ANNALES
POLONICI MATHEMATICI
LII (1990)

Multiple solutions for a nonlinear second order differential equation

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Abstract. It is proved that a certain nonlinear periodic second order boundary value problem has at least two solutions.

Consider the following periodic boundary value problem:

$$x''(t)+e\cos t\cdot x''-2e\sin t\cdot x'+\alpha\sin x = 4e\sin t, \tag{1}$$

$$x(0)-x(2\pi) = x'(0)-x'(2\pi) = 0. \tag{2}$$

For $0<e<1$ and $|\alpha|\leqslant 3$, equation (1) governs the periodic motions of a satellite in the plane of its elliptical orbit (see [1]). The problem was recently treated by Petryshin and Yu in [5], where the existence of a solution of (1)–(2) was established for

$$0\leqslant e<(2/\pi)|\alpha| \quad \text{and} \quad (8\sqrt{2}+3)e+2|\alpha|<1,$$

using the degree theory for A-proper mappings. In [4], we proved that in fact (1)–(2) has at least one solution for $|e|<1$ and α arbitrary. The purpose of this note is to establish the following multiplicity result:

THEOREM 1. *Let $|e|<1$ and let α be any real number. Then* (1)–(2) *has at least two solutions not differing by a multiple of* 2π.

Proof. Let H be the space of absolutely continuous 2π-periodic function u such that $u'\in L^2(0, 2\pi)$, with the inner product

$$(u, v) = u(0)v(0)+\int_0^{2\pi} a^2(t)u'v'dt,$$

where $a(t) = 1+e\cos t$. Then H is a Hilbert space. We denote its norm by $|\cdot|_H$. Define the C^1-functional Φ: $H\to R$ by

$$\Phi(u) = \int_0^{2\pi} [a^2(t)u'^2/2+\alpha a(t)\cos u+4ea(t)\sin t\cdot u]dt.$$

Then Φ is weakly lower semicontinuous. Indeed, let (u_n) be a sequence in H such that $u_n \rightharpoonup u$ ($\rightharpoonup$ denotes the weak convergence). Then (see [2])

$$a(t)u_n' \rightharpoonup a(t)u' \quad \text{in } L^2(0, 2\pi) \quad \text{and} \quad u_n \to u \quad \text{uniformly on } [0, 2\pi]$$

from which it follows that $\liminf \Phi(u_n) \geqslant \Phi(u)$, as claimed.

Now, we claim that Φ attains its minimum on some open ball $B(0, R)$ in H. Let $R > 0$. Since Φ is weakly lower semicontinuous, Φ attains its minimum on the closed ball $B'(0, R)$. Let u be its minimum point. Since Φ is 2π-periodic, there exists $x \in H$ such that

$$\Phi(x) = \Phi(u) \quad \text{and} \quad |x(0)| < 2\pi.$$

Integrating by parts now gives

$$\Phi(x) = \int_0^{2\pi} [a^2(t)x'^2/2 + 2a^2(t)x' + \alpha a(t)\cos x]dt \tfrac{1}{4}((|x|_H^2 - |x(0)|^2)$$
$$-4\int_0^{2\pi} a^2(t)dt - 2\pi|\alpha|(1+|e|).$$

Since $\Phi(x) \leqslant \Phi(0)$, this implies that $x \in B(0, R)$ if R is chosen sufficiently large, which proves the claim. For this R, it is easy to see that x is a solution of (1)$-$(2). We now produce a second solution that does not differ from x by a multiple of 2π. We can (and shall) assume that the local minimum at x is a strict minimum (for otherwise there is nothing to prove). We shall apply the following Brezis–Coron–Nirenberg variant of the mountain pass lemma:

THEOREM A [3]. *Assume F is a Gateaux differentiable function on a Banach space E and DF: $E \to E^*$ is continuous from the strong topology of E into the weak* topology of E^*. Assume $x_0 \in E$ and:*

(i) *There exist a neighborhood U of x_0 and a constant ϱ such that $F(u) \geqslant \varrho$ for every u on the boundary of U and $F(x_0) < \varrho$.*

(ii) *There exists $y \notin U$ such that $F(y) < \varrho$.*

(iii) *F satisfies the condition:*

(PS)$_c$ *Whenever a sequence (u_n) in E is such that $F(u_n) \to c$ and $DF(u_n) \to 0$ in E^*, then c is a critical value of F, where $c = \inf_{P \in \mathscr{P}} \max_{u \in P} F(u) \geqslant \varrho$, $\mathscr{P}$ denotes the class of paths joining x_0 to y, and DF is the derivative of F.*

CONCLUSION. *c is a critical value of F.*

We shall now verify the assumptions of Theorem A for $F = \Phi$ and $x_0 = x$. Once this is done, it follows that there exists a critical value c of Φ with $c > \Phi(x)$ and the corresponding critical point is a solution of (1)-(2) that does not differ from x by a multiple of 2π, completing the proof of Theorem 1.

Verification of (i). Let $r > 0$ be chosen such that $\Phi(u) > \Phi(x)$ for $u \in B'(x, r)$, $u \neq x$. Let $\varrho = \operatorname{Inf}\{\Phi(u)\colon |u-x|_H = r\}$ and $U = B(x, r)$. We claim that U and ϱ satisfy (i). Indeed, if $\varrho = \Phi(x)$, then there is a sequence (u_k) with $|u_k - x|_H = r$ and $\Phi(u_k) \to \Phi(x)$. Going if necessary to a subsequence, we may assume that $u_k \rightharpoonup u$ in H and $u_k \to u$ uniformly on $[0, 2\pi]$. By the weak lower semicontinuity of Φ, we have $\Phi(x) \geqslant \Phi(u)$. Since $u \in B'(x, r)$, this implies that $u = x$. Hence

$$\int_0^{2\pi} a^2(t) u_k'^2\, dt \to \int_0^{2\pi} a^2(t) x'^2\, dt$$

from which it follows that $|u_k|_H \to |x|_H$, and $u_k \to x$ in H, a contradiction.

Verification of (ii). Let $y = x + 2\pi$. Then $y \notin U$ and $\Phi(y) = \Phi(x) < \varrho$.

Verification of (iii). Let $c \in \boldsymbol{R}$ and let (u_k) be a sequence in H such that $\Phi(u_k) \to c$ and $D\Phi(u_k) \to 0$ in H^*. Since Φ is 2π-periodic, we may assume without loss of generality that $|u_k(0)| < 2\pi$. We then have

$$\Phi(u_k) \geqslant \tfrac{1}{4}(|u_k|_H^2 - 4\pi^2) - 8\pi(1+|e|)^2 - 2\pi|\alpha|(1+|e|)$$

which implies that (u_k) is bounded in H. Consequently, (u_k) has a subsequence, still denoted by (u_k), such that

$$u_k \rightharpoonup u \quad \text{in } H \quad \text{and} \quad u_k \to u \quad \text{uniformly on } [0, 2\pi]. \tag{3}$$

We have

$$D\Phi(u_k)v = \int_0^{2\pi} [a^2(t) u_k' v' - \alpha a(t) \sin u_k \cdot v + 4ea(t) \sin t \cdot v]\, dt$$

from which it follows by (3) that

$$\int_0^{2\pi} a^2(t) u_k' v'\, dt \to \int_0^{2\pi} a^2(t) u' v'\, dt$$

uniformly for $|v|_H \leqslant 1$.

Since $u_k(0) \to u(0)$, this implies that $u_k \to u$ in H. Hence $\Phi(u) = c$ and $D\Phi(u) = 0$, i.e., c is a critical value.

References

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Reçu par la Rédaction le 28.02.1988