfy the assumptions of a well-known theorem on implicit functions (see e.g. [3], vol. I, p. 454), we have

$$\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial w} + \frac{\partial f}{\partial\bar{z}}\cdot\frac{\partial\bar{z}}{\partial w} = 1 \,, \tag{20}$$

$$\frac{\partial\bar{f}}{\partial z}\cdot\frac{\partial z}{\partial w} + \frac{\partial\bar{f}}{\partial\bar{z}}\cdot\frac{\partial\bar{z}}{\partial w} = 0 \,, \tag{21}$$

where (21), in view of (19), may be replaced by

$$\overline{\frac{\partial f}{\partial\bar{z}}}\cdot\frac{\partial z}{\partial w} + \overline{\frac{\partial f}{\partial z}}\frac{\partial\bar{z}}{\partial w} = 0 \,. \tag{22}$$

Equations (20) and (22) yield

$$\frac{\partial z}{\partial w} = \overline{\frac{\partial f}{\partial z}} \Big/ \left(\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial\bar{z}}\right|^2\right), \tag{23}$$

$$\frac{\partial\bar{z}}{\partial w} = -\overline{\frac{\partial f}{\partial\bar{z}}} \Big/ \left(\left|\frac{\partial f}{\partial z}\right|^2 - \left|\frac{\partial f}{\partial\bar{z}}\right|^2\right), \tag{24}$$

whence by (19) we get

$$\frac{\partial \bar z}{\partial \bar w}=\frac{\partial f}{\partial z}\Big/\left(\left|\frac{\partial f}{\partial z}\right|^2-\left|\frac{\partial f}{\partial \bar z}\right|^2\right), \tag{25}$$

$$\frac{\partial z}{\partial \bar w}=-\frac{\partial f}{\partial \bar z}\Big/\left(\left|\frac{\partial f}{\partial z}\right|^2-\left|\frac{\partial f}{\partial \bar z}\right|^2\right). \tag{26}$$

The relations obtained above permit differentiation in formula (17). In view of (15) we have

$$\begin{aligned}\frac{\partial}{\partial w}G(w,\Delta\vartheta)\\ &= \{e^{i\Delta\theta}/f(e^{i\Delta\theta})\}\left(\frac{\partial}{\partial s}f(s)\right)_{s=z^*}\frac{\partial z}{\partial w}+\{e^{-i\Delta\theta}/f(e^{i\Delta\theta})\}\left(\frac{\partial}{\partial \bar s}f(s)\right)_{s=z^*}\frac{\partial \bar z}{\partial w}\end{aligned} \tag{27}$$

and

$$\begin{aligned}\frac{\partial}{\partial \bar w}G(w,\Delta\vartheta)\\ &= \{e^{i\Delta\theta}/f(e^{i\Delta\theta})\}\left(\frac{\partial}{\partial s}f(s)\right)_{s=z^*}\frac{\partial z}{\partial \bar w}+\{e^{-i\Delta\theta}/f(e^{i\Delta\theta})\}\left(\frac{\partial}{\partial \bar s}f(s)\right)_{s=z^*}\frac{\partial \bar z}{\partial \bar w},\end{aligned} \tag{28}$$

where $z^* = z\exp(i\Delta\vartheta)$, $z=f^{-1}(w)$. Now, applying relations (23) and (24) to formula (27), and relations (25) and (26) to formula (28), we obtain

$$\begin{aligned}f(e^{i\Delta\theta})\left\{\left|\frac{\partial}{\partial z}f(z)\right|^2-\left|\frac{\partial}{\partial \bar z}f(z)\right|^2\right\}\frac{\partial}{\partial w}G(w,\Delta\vartheta)\\ &= e^{i\Delta\theta}\left(\frac{\partial}{\partial s}f(s)\right)_{s=z^*}\overline{\frac{\partial}{\partial z}f(z)}-e^{-i\Delta\theta}\left(\frac{\partial}{\partial \bar s}f(s)\right)_{s=z^*}\overline{\frac{\partial}{\partial \bar z}f(z)}\end{aligned}$$

and

$$\begin{aligned}f(e^{i\Delta\theta})\left\{\left|\frac{\partial}{\partial z}f(z)\right|^2-\left|\frac{\partial}{\partial \bar z}f(z)\right|^2\right\}\frac{\partial}{\partial \bar w}G(w,\Delta\vartheta)\\ &= -e^{i\Delta\theta}\left(\frac{\partial}{\partial s}f(s)\right)_{s=z^*}\frac{\partial}{\partial \bar z}f(z)+e^{-i\Delta\theta}\left(\frac{\partial}{\partial \bar s}f(s)\right)_{s=z^*}\frac{\partial}{\partial z}f(z).\end{aligned}$$

Hence

$$\mu_1(w,\Delta\vartheta)=\frac{-e^{i\Delta\theta}\left(\frac{\partial}{\partial s}f(s)\right)_{s=z^*}\frac{\partial}{\partial \bar z}f(z)+e^{-i\Delta\theta}\left(\frac{\partial}{\partial \bar s}f(s)\right)_{s=z^*}\frac{\partial}{\partial z}f(z)}{e^{i\Delta\theta}\left(\frac{\partial}{\partial s}f(s)\right)_{s=z^*}\overline{\frac{\partial}{\partial z}f(z)}-e^{-i\Delta\theta}\left(\frac{\partial}{\partial \bar s}f(s)\right)_{s=z^*}\overline{\frac{\partial}{\partial \bar z}f(z)}}.$$

Let us note finally that, according to the notation of our Theorem, by virtue of the Beltrami equation for $w=f(z)$, we have

$$\frac{\partial}{\partial \bar z}f(z)\Big/\frac{\partial}{\partial z}f(z)=\mu(z). \tag{29}$$

Hence

$$\mu_1(w, \Delta\vartheta) = \frac{\mu(ze^{i\Delta\vartheta}) - e^{2i\Delta\vartheta}\mu(z)}{e^{2i\Delta\vartheta} - \mu(ze^{i\Delta\vartheta})\overline{\mu(z)}} \exp\left(2i\arg\frac{\partial}{\partial z}f(z)\right). \tag{30}$$

Dividing both sides of (30) by $\Delta\vartheta$ and letting $\Delta\vartheta \to 0+$, in view of (18) and the assumed existence of $\partial\mu/\partial\vartheta$, where $\vartheta = \arg z$, we easily obtain

$$\varphi_1(w) = \frac{(\partial/\partial\vartheta)\mu(z) - 2i\mu(z)}{1-|\mu(z)|^2} \exp\left(2i\arg\frac{\partial}{\partial z}f(z)\right). \tag{31}$$

Formulae (30) and (31), where $z = f^{-1}(w)$, give the expressions for μ_1 and φ_1, as desired. In Step D it will be proved that μ_1 can also be expressed by formula (13).

Step C. Verification of some properties of the function G. It remains to verify only the following properties of G, which concern the corresponding complex dilatation μ_1 considered in $\overline{K} \times \{\Delta\vartheta\colon 0 < \Delta\vartheta \leqslant \Delta\vartheta^*\}$, where $\Delta\vartheta^*$ is chosen sufficiently small:

$$\mu_1 \in C^1, \tag{32}$$

$$|\mu_1(w, \Delta\vartheta)| \leqslant Q_1^*(\Delta\vartheta) \leqslant 1, \tag{33}$$

$$(1/\Delta\vartheta)\mu_1(w, \Delta\vartheta) \rightrightarrows \varphi_1(w) \quad \text{for} \quad \Delta\vartheta \to 0+ \quad (|\varphi_1(w)| \leqslant \varphi_0 < +\infty), \tag{34}$$

$$(1/\Delta\vartheta)|(\partial/\partial w)\mu_1(w, \Delta\vartheta)| \leqslant k(w) \leqslant k_0 < +\infty. \tag{35}$$

In fact, it has been assumed that $\mu \in C^2$ and proved that $f \in C^2$, and so, by (31), relation (32) holds. Similarly, relation (33) is obvious, because, as can easily be seen,

$$Q_1^*(\Delta\vartheta) = \sup_{|w|\leqslant 1} \frac{|\mu(ze^{i\Delta\vartheta}) - e^{2i\Delta\vartheta}\mu(z)|}{|e^{2i\Delta\vartheta} - \mu(ze^{i\Delta\vartheta})\overline{\mu(z)}|} < 1,$$

provided $\Delta\vartheta^*$ is chosen sufficiently small.

Now, let us observe that to prove (34), in view of (31) and (18), it remains to prove that we have

$$(1/\Delta\vartheta)\mu_1(w, \Delta\vartheta) \to \varphi_1(w) \tag{36}$$

uniformly in $\overline{K}$. So, let ε, w, $\Delta\vartheta$ be arbitrary numbers fulfilling the conditions $\varepsilon > 0$, $|w| \leqslant 1$, $0 \leqslant \Delta\vartheta \leqslant \Delta\vartheta^*$, respectively, and let $z = f^{-1}(w)$, $\vartheta = \arg z$. Clearly, we have

$$1/(1-|\mu(z)|^2) \leqslant (Q+1)^2/4Q. \tag{37}$$

Moreover, from the assumption of the existence of $\partial\mu/\partial\vartheta$ we easily infer that for a certain η we have

$$\left|\frac{1}{e^{2i\Delta\vartheta} - \mu(ze^{2i\Delta\vartheta})\overline{\mu(z)}} - \frac{1}{1-|\mu(z)|^2}\right| \leqslant \frac{\varepsilon}{2M} \quad (0 < \Delta\vartheta < \eta). \tag{38}$$

On the other hand, it can easily be seen that $|(\partial/\partial\vartheta)\mu(z) - 2i\mu(z)| \leqslant M^*$.

Here M, $M^* < +\infty$, and we can suppose that $M = M^*$. Hence, by a well-known theorem,

$$(39)\qquad |\mu(ze^{i\Delta\vartheta}) - e^{2i\Delta\vartheta}\mu(z)| \leqslant M\Delta\vartheta \qquad (0 < \Delta\vartheta < \eta)\,.$$

Moreover, for a certain η^* we have

$$(40)\quad \left|\frac{\mu(ze^{i\Delta\vartheta}) - e^{2i\Delta\vartheta}\mu(z)}{\Delta\vartheta} - \frac{\partial}{\partial\vartheta}\mu(z) + 2i\mu(z)\right| \leqslant \frac{2Q\varepsilon}{(Q+1)^2} \qquad (0 < \Delta\vartheta < \eta^*)\,.$$

From inequalities (37)-(40) we immediately obtain

$$\left|(1/\Delta\vartheta)\frac{\mu(ze^{i\Delta\vartheta}) - e^{2i\Delta\vartheta}\mu(z)}{e^{2i\Delta\vartheta} - \mu(ze^{i\Delta\vartheta})\overline{\mu(z)}} - \frac{(\partial/\partial\vartheta)\mu(z) - 2i\mu(z)}{1 - |\mu(z)|^2}\right|$$
$$\leqslant \left|\frac{\mu(ze^{i\Delta\vartheta}) - e^{2i\Delta\vartheta}\mu(z)}{\Delta\vartheta}\right| \left|\frac{1}{e^{2i\Delta\vartheta} - \mu(ze^{i\Delta\vartheta})\overline{\mu(z)}} - \frac{1}{1 - |\mu(z)|^2}\right| +$$
$$+ \left|\frac{\mu(ze^{i\Delta\vartheta}) - e^{2i\Delta\vartheta}\mu(z)}{\Delta\vartheta} - \frac{\partial}{\partial\vartheta}\mu(z) + 2i\mu(z)\right| \frac{1}{1 - |\mu(z)|^2} < \varepsilon$$

in the interval $0 < \Delta\vartheta < \min(\eta, \eta^*)$. Hence, by (30) and (31), in the same interval we have

$$|(1/\Delta\vartheta)\mu_1(w, \Delta\vartheta) - \varphi_1(w)| < \varepsilon \qquad (|w| \leqslant 1)\,,$$

i.e. uniform convergence takes place in (36). Thus, there exists a function φ_1 which fulfils (34); it is uniquely determined and is expressed by (31).

In connection with the question of the existence of the function k which fulfils (35) let us notice first that the left-hand side of this inequality exists in view of (30) and of the previously shown fact that the function f belongs to C^2. The existence of the function k fulfilling (35) follows from the fact that the function f belongs to C^2, which implies in particular the existence of $\partial^2\mu/\partial z\partial\vartheta$. In fact, if for $|z| \leqslant 1$ the derivative $\partial^2\mu/\partial z\partial\vartheta$ exists, then, in view of (30) and $0 \leqslant |\mu(z)| \leqslant Q^*$, the derivative $\partial^2\mu_1/\partial w\partial\vartheta$ also exists, and we have $|(\partial^2/\partial w\partial\vartheta)\mu_1(w, \Delta\vartheta)| \leqslant M^{**}$ for $0 < \Delta\vartheta \leqslant \Delta\vartheta^*$, where $M^{**} < +\infty$. Hence we infer that in $\bar{K} \times \{\Delta\vartheta\colon 0 < \Delta\vartheta \leqslant \Delta\vartheta^*\}$ takes place an estimate $|(\partial/\partial w)\mu_1(w, \Delta\vartheta) - (\partial/\partial w)\mu_1(w, 0)| \leqslant M^{**}\Delta\vartheta$, where the existence of $\partial\mu_1/\partial w$ for $\partial\vartheta = 0$, considered as a corresponding limit, follows immediately from (30) and, as is easily seen, we have $(\partial/\partial w)\mu_1(w, 0) = 0$. Thus we may write the last inequality in the form $|(1/\Delta\vartheta)(\partial/\partial w) \times \times \mu_1(w, \Delta\vartheta)| \leqslant M^{**}$. This means that estimate (35) holds, and that we may put $k(w) = k_0 = M^{**}$ identically.

In this way we have proved that the function G, constructed in Step A of our proof, satisfies all the assumptions of Lemma 1.

Step D. The differential equation for the class considered. In the previous parts of our proof we have constructed the function G determined by formula (15) and fulfilling condition (14), and we have verified that this functions satisfie the assumptions of Lemma 1. Therefore, applying this lemma to the function G, we obtain by (14), (15) and (16), equation (10), as desired.

According to Steps B and C of our proof, the function φ_1 which appears in equation (10) is determined by (31) where $z = f^{-1}(w)$. To complete the proof of our theorem it remains to reduce the formula obtained for the function φ_1 to form (13). To this end it is sufficient to verify that if $z = f^{-1}(w)$, then

$$\arg(\partial/\partial z)f(z) = -\arg(\partial/\partial w)f^{-1}(w) ,$$

i.e.

$$\arg(\partial/\partial z)w = -\arg(\partial/\partial w)f^{-1}(w) . \tag{41}$$

Applying a well-known theorem on implicit functions to the functions $z-f^{-1}$ and $\bar{z}-\overline{f^{-1}}$, considered as functions of the variables w, $\overline{w}$, z, $\bar{z}$, we obtain, as in Step B of our proof, the formula

$$\frac{\partial w}{\partial z} = \overline{\frac{\partial}{\partial w} f^{-1}(w)} \Big/ \left\{ \left| \frac{\partial}{\partial w} f^{-1}(w) \right|^2 - \left| \frac{\partial}{\partial \overline{w}} f^{-1}(w) \right|^2 \right\}, \tag{42}$$

which is analogous to (23). Since, as is easily seen, f^{-1} maps Q-quasiconformally $\bar{K}$ onto itself, it satisfies in it the Beltrami equation, and, consequently, we have

$$\left| \frac{\partial}{\partial w} f^{-1}(w) \right|^2 - \left| \frac{\partial}{\partial \overline{w}} f^{-1}(w) \right|^2 \geqslant \left(\frac{Q+1}{Q-1} - 1 \right) \left| \frac{\partial}{\partial \overline{w}} f^{-1}(w) \right| = 4Q(Q-1)^{-2} \left| \frac{\partial}{\partial \overline{w}} f^{-1}(w) \right| \geqslant 0.$$

Thus, by virtue of (42), we obtain the formula

$$\arg(\partial/\partial z)w = \arg\overline{(\partial/\partial w)f^{-1}(w)} ,$$

equivalent to (41). In this way the proof of Theorem 3 is completed.

§ 3. The second functional equation in the unit disc.

Now we make the same assumptions as in Theorem 3, and we assume additionally that f can be continued onto a larger disc $\{z: |z| \leqslant 1+\varepsilon\}$, $\varepsilon = \varepsilon(f)$, with all the properties preserved, so that $|f(z)| = \text{const}$ on any circle $|z| = r_n$ $(n = 1, 2, \ldots)$, where $1+\varepsilon = r_1 > r_2 > \ldots > 1$. Clearly, the class of the functions considered is non-empty, because it includes the class of all function defined in the whole plane, satisfying the assumptions of Theorem 3 on $\bar{K}$, preserving all points belonging to the unit circle ∂K unchanged, and defined as the identity inside $\bar{K}$. The subclass mentioned seems to be interesting in itself, and possibly has several interesting extremal properties.

THEOREM 4. *Suppose that $\sigma = \mu(z)$ is a function of the class C^2 defined in $\bar{K}$ and bounded in it by $Q^* < 1$ in absolute value. Suppose moreover that*

(i) *$w = f(z)$ is a Q-quasiconformal mapping* [3] *of $\bar{K}$ onto itself generated by the complex dilatation μ so that $f(0) = 0$ and $f(1) = 1$,*

(ii) *$w = f(z)$ can be continued onto a larger disc $\{z\colon\ |z| \leqslant 1+\varepsilon\}$, $\varepsilon = \varepsilon(f)$, with all properties preserved* [4], *so that $|f(z)| = \mathrm{const}$ on any circle $|z| = r_n$ $(n = 1, 2, \ldots)$, where $1+\varepsilon = r_1 > r_2 > \ldots > 1$.*

Then $w = f(z)$ satisfies in K the equation

$$\partial w/\partial\varrho = w(\partial w/\partial\varrho)_{z=1} + \Phi_2(w)\,, \tag{43}$$

where

$$\Phi_2(w) = \frac{w(1-w)}{\pi}\iint\limits_{|\zeta|\leqslant 1}\left\{\frac{\varphi_2(\zeta)}{\zeta(1-\zeta)(w-\zeta)} + \frac{\overline{\varphi_2(\zeta)}}{\bar{\zeta}(1-\bar{\zeta})(1-w\bar{\zeta})}\right\}d\xi\, d\eta\,, \tag{44}$$

$$\varrho = \log|z|\,,\qquad \xi = \mathrm{re}\,\zeta\,,\qquad \eta = \mathrm{im}\,\zeta\,, \tag{45}$$

$$\varphi_2(w) = \frac{1}{1-\left|\mu\big(f^{-1}(w)\big)\right|^2}\left(\frac{\partial}{\partial\varrho}\mu(z)\right)_{z=f^{-1}(w)}\exp\left(-2i\arg\frac{\partial}{\partial w}f^{-1}(w)\right). \tag{46}$$

Proof. We can obviously modify Lemma 1 in such a manner that the parameter t tends to 0 over a sequence $\{t_\nu\}$ strictly decreasing to 0. We apply the modified lemma to the function

$$F(w, \Delta\varrho) = f\big(e^{\Delta\varrho}f^{-1}(w)\big)\big/f(e^{\Delta\varrho}) \tag{47}$$

defined in $\bar{K}\times\{\log r_\nu\}$. Further reasoning is analogous to the proof of Theorem 3.

Theorems 3 and 4 imply

THEOREM 5. *Suppose that $\sigma = \mu(z)$ and $w = f(z)$ satisfy the assumptions of Theorem 4* [4]. *Then $w = f(z)$ satisfies in K the system of equations*

$$\begin{aligned}\partial w/\partial z &= w(\partial w/\partial z)_{z=1} + \Psi_1(w)\,,\\ \partial w/\partial\bar{z} &= w(\partial w/\partial\bar{z})_{z=1} + \Psi_2(w)\,,\end{aligned} \tag{48}$$

where

$$\Psi_n(w) = \frac{w(1-w)}{\pi}\iint\limits_{|\zeta|\leqslant 1}\left\{\frac{\psi_n(\zeta)}{\zeta(1-\zeta)(w-\zeta)} + \frac{\overline{\psi_n(\zeta)}}{\bar{\zeta}(1-\bar{\zeta})(1-w\bar{\zeta})}\right\}d\xi\, d\eta\qquad (n = 1, 2)\,, \tag{49}$$

$$\xi = \mathrm{re}\,\zeta\,,\qquad \eta = \mathrm{im}\,\zeta\,, \tag{50}$$

---

[3] The connection between Q and Q^* is the same as in Theorem 3.

[4] We remark that the theorem is also true if the continued mapping is $\hat{Q}$-quasiconformal with a $\hat{Q} > Q$.

$$\psi_1(w) = \frac{1}{1-|\mu(f^{-1}(w))|^2}\left\{\left(\frac{\partial}{\partial z}\mu(z)\right)_{z=f^{-1}(w)} - (2/f^{-1}(w))\,\mu(f^{-1}(w))\right\}\times$$

$$\times\exp\left(-2i\arg\frac{\partial}{\partial w}f^{-1}(w)\right),\tag{51}$$

$$\psi_2(w) = \frac{1}{1-|\mu(f^{-1}(w))|^2}\left\{\left(\frac{\partial}{\partial \bar z}\mu(z)\right)_{z=f^{-1}(w)} + (2/\overline{f^{-1}(w)})\,\mu(f^{-1}(w))\right\}\times$$

$$\times\exp\left(-2i\arg\frac{\partial}{\partial w}f^{-1}(w)\right).$$

Proof. Let $\varrho = \log z$, $\vartheta = \arg z$. Hence

$$\partial\varrho/\partial z = 1/2z\,, \qquad \partial\varrho/\partial\bar z = 1/2\bar z\,,$$

$$\partial\vartheta/\partial z = 1/2iz\,, \qquad \partial\vartheta/\partial\bar z = -1/2i\bar z\,.$$

Concequently,

$$\frac{\partial w}{\partial z} = \frac{1}{2z}\left(\frac{\partial w}{\partial\varrho} - i\frac{\partial w}{\partial\vartheta}\right), \qquad \frac{\partial w}{\partial\bar z} = \frac{1}{2\bar z}\left(\frac{\partial w}{\partial\varrho} + i\frac{\partial w}{\partial\vartheta}\right).$$

Applying now Theorems 3 and 4, we obtain

$$\partial w/\partial z = w(\partial w/\partial z)_{z=1} + (1/2z)\{\Phi_2(w) - i\Phi_1(w)\}\,,$$

$$\partial w/\partial\bar z = w(\partial w/\partial\bar z)_{z=1} + (1/2z)\{\Phi_2(w) + i\Phi_1(w)\}\,.$$

Next we verify directly that

$$(1/2z)\{\varphi_2(w) - i\varphi_1(w)\} = \psi_1(w)\,,$$

$$(1/2\bar z)\{\varphi_2(w) + i\varphi_1(w)\} = \psi_2(w)\,.$$

Consequently,

$$(1/2z)\{\Phi_2(w) - i\Phi_1(w)\} = \Psi_1(w)\,,$$

$$(1/2\bar z)\{\Phi_2(w) + i\Phi_1(w)\} = \Psi_1(w)\,.$$

Theorem 5 is proved.

§ 4. The functional equations in an annulus. Here we present the analogues of Theorems 3, 4 and 5 in the case of an annulus. The proofs are omitted as completely analogous to that of the theorems mentioned. Clearly, Lemma 1 and Theorem 1 must be replaced in these proofs by Lemma 2 and Theorem 2, respectively.

THEOREM 6. *Let $\sigma = \mu(z)$ be a function of the class C^2 defined in $\bar K_r$ and bounded in it by $Q^* < 1$ in absolute value. Let $Q = (1+Q^*)/(1-Q^*)$. Then the Q-quasiconformal mapping $w = f(z)$ of $\bar K_r$ onto $\bar K_R$, generated by the complex dilatation μ so that $f(1) = 1$, where R is determined uniquely, satisfies in K the equation* (2)

$$\partial w/\partial\vartheta = w(\partial w/\partial\vartheta)_{z=1} + \Phi_{1R}(w)\,,\tag{52}$$

where

$$(53)\qquad \Phi_{1R}(w) = (1/2\pi)\iint\limits_{R\leqslant|\zeta|\leqslant 1} w\sum_{\nu=-\infty}^{+\infty}\left\{\frac{\varphi_1(\zeta)}{\zeta^2}\left(\frac{w+R^{2\nu}\zeta}{w-R^{2\nu}\zeta}-\frac{1+R^{2\nu}\zeta}{1-R^{2\nu}\zeta}\right)-\right.$$
$$\left.-\frac{\overline{\varphi_1(\zeta)}}{\bar\zeta^2}\left(\frac{1+R^{2\nu}w\bar\zeta}{1-R^{2\nu}w\bar\zeta}-\frac{1+R^{2\nu}\bar\zeta}{1-R^{2\nu}\bar\zeta}\right)\right\}d\xi\,d\eta$$

and ϑ, ξ, η, $\varphi=\varphi_1(w)$ *are given by* (12) *and* (13). *Moreover,*

$$(54)\qquad \iint\limits_{R\leqslant|\zeta|\leqslant 1}\{\varphi_1(\zeta)/\zeta^2+\overline{\varphi_1(\zeta)}/\bar\zeta^2\}\,d\xi\,d\eta = 0\,.$$

THEOREM 7. *Suppose that* $\sigma=\mu(z)$ *is a function of the class* C^2 *defined in* $\bar K_r$ *and bounded in it by* $Q^*<1$ *in absolute value. Suppose moreover that*

(i) $w=f(z)$ *is a* Q*-quasiconformal mapping* (3) *of* $\bar K_r$ *onto* $\bar K_R$ *generated by the complex dilatation* μ *so that* $f(1)=1$, *where* R *is determined uniquely,*

(ii) $w=f(z)$ *can be continued onto a larger annulus* $\{z\colon (1-\varepsilon^*)r\leqslant|z|\leqslant 1+\varepsilon\}$, $\varepsilon=\varepsilon(f)$, $\varepsilon^*=\varepsilon^*(f)$, *with all the properties preserved* (4), *so that* $|f(z)|=\text{const}$ *on any circle* $|z|=r_n$ *and* $|z|=r_n^*$ $(n=1,2,\ldots,)$ *where* $1+\varepsilon=r_1>r_2>\ldots>1$, $(1-\varepsilon^*)r=r_1^*<r_2^*<\ldots<r$.

Then $w=f(z)$ *satisfies in* K_r *the equation* (2)

$$(55)\qquad \partial w/\partial\varrho = w(\partial w/\partial\varrho)_{z=1}+\Phi_{2R}(w)\,,$$

where

$$(56)\qquad \Phi_{2R}(w) = (1/2\pi)\iint\limits_{R\leqslant|\zeta|\leqslant 1} w\sum_{\nu=-\infty}^{+\infty}\left\{\frac{\varphi_2(\zeta)}{\zeta^2}\left(\frac{w+R^{2\nu}\zeta}{w-R^{2\nu}\zeta}-\frac{1+R^{2\nu}\zeta}{1-R^{2\nu}\zeta}\right)-\right.$$
$$\left.-\frac{\overline{\varphi_2(\zeta)}}{\bar\zeta^2}\left(\frac{1+R^{2\nu}w\bar\zeta}{1-R^{2\nu}w\bar\zeta}-\frac{1+R^{2\nu}\bar\zeta}{1-R^{2\nu}\bar\zeta}\right)\right\}d\xi\,d\eta$$

and ϱ, ξ, η, $\varphi=\varphi_2(w)$ *are given by* (45) *and* (46). *Moreover,*

$$(57)\qquad \iint\limits_{R\leqslant|\zeta|\leqslant 1}\{\varphi_2(\zeta)/\zeta^2+\overline{\varphi_2(\zeta)}/\bar\zeta^2\}\,d\xi\,d\eta = 0\,.$$

THEOREM 8. *Suppose that* $\sigma=\mu(z)$ *and* $w=f(z)$ *satisfy the assumptions of Theorem* 7 (4). *Then* $w=f(z)$ *satisfies in* K_r *the system of equations* (2)

$$(58)\qquad \begin{aligned}\partial w/\partial z &= w(\partial w/\partial z)_{z=1}+\Psi_{1R}(w)\,,\\ \partial w/\partial\bar z &= w(\partial w/\partial\bar z)_{z=1}+\Psi_{2R}(w)\,,\end{aligned}$$

where

$$(59)\qquad \Psi_{nR}(w) = (1/2\pi)\iint\limits_{R\leqslant|\zeta|\leqslant 1}\sum_{\nu=-\infty}^{+\infty} w\left\{\frac{\psi_n(\zeta)}{\zeta^2}\left(\frac{w+R^{2\nu}\zeta}{w-R^{2\nu}\zeta}-\frac{1+R^{2\nu}\zeta}{1-R^{2\nu}\zeta}\right)-\right.$$
$$\left.-\frac{\overline{\psi_n(\zeta)}}{\zeta^2}\left(\frac{1+R^{2\nu}w\bar\zeta}{1-R^{2\nu}w\bar\zeta}-\frac{1+R^{2\nu}\bar\zeta}{1-R^{2\nu}\bar\zeta}\right)\right\}d\xi\,d\eta\qquad (n=1,2)$$

and ξ, η, $\psi = \psi_1(w)$, $\psi = \psi_2(w)$ *are given by* (50) *and* (51). *Moreover,*

$$\iint\limits_{R\leqslant|\zeta|\leqslant 1} \{\psi_n(\zeta)/\zeta^2 + \overline{\psi_n(\zeta)}/\bar{\zeta}^2\}\, d\xi\, d\eta = 0 \qquad (n = 1, 2)\,. \tag{60}$$

References

[1] L. V. Ahlfors, *Lectures on Quasiconformal Mappings*, Princeton 1966.

[2] L. Bers, *On a theorem of Mori and the definition of quasiconformality*, Trans. Amer. Math. Soc. 84 (1957), pp. 78-84.

[3] G. M. Fichtenholz, *A Course of Differential and Integral Calculus* (in Russian), 3 volumes, Moscow 1962/3.

[4] O. Lehto and K. I. Virtanen, *Quasikonforme Abbildungen*, Berlin–Heidelberg–New York 1965.

[5] J. Ławrynowicz, *On the parametrization of quasiconformal mappings in an annulus*, Ann. Univ. M. Curie-Skłodowska, Sectio A, 18 (1966), to appear.

[6] — *On certain functional equations for quasiconformal mappings*, Proceedings of the Conference on Analytic Functions, Łódź 1966, p. 14.

[7] Shah Tao-shing, *Parametrical representation of quasiconformal mappings* (in Russian), Science Record N. S. 3 (1959), pp. 400-407.

[8] L. I. Volkoviskii, *Quasiconformal Mappings* (in Russian), Lvov 1954.

INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES
LABORATORY IN ŁÓDŹ
DEPARTMENT OF MATHEMATICS, IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY
UNIVERSITY OF LONDON

Reçu par la Rédaction le 20. 12. 1966