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Characterization of the solutions of a $2b$-parabolic operator with initial data in BMO

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Abstract. In this paper we show that the solution $u(x, y)$ of $2b$-parabolic equation $H_b u = \sum_{|\alpha|=2b} a_\alpha D^\alpha u - \partial u/\partial t = 0$, satisfying a suitable integral condition, turns out to be of the form $u(x, y) = [G(\cdot, t) * f](x)$ with $f \in \mathrm{BMO}(\boldsymbol{R}^n)$ and $G(x, t)$ the fundamental solution associated to H_b.

Introduction. Letting $H_b = \sum_{|\alpha|=2b} a_\alpha D^\alpha - (\partial/\partial t)$ the $2b$-parabolic operator on $\boldsymbol{R}_+^{n+1} = \boldsymbol{R}^n x\,(0, +\infty)$, we wish to characterize the solutions $u(x, t)$ of the equation $H_b u = 0$, which for $t = 0$ have initial values in $\mathrm{BMO}(\boldsymbol{R}^n)$. If we identify two solutions u_1 and u_2 which differ by a polynomial in x of degree $\leqslant b-1$, each $u(x, t)$ satisfying a suitable integral condition turns out to be of the form

$$u(x, t) = [G(\cdot, t) * f](x)$$

with $f \in \mathrm{BMO}(\boldsymbol{R}^n)$ and $G(x, t)$ the "fundamental solution" associated to H_b. For $b = 1$, we reobtain the results of E. B. Fabes and U. Neri [3]. The methods used by us extend, in a natural way, those employed in [3].

In Section 1, we define the space $T^b\,\mathrm{MO}(\boldsymbol{R}_+^{n+1})$ of all those solutions of $H_b u = 0$ satisfying a given global condition (modulo polynomials as above). Taking momentarily for granted certain estimates on the $D^\alpha u$, we then prove the characterization $T^b\,\mathrm{MO} = G * \mathrm{BMO}$. In Section 2, we give the proofs of the estimates on the $D^\alpha u$.

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1. Definitions and main result. Let us consider the equation

$$H_b u = \sum_{|\alpha|=2b} a_\alpha D^\alpha u - (\partial u/\partial t) = 0 \tag{1}$$

where the a_α are constants and $D^\alpha = (-i)^{|\alpha|}(\partial^{|\alpha|}/\partial x_1^{\alpha_1}\,\partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n})$, $|\alpha| = \alpha_1 +$

$+\alpha_2+\ldots+\alpha_n$. We assume that the equation be parabolic: for each $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$

$$\text{(P)} \qquad \operatorname{Re}\Big(\sum_{|\alpha|=2b} a_\alpha \sigma^\alpha\Big) \leqslant -\delta|\sigma|^{2b}$$

where $\sigma^\alpha = \sigma_1^{\alpha_1}\sigma_2^{\alpha_2}\ldots\sigma_n^{\alpha_n}$.

If $u_0 \in C_0^\infty(\boldsymbol{R}^n)$, say, the solution $u(x, t)$ of the Cauchy problem

$$\text{(2)} \qquad \begin{aligned} &H_b u = 0 \quad \text{on } \boldsymbol{R}_+^{n+1},\\ &\lim_{t\to 0^+} u(x, t) = u_0(x) \end{aligned}$$

has the form

$$\text{(3)} \qquad u(x, t) = [G(\cdot, t) * u_0](x),$$

(see [1]). Here, G is the inverse Fourier transform

$$\text{(4)} \qquad G(x, t) = \mathscr{F}_\xi^{-1}[Q(\xi, t)](x)$$

and $Q(\xi, t)$ satisfies the bounds

$$|Q(\xi, t)| \leqslant C\exp[-\delta|\xi|^{2b}]$$

with C and δ strictly positive reals.

The fundamental solution G, and its derivatives, satisfy the following estimates:

$$\text{(5)} \qquad |D^\alpha G(x, t)| \leqslant C_\alpha t^{-(n+|\alpha|)/2b}\exp\Big[-c\sum_{k=1}^n |x_k|^{2b/(2b-1)} t^{-1/(2b-1)}\Big],$$

$$\text{(5')} \qquad |D^\alpha G(x, t)| \leqslant C_\alpha |x|^{-(n+|\alpha|)},$$

$$\text{(6)} \qquad |(\partial/\partial t)\, G(x, t)| \leqslant C t^{-1-n/2b}\exp\Big[-c\sum_{k=1}^n |x_k|^{2b/(2b-1)} t^{-1/(2b-1)}\Big],$$

where the constants c, C and C_α are (as always) strictly positive and may differ from line to line. For each t, the rapid decrease of $G(x, t)$ and of its derivatives (as $|x| \to \infty$) yields the inversion formula

$$[\mathscr{F}_x G(x, t)](\xi) = Q(\xi, t)$$

on account of (4).

For every $u \in C^\infty(\boldsymbol{R}_+^{n+1})$ we associate the function $\gamma = \gamma_u$ on $\boldsymbol{R}^n$, $0 \leqslant \gamma_u(x) \leqslant +\infty$ given by

$$\text{(7)} \qquad [\gamma(x)]^2 = \int_0^\infty \sum_{|\alpha|=b} |D^\alpha u(x, t)|^2\, dt.$$

Then, if $u(x, t) = [G(\cdot, t) * f](x)$, with $f \in L^2(\boldsymbol{R}^n)$ we obtain the inequality

$$\text{(7')} \qquad \|\gamma\|_2 \leqslant B\|f\|_2$$

where B depends on δ of (P).

To prove (7′), we follow the argument of [3], p. 2. Thus, using (twice) Plancherel's formula in the x variables, we have

$$\|\gamma\|_2^2 = \int_{R^n} \Big(\int_0^\infty \sum_{|\alpha|=b} |D^\alpha u(x,t)|^2 \, dt \Big) dx = \int_0^\infty \Big(\int_{R^n} \sum_{|\alpha|=b} |\mathscr{F}_x D^\alpha u(x,t)|^2 \, dx \Big) dt$$

$$= \int_0^\infty \Big[(2\pi)^{2b} \sum_{|\alpha|=b} \int_{R^n} |\xi^\alpha|^2 \, |\mathscr{F}_x u(x,t)|^2 \, dx \Big] dt$$

$$= (2\pi)^{2b} \int_0^\infty \Big(\sum_{|\alpha|=b} \int_{R^n} |\xi^\alpha|^2 \, |\mathscr{F}_x G * \mathscr{F}_x f|^2 \, d\xi \Big) dt.$$

Since $\mathscr{F}_x G = Q$, we see that

$$\|\gamma\|_2^2 = (2\pi)^{2b} \int_0^\infty \Big(\sum_{|\alpha|=b} \int_{R^n} |\xi^\alpha|^2 \, |Q|^2 \, |\hat{f}|^2 \, d\xi \Big) dt$$

$$= (2\pi)^{2b} \int_{R^n} |\hat{f}|^2 \Big(\int_0^\infty \sum_{|\alpha|=b} |\xi^\alpha|^2 \, |Q(\xi,t)|^2 \, dt \Big) d\xi$$

$$\leqslant C \int_{R^n} |\hat{f}|^2 \Big(\int_0^\infty |\xi|^{2b} e^{-2\delta|\xi|^{2b}} \, dt \Big) d\xi.$$

Using the change $s = t|\xi|^{2b}$ we conclude

$$\|\gamma\|_2^2 \leqslant C \int_{R^n} |\hat{f}|^2 \Big(\int_0^\infty e^{-2\delta s} \, ds \Big) d\xi \leqslant B^2 \|\hat{f}\|_2^2.$$

The space T^bMO. Denoting by $Q_\delta \subset R^n$ the cube centered at x_0 having sides (parallel to the axes) of length δ, we associate with each $u(x,t)$ on R_+^{n+1} the "average" $[u]_{x_0,\delta}$ where

$$[u]_{x_0,\delta}^2 = |Q_\delta|^{-1} \int_{Q_\delta} \int_0^{\delta^{2b}} \sum_{|\alpha|=b} |D^\alpha u(x,t)|^2 \, dt \, dx \tag{8}$$

and

$$\|u\|_{**} = \sup\{[u]_{x_0,\delta} \colon x_0 \in R^n, \ \delta > 0\}. \tag{8′}$$

The linear space of all solutions $u(x,t)$ of (1) such that $\|u\|_{**} < \infty$ will be denoted by $T^b\mathrm{MO} = T^b\mathrm{MO}(R_+^{n+1})$. In T^bMO we identify two solutions u_1 and u_2 which differ by a polynomial in x (independent of t) of degree $\leqslant b-1$. In this way, the map $u \to \|u\|_{**}$ becomes a norm.

Recall that a locally integrable function f is in $\mathrm{BMO} = \mathrm{BMO}(R^n)$ if it has bounded mean oscillation on R^n:

$$\|f\|_* = \sup\Big\{|Q_\delta|^{-1} \int_{Q_\delta} |f(x) - f_{Q_\delta}| \, dx \colon x_0 \in R^n, \ \delta > 0\Big\} < +\infty \tag{9}$$

where f_{Q_δ} is the integral average (or mean value) of f on Q_δ. With the identification $f \sim f+C$ (for any constant C), $f \to \|f\|_*$ becomes a norm. It is well known that the L^1-averages above (see [6]) may be replaced by equivalent L^2-averages $[f]_{x_0,\delta}$ where

$$(9') \qquad [f]^2_{x_0,\delta} = |Q_\delta|^{-1} \int_{Q_\delta} |f(x) - f_{Q_\delta}|^2 \, dx.$$

If we denote by $G * \mathrm{BMO} = \{u = G * f : f \in \mathrm{BMO}\}$ our result can be stated as follows:

THEOREM. $T^b\mathrm{MO} = G * \mathrm{BMO}$, *with equivalence of norms.*

We shall prove the theorem through several lemmas.

LEMMA 1.1. $G * \mathrm{BMO} \subset T^b\mathrm{MO}$ *and the inclusion is continuous.*

Proof. We fix any cube $Q_\delta \subset \boldsymbol{R}^n$ centered at x_0 having sides of length $\delta = 4h$. By translation invariance of $\|u\|_{**}$, we may suppose that $x_0 \equiv 0$. We denote by χ_δ the characteristic function of Q_δ and we choose any $f \in \mathrm{BMO}(\boldsymbol{R}^n)$. We have that

$$f \sim [f - f_{Q_\delta}] = [f - f_{Q_\delta}]\chi_\delta + [f - f_{Q_\delta}](1 - \chi_\delta) = f_1 + f_2$$

and we let

$$u_1 = G * f_1 \quad \text{and} \quad u_2 = G * f_2.$$

Since $f_1 \in L^2(\boldsymbol{R}^n)$, we associate with u_1 by (7) the corresponding function γ_1. On account of (7′) and (8), it follows that

$$h^n \|u_1\|^2_{0,h} \leqslant \|\gamma_1\|^2_2 \leqslant B^2 \|f_1\|^2_2 = B^2 (4h)^n [f]^2_{0,4h}.$$

Hence $\|u_1\|_{**} \leqslant B\|f\|_*$.

In order to estimate $\|u_2\|_{**}$ we observe that if $x \in Q_h$ and $y \in \boldsymbol{R}^n - Q_{4h}$ we have $|x-y| \geqslant 3h/2$ and $|x-y| \approx |y|$ for all large $|y|$. Moreover, for $|\alpha| = b$, we have

$$|D^\alpha G(x-y, t)| \leqslant C_\alpha |x-y|^{-(n+b)} \leqslant 2C_\alpha [h^{n+b} + |x-y|^{n+b}]^{-1}$$

and

$$|D^\alpha u_2(x, t)| \leqslant 2C_\alpha \int_{\boldsymbol{R}^n - Q_\delta} |f(y) - f_{Q_\delta}| \, [h^{n+b} + |x-y|^{n+b}]^{-1} \, dy.$$

Letting now $\eta = y/h$ and integrating on all of $\boldsymbol{R}^n$ (see [5] or [4]), we deduce that

$$|D^\alpha u_2(x, t)| \leqslant C h^{-b} \|f\|_*$$

since the BMO-norm is dilatation-invariant. Consequently,

$$[u_2]^2_{0,h} \leqslant C^2 h^{-2b} \|f\|^2_* h^{2b}$$

and hence $\|u_2\|_{**} \leqslant C\|f\|_*$. This completes the proof of Lemma 1.1. To show the inverse inclusion, we shall make use of some estimates for $D^\beta u$ whose proof is deferred to § 2.

LEMMA 1.2. *There is a constant $C > 0$ such that for all $u \in T^b \mathrm{MO}$, $|\beta| \geqslant b$ and all $(x, t) \in \boldsymbol{R}_+^{n+1}$:*

$$|D^\beta u(x, t)| \leqslant C t^{-(1/2)-(|\beta|-b)/2b} \|u\|_{**}, \tag{10}$$

$$|(\partial/\partial t) u(x, t)| \leqslant C t^{-1} \|u\|_{**}. \tag{10'}$$

LEMMA 1.3. *Let $u \in T^b \mathrm{MO}$. If k is any positive integer, then the convolution $G * u(\cdot, k^{-2})$ exists everywhere on $\boldsymbol{R}_+^{n+1}$.*

Proof. Let us consider the Taylor expansion of u about $y = 0$: $u(y, t) = P(y, t) +$(error term of order b) where

$$P(y, t) = u(0, t) + \sum_{|\gamma| \leqslant b-1} c_\gamma D^\gamma u(0, t) y^\gamma.$$

Replacing (as we may do) u by the representative $u - P$, we have

$$\Big| \int_{\boldsymbol{R}^n} G(x-y, t)\, u(y, k^{-2})\, dy - \int_{\boldsymbol{R}^n} G(x-y, t)\, P(y, k^{-2})\, dy \Big|$$
$$\leqslant M \int_{\boldsymbol{R}^n} \sum_{|\gamma| = b} |y^\gamma D^\gamma u(\theta y, k^{-2})|\, dy \leqslant C \|u\|_{**}$$

on account of the estimates (5) and (10), where the constant C depends only on k and $0 < \theta < 1$.

LEMMA 1.4. *Let $u \in T^b \mathrm{MO}$. Then, for each $k \in N$ and all $(x, t) \in \boldsymbol{R}_+^{n+1}$*

$$[G * u(\cdot, k^{-2})](x) = u(x, t + k^{-2}). \tag{11}$$

Moreover, setting from now on $u_k(x, t) = u(x, t + k^{-2})$, we have the uniform estimates

$$\|u_k\|_{**} \leqslant C \|u\|_{**}. \tag{12}$$

Proof. Both sides of (11) are solutions on $\boldsymbol{R}_+^{n+1}$ of (1) having pointwise limit $u(x, k^{-2})$ as $t \to 0^+$. Now, for all $(x, t) \in \overline{\boldsymbol{R}_+^{n+1}}$, estimates (10) show that

$$\sum_{|\beta| = b} |D^\beta u(x, t + k^{-2})| \leqslant C (t + k^{-2})^{-1/2} \|u\|_{**}.$$

Hence, for each k, $\sum_{|\beta| = b} D^\beta u(x, t + k^{-2})$ is a bounded function on $\boldsymbol{R}_+^{n+1}$ and as $t \to 0^+$ with pointwise limit $\sum_{|\beta| = b} D^\beta u(x, k^{-2})$. Therefore, by the uniqueness of the solution of the Cauchy problem on R_+^{n+1} with trace $\sum_{|\beta|=b} D^\beta u(x, k^{-2})$ on R^n (see [1]) we have that

$$\sum_{|\beta|=b} D^\beta u(x, t+k^{-2}) = [G(\cdot, t) * \sum_{|\beta|=b} D^\beta u(\cdot, k^{-2})](x)$$
$$= \sum_{|\beta|=b} D^\beta [G(\cdot, t) * u(\cdot, k^{-2})]$$

which shows that $u(x, t+k^{-2})$ and $G*u(x, k^{-2})$ are equal up to a polynomial in x of degree $\leqslant b-1$. Since they have the same limit as $t \to 0^+$, this polynomial is zero.

Estimate (12) is then verified exactly like Lemma 1.4 of [2], by examining $[u_k]^2_{x_0,\delta}$ separately: for $\delta^b \geqslant 1/k$ and $\delta^b < 1/k$.

LEMMA 1.5. *Let $u \in T^b\mathrm{MO}$ and set $f_k = u(x, k^{-2})$. We prove that*

$$\|f_k\|_* \leqslant C\|u\|_{**}.$$

Proof. Given $u \in T^b\mathrm{MO}$ and setting

$$u_k(x, t) = u(x, t+k^{-2}),$$

we consider the identity

$$(\partial/\partial t)[u_k - a_k]^2 = 2[u_k - a_k](\partial/\partial t)u_k = 2[u_k - a_k] \sum_{|\beta|=2b} a_\beta D^\beta u_k \tag{13}$$

where the a_k are polynomials in x of degree $\leqslant b-1$, to be determined later. Integrating the left-hand side of (13), we have

$$\int_{Q_\delta} \Big(\int_0^{\delta^{2b}} (\partial/\partial t)[u_k - a_k]^2\, dt\Big)\, dx = \int_{Q_\delta} [u_k(x, \delta^{2b}) - a_k]^2 - \int_{Q_\delta} [u_k(x, 0) - a_k]^2\, dx.$$

Hence

$$\int_{Q_\delta} [u_k(x, 0) - a_k]^2\, dx$$
$$= \int_{Q_\delta} [u_k(x, \delta^{2b}) - a_k]^2\, dx - 2 \int_{Q_\delta} \int_0^{\delta^{2b}} [u_k - a_k] \sum_{|\beta|=2b} a_\beta D^\beta u_k\, dt\, dx.$$

We now suppose, as we may do, that $Q_\delta = \{x : |x - x_0| \leqslant \delta\}$ and we evaluate each term of the last equality choosing the polynomials

$$a_k = \sum_{|\alpha| \leqslant b-1} c_\alpha D^\alpha u_k(x_0, \delta^{2b})(x - x_0)^\alpha$$

where c_α are the Taylor coefficients. We obtain then

$$\int_{Q_\delta} [u_k(x, \delta^{2b}) - a_k]^2\, dx = \int_{Q_\delta} [u_k(x, \delta^{2b}) - \sum_{|\alpha| \leqslant b-1} c_\alpha D^\alpha u_k(x_0, \delta^{2b})(x - x_0)^\alpha]^2\, dx$$
$$\leqslant c \int_{Q_\delta} |x - x_0|^{2b} [\sum_{|\alpha|=b} D^\alpha u_k(x, \delta^{2b})]^2\, dx$$

so, by (10), we may write

$$\int_{Q_\delta} [u_k(x, \delta^{2b}) - a_k]^2 \, dx \leqslant C \|u\|_{**}^2 \int_{Q_\delta} |x - x_0|^{2b} (\delta^{2b} + k^{-2})^{-1} \, dx$$

$$\leqslant C \|u\|_{**}^2 \delta^{-2b} \int_{Q_\delta} |x - x_0|^{2b} \, dx \leqslant C \|u\|_{**}^2 |Q_\delta|$$

with the constant C which is independent of Q_δ and k. We consider now the terms of the form

$$\int_{Q_\delta} \Big\{ \int_0^{\delta^{2b}} [u_k - a_k] D^\beta u_k \, dt \Big\} dx, \qquad |\beta| = 2b.$$

Adding and subtracting $u_k(x, \delta^{2b})$, we may write

$$\Big| \int_{Q_\delta} \Big\{ \int_0^{\delta^{2b}} |u_k - a_k| D^\beta u_k \, dt \Big\} dx \Big|$$

$$\leqslant \Big| \int_{Q_\delta} \int_0^{\delta^{2b}} [u_k(x, t) - u_k(x, \delta^{2b})] D^\beta u_k(x, t) \, dt \, dx \Big| +$$

$$+ \Big| \int_{Q_\delta} \int_0^{\delta^{2b}} [u_k(x, \delta^{2b}) - \sum_{|\alpha| \leqslant b-1} c_\alpha D^\alpha u_k(x_0, \delta^{2b})] D^\beta u_k(x, t) \, dt \, dx \Big|$$

$$= |I_1(k)| + |I_2(k)|.$$

Let us introduce the Taylor remainder of order b:

$$R_b(x) = u_k(x, \delta^{2b}) - \sum_{|\alpha| \leqslant b-1} c_\alpha (x - x_0)^\alpha D^\alpha u_k(x_0, \delta^{2b}).$$

For every $x \in Q_\delta$ and $|\mu| \geqslant 0$ we claim that

$$(14) \qquad |D^\mu R_b(x)| \leqslant C \delta^{-\mu} \|u\|_{**}.$$

If $|\mu| = 1$ we obtain

$$D^\mu R_b(x) = D^\mu u_k(x, \delta^{2b}) - \sum_{|\alpha| \leqslant b-2} c_\alpha (x - x_0)^\alpha D^{\mu+\alpha} u_k(x_0, \delta^{2b})$$

which is the remainder of order $b-1$ of $D^\mu u_k(x, \delta^{2b})$.

The estimates (10) show that $(x - x_0)^\alpha D^{\mu+\alpha} u_k$ is domainated by $\delta^{-\mu} \|u\|_{**}$. By iteration we obtain (14).

First, we estimate $I_2(k)$. For a fixed t between 0 and δ^{2b}, integrating by parts in x, we have

$$|I_2(k)| = \Big| \int_0^{\delta^{2b}} \int_{Q_\delta} R_b(x) D^{\beta_1} u_k(x, t) \, dx \, dt \Big|$$

$$\leqslant \int_0^{\delta^{2b}} \int_{S_\delta} |R_b(x)| \, |D^{\beta_1} u_k(x, t)| \, dt \, dx + \Big| \int_0^{\delta^{2b}} \int_{Q_\delta} D^{\mu_1} R_b(x) D^{\beta_1} u_k(x, t) \, dx \, dt \Big|$$

where $|\beta_1| = 2b-1$, $|\mu_1| = 1$ and $S_\delta = \{x \colon |x - x_0| = \delta\}$.

From estimates (10) and (14) it follows that

$$|I_2(k)| < C\,\|u\|_{**}^2 \left\{ \int_{S_\delta} \int_0^{\delta^{2b}} (t+k^{-2})^{-1+(1/2b)}\,dt\,d\sigma + \right.$$

$$\left. + \int_{Q_\delta} \int_0^{\delta^{2b}} \delta^{-1}(t+k^{-2})^{-1+(1/2b)}\,dt\,dx \right\}$$

$$\leqslant C\,\|u\|_{**}^2 \{\delta\,|S_\delta| + \delta^{-1}\,\delta\,|Q_\delta|\}.$$

Hence

$$|I_2(k)| \leqslant C\|u\|_{**}^2\,|Q_\delta|.$$

Similarly we estimate $I_1(k)$ as follows:

$$|I_1(k)| \leqslant \int_0^{\delta^{2b}} \int_{S_\delta} |u_k(x,t) - u_k(x,\delta^{2b})|\,|D^{\beta_1} u_k(x,t)|\,dx\,dt +$$

$$+\left| \int_0^{\delta^{2b}} \int_{Q_\delta} D^{\mu_1}[u_k(x,t) - u_k(x,\delta^{2b})]\,|D^{\beta_1} u_k(x,t)|\,dx\,dt \right|$$

$$= |I_{11}(k)| + |I_{12}(k)|.$$

For the first addendum, if $\delta^b \leqslant 1/k$, we see that

$$|I_{11}(k)| \leqslant C\,\|u\|_{**}^2 \int_{S_\delta} \int_0^{\delta^{2b}} (\delta^{2b}-t)\,(t+k^{-2})^{-1}\,(t+k^{-2})^{-1+(1/2b)}\,d\sigma\,dt$$

$$\leqslant C\,\|u\|_{**}^2\,|Q_\delta| \quad \text{since } \delta^{2b}-t \leqslant \delta^{2b}+t \leqslant k^{-2}+t.$$

If $\delta^b > 1/k$,

$$|I_{11}(k)| \leqslant \int_{S_\delta} \int_0^{\delta^{2b}} \left\{ \int_{t+k^{-2}}^{\delta^{2b}+k^{-2}} |(d/ds)\,u(x,s)|\,ds \right\} |D^{\beta_1} u_k(x,t)|\,dt\,d\sigma$$

$$\leqslant C\,\|u\|_{**}^2 \int_{S_\delta} \left\{ \int_0^{\delta^{2b}} \log \frac{\delta^{2b}+k^{-2}}{t+k^{-2}}\,(t+k^{-2})^{-1+(1/2b)}\,dt \right\} d\sigma$$

and with an easy computation we deduce that

$$\left| \int_0^{\delta^{2b}} \log \frac{\delta^{2b}+k^{-2}}{t+k^{-2}}(t+k^{-2})^{-1+(1/2b)}\,dt \right| \leqslant C\cdot\delta.$$

Therefore, for both the cases, we can conclude that

$$|I_{11}(k)| \leqslant C\,\|u\|_{**}^2\,|Q_\delta|.$$

In order to estimate $I_{12}(k)$, integrating by parts, we see that

$$|I_{12}(k)| \leqslant \int_0^{\delta^{2b}} \int_{S_\delta} |D^{\mu_1} u_k(x,t) - D^{\mu_1} u_k(x,\delta^{2b})|\,|D^{\beta_2} u_k(x,t)|\,d\sigma\,dt +$$

$$+\left|\int_0^{\delta^{2b}}\int_{Q_\delta}[D^{\mu_2}u_k(x,t)-D^{\mu_2}u_k(x,\delta^{2b})]\,D^{\beta_2}u_k(x,t)\,dx\,dt\right|$$
$$=|I_{13}(k)|+|I_{14}(k)|;\qquad |\beta_2|=2b-2,\ |\mu_2|=2.$$

If $\delta^b\leqslant 1/k$, since u_k are solutions of (1), we have

$$|I_{13}(k)|\leqslant\int_0^{\delta^{2b}}\int_{S_\delta}\left|(\delta^{2b}-t)(d/dt)D^{\mu_1}u_k(x,\theta(\delta^{2b}-t))\right||D^{\beta_2}u_k(x,t)|\,d\sigma\,dt$$
$$=\int_0^{\delta^{2b}}\int_{S_\delta}\left|(\delta^{2b}-t)\sum_{|\alpha|=2b}a_\alpha D^{\alpha+\mu_1}u_k(x,\theta(\delta^{2b}-t))\right||D^{\beta_2}u_k(x,t)|\,d\sigma\,dt.$$

For every addendum of the last integral we have

$$a_\alpha\int_0^{\delta^{2b}}\int_{S_\delta}(\delta^{2b}-t)\left|D^{\alpha+\mu_1}u_k(x,\theta(\delta^{2b}-t))\right|\,|D^{\beta_2}u_k(x,t)|\,d\sigma\,dt$$
$$\leqslant C\,\|u\|_{**}^2\,|S_\delta|\int_0^{\delta^{2b}}(\delta^{2b}-t)(t+k^{-2})^{-1}(t+k^{-2})^{-1+(1/2b)}(t+k^{-2})^{-1+(2/2b)}\,dt.$$

Hence, reasonning as for $I_{11}(k)$, we see that last expression is dominated by

$$\|u\|_{**}^2\,|S_\delta|\int_0^{\delta^{2b}}(t+k^{-2})^{-1+(1/2b)}\,dt\leqslant\|u\|_{**}^2\,|Q_\delta|.$$

Therefore, if $\delta^b\leqslant 1/k$ we have

$$|I_{13}(k)|\leqslant C\,\|u\|_{**}^2\,|Q_\delta|.$$

If $\delta^b>1/k$,

$$|I_{13}(k)|\leqslant\int_0^{\delta^{2b}}\int_{S_\delta}\left\{\int_{t+k^{-2}}^{\delta^{2b}+k^{-2}}|(d/ds)D^{\mu_1}u_k(x,s)|\,ds\right\}|D^{\beta_2}u_k(x,t)|\,d\sigma\,dt$$
$$=\int_0^{\delta^{2b}}\int_{S_\delta}\left\{\sum_{|\alpha|=2b}a_\alpha\left(\int_{t+k^{-2}}^{\delta^{2b}+k^{-2}}|D^{\mu_1+\alpha}u_k(x,s)|\right)ds\,D^{\beta_2}u_k(x,t)|\right\}d\sigma\,dt.$$

For each term of the sum, we may write

$$\int_0^{\delta^{2b}}\int_{S_\delta}\left\{\int_{t+k^{-2}}^{\delta^{2b}+k^{-2}}|D^{\mu_1+\alpha}u_k(x,s)|\,ds\,|D^{\beta_2}u_k(x,t)|\right\}d\sigma\,dt$$
$$\leqslant c\,\|u\|_{**}^2\,|S_\delta|\int_0^{\delta^{2b}}(t+k^{-2})^{-1+(2/2b)}\left[s^{-1/2b}\right]_{t+k^{-2}}^{\delta^{2b}+k^{-2}}dt$$
$$\leqslant C\,\|u\|_{**}^2\,|S_\delta|\left\{\int_0^{\delta^{2b}}(t+k^{-2})^{1-(2/2b)}(\delta^{2b}+k^{-2})^{1/2b}\,dt+\right.$$

$$+\int_0^{\delta^{2b}} (t+k^{-2})^{-1+(2/2b)}(t+k^{-2})^{-1/2b}\,dt\}$$

$$\leqslant C\|u\|_{**}^2 |S_\delta| \{(\delta^{2b}+k^{-2})^{(2/2b)-(1/2b)}+$$

$$+(k^{-2})^{2/2b}(\delta^{2b}+k^{-2})^{1/2b}+(\delta^{2b}+k^{-2})^{1/2b}+(k^{-2})^{1/2b}\}$$

$$\leqslant C\|u\|_{**}^2 |S_\delta| \cdot \delta.$$

In order to estimate

$$\left|\int_0^{\delta^{2b}} \int_{Q_\delta} [D^{\mu_2} u_k(x, t)-D^{\mu_2} u_k(x, \delta^{2b})]\, D^{\beta_2} u_k(x, t)\, dx\, dt\right|$$

let us make an integration by parts. We can estimate the term relative to the surface integral as in the previous case and for the term relative to the volume integral we make a new integration by parts. If we iterate this process b times we obtain the volume integral

$$\left|\int_0^{\delta^{2b}} \int_{Q_\delta} [D^{\mu_b} u_k(x, t)-D^{\mu_b} u_k(x, \delta^{2b})]\, D^{\beta_b} u_k(x, t)\, dx\, dt\right| \equiv J$$

where $|\beta_b| = 2b-2$ and $|\mu_b| = b$.

By the elementary inequality $ab \leqslant a^2+b^2$, J is majorized by

$$\int_0^{\delta^{2b}} \int_{Q_\delta} [D^{\mu_b} u_k(x, t)]^2\, dx\, dt + \int_0^{\delta^{2b}} \int_{Q_\delta} |D^{\mu_b} u_k(x, \delta^{2b})|^2\, dx\, dt +$$

$$+\int_0^{\delta^{2b}} \int_{Q_\delta} |D^{\beta_b} u_k(x, t)|^2\, dx\, dt.$$

For the second integral, we obtain

$$\int_0^{\delta^{2b}} \int_{Q_\delta} |D^{\mu_b} u_k(x, \delta^{2b})|^2\, dx\, dt \leqslant C\|u\|_{**}^2 |Q_\delta| \int_0^{\delta^{2b}} [(\delta^{2b}+k^{-2})^{-1/2}]^2\, dt$$

$$\leqslant C\|u\|_{**}^2 |Q_\delta|.$$

The first and third integrals are dominated by $\|u\|_{**}^2 |Q_\delta|$ in view of the definition of $T^b\mathrm{MO}$. Hence the lemma is proved.

The proof of our theorem may then be concluded as follows. For any fixed $t > 0$ and $|\alpha| = b$ the function $x \mapsto D^\alpha G(x, t)$ belongs to the space $H^1 = H^1(\mathbf{R}^n)$ where $H^1 = \{f \in L^1 : R_j(f) \in L^1, j = 1, 2, \ldots, n\}$ and R_j denotes the Riesz transform (see [7]). For this proof the reader can see Lemma 1.5 of [2]. Now, for each $(x, t) \in \mathbf{R}_+^{n+1}$ and $k \in N$, formula (11) implies that

$$[(D^\alpha G) * f_k(\cdot)](x) = D^\alpha u(x, t+k^{-2}).$$

On account of the duality BMO $=(H^1)^*$, reasoning as in [3] we deduce the existence of some $f \in \mathrm{BMO}$ such that

$$(D^\alpha G * f)(x) = D^\alpha u(x, t) \quad \text{for any } |\alpha| = b.$$

Hence $u(x, t) = G * f$ modulo a polynomial in x of degree $\leqslant b-1$, that is, $u = G * f$ as element of $T^b\mathrm{MO}$. The proof of the theorem is complete.

2. Proof of the pointwise estimates (10) and (10′). We denote by H_b the operator $\sum_{|\beta|=2b} a_\beta D^\beta - \partial/\partial t$. Fixing any $(x_0, t_0) \in R_+^{n+1}$ we choose some $\varphi \in C_0^\infty(R^n)$ such that, $0 \leqslant \varphi(x) \leqslant 1$,

$$\varphi(x) = \begin{cases} 1 & \text{if } |x| \leqslant 1, \\ 0 & \text{if } |x| \geqslant 2, \end{cases} \quad \text{and} \quad |D^\beta \varphi(x)| < K, \text{ say, if } |\beta| = 2b.$$

We then choose $\psi \in C^\infty(R)$ with $0 \leqslant \psi(t) \leqslant 1$,

$$\psi(t) = \begin{cases} 0 & \text{if } t \leqslant 1 \\ 1 & \text{if } t \geqslant 2 \end{cases} \quad \text{and} \quad |\psi'(t)| < K$$

for some absolute constant $K > 0$.

If ε, r are > 0 we let $\varphi_r = \varphi(x/r)$, $\psi_\varepsilon(t) = \psi(t/\varepsilon)$ and we examine the smooth function

$$v(x, t) = \varphi_r(x - x_0)\, \psi_\varepsilon(t)\, D^\alpha u(x, t), \quad \text{where} \quad |\alpha| \geqslant b.$$

For all $(x, t) \in R_+^{n+1}$ we have the representation formula (see [1], [3])

$$v(x, t) = \int_0^t \int_{R^n} G(x-y, t-s)[H_b v](y, s)\, dy\, ds$$

where $G(x, t)$ is the fundamental solution associated to $H_b u = 0$. Moreover, if $0 < \varepsilon \leqslant t_0/\varepsilon$, on account of the previous definitions we have

$$v(x_0, t_0) = D^\alpha u(x_0, t_0).$$

Consequently

$$D^\alpha u(x_0, t_0) = \int_0^{t_0} \int_{R^n} G(x_0 - y, t_0 - s)[H_b v](y, s)\, dy\, ds.$$

We observe that

$$\begin{aligned} H_b v &= H_b[\varphi_r(x - x_0)\, \psi_\varepsilon(t)\, D^\alpha u] \\ &= \sum_{|\beta|=2b} a_\beta\, \psi_\varepsilon(t)\, [\Sigma^\beta D^{\gamma_1} \varphi_r\, D^{\gamma_2}(D^\alpha u)] - \\ &\quad - \varphi_r(\partial/\partial t)\, \psi_\varepsilon(t)\, D^\alpha u - \varphi_r \psi_\varepsilon(t)\, (\partial/\partial t)\, D^\alpha u \end{aligned}$$

where the Σ^{β} means that the sum is taken for all γ_1 and γ_2 such that $\gamma_1+\gamma_2 = \beta$. Since $H_b(D^{\alpha}u)=0$, we have

$$(15)\qquad H_b v = \sum_{|\beta|=2b} a_{\beta}\psi_{\varepsilon}(t)\left(\Sigma^{\beta'} D^{\gamma_1}\varphi_r D^{\gamma_2}(D^{\alpha}u) - \varphi_r(\partial/\partial t)\psi_{\varepsilon}D^{\alpha}u\right)$$

where the $\Sigma^{\beta'}$ is taken for all $\gamma_1+\gamma_2=\beta$ but $\gamma_1\neq 0$.

Now, (15) implies that

$$(16)\quad D^{\alpha}u(x_0,t_0) = \sum_{|\beta|=2b} a_{\beta}\Sigma^{\beta'}\int_0^{t_0}\int_{\mathbf{R}^n}[G(x_0-y,t_0-s)\psi_{\varepsilon}D^{\gamma_1}\varphi_r D^{\gamma_2}(D^{\alpha}u)]\,dy\,ds - \\ - \int_0^{t_0}\int_{\mathbf{R}^n}[G(x_0-y,t_0-s)\varphi_r(\partial/\partial t)\psi_{\varepsilon}D^{\alpha}u]\,dy\,ds.$$

In (16), we consider first the term

$$\int_0^{t_0}\int_{\mathbf{R}^n}[G(x_0-y,t_0-s)\varphi_r(\partial/\partial t)\psi_{\varepsilon}D^{\alpha}u]\,dy\,ds \\ \leqslant CK\varepsilon^{-1}\int_{\varepsilon}^{2\varepsilon}\int_{|y-x_0|\leqslant 2r}(t_0-s)^{-n/2b}|D^{\alpha}u|\,dy\,ds \\ \leqslant CK\varepsilon^{-1}(t_0-2\varepsilon)^{-n/2b}\int_0^{2\varepsilon}\int_{|y-x_0|\leqslant 2r}|D^{\alpha}u|\,dy\,ds \\ \leqslant C_n(t_0-2\varepsilon)^{-n/2b}\varepsilon^{-1}\varepsilon^{1/2}r^{n/2}\left\{\int_0^{2\varepsilon}\int_{Q_{2r}}|D^{\alpha}u|^2\,dy\,ds\right\}^{1/2}$$

by the Schwarz inequality and where Q_{2r} is the cube with center x_0 and side $2r$. Letting $\varepsilon = t_0/4$, $2r = t_0^{1/2b}$, the last expression is majorized by

$$C_n t_0^{-1/2}\left\{r^{-n}\int_0^{(2r)^{2b}}\int_{Q_{2r}}|D^{\alpha}u|^2\,dy\,ds\right\}^{1/2}\leqslant Ct_0^{-1/2}\|u\|_{**}$$

for any α with $|\alpha|=b$.

Let us estimate the terms of the form

$$\int_0^{t_0}\int_{\mathbf{R}^n}G(x_0-y,t_0-s)\psi_{\varepsilon}D^{\gamma_1}\varphi_r D^{\gamma_2}(D^{\alpha}u)\,dy\,ds.$$

If we integrate by parts in space $|\gamma_2|$ times (where $|\gamma_2|\leqslant 2b-1$), we have

$$\left|\int_0^{t_0}\int_{\mathbf{R}^n}G(x_0-y,t_0-s)\psi_{\varepsilon}(t)D^{\gamma_1}\varphi_r D^{\gamma_2}(D^{\alpha}u)\,dy\,ds\right| \\ \leqslant \Sigma^{\gamma_2}\int_0^{t_0}\int_{\mathbf{R}^n}|D^{\nu}G(x_0-y,t_0-s)D^{\gamma_1+\mu}\varphi_r\psi_{\varepsilon}(D^{\alpha}u)|\,dy\,ds$$

where Σ^{γ_2} means that the sum is taken for all ν and μ with $\nu+\mu = \gamma_2$. Using (5') and remarking that $|D^{\gamma_1+\mu}\varphi_r| < Kr^{-|\gamma_1|-|\mu|}$, we may write that

$$\int_0^{t_0}\int_{R^n} |D^\nu G(x_0-y,\, t_0-s)\psi_\varepsilon D^{\gamma_1+\mu}\varphi_r(D^\alpha u)|\,dy\,ds$$
$$\leqslant Kr^{-|\gamma_1|-|\mu|}\int_\varepsilon^{t_0}\int_{r\leqslant|y_0-y|\leqslant 2r} |x_0-y|^{-n-|\nu|}|D^\alpha u|\,dy\,ds$$
$$\leqslant Kr^{-|\gamma_1|-|\gamma_2|}r^{-n}\int_0^{t_0}\int_{Q_{2\delta}} |D^\alpha u|\,dy\,ds$$
$$\leqslant Kr^{-|\beta|}r^{-n}t_0^{1/2}\{r^{-n}\int_0^{t_0}\int_{Q_{2r}} |D^\alpha u|^2\,dy\,ds\}^{1/2}$$
$$\leqslant C_n t_0^{-|\beta|/2b}t_0^{1/2}\big[\delta^{-n}\int_0^{\delta^{2b}}\int_{Q_\delta}|D^\alpha u|^2\,dy\,ds\big]^{1/2} \leqslant C_n t_0^{-1/2}\|u\|_{**}$$

where $2r = t_0^{1/2b} = \delta$ and $|\alpha| = b$. Hence, for $|\alpha| = b$,

$$|D^\alpha u(x_0, t_0)| < Ct_0^{-1/2}\|u\|_{**}. \tag{17}$$

For $|\alpha| > b$, we can obtain the estimate integrating $|\alpha|-b$ times by parts and reasoning in the same manner as for $|\alpha| = b$. Therefore, we have

$$|D^\alpha u(x, t)| \leqslant Ct_0^{(-1/2)-(|\alpha|-b)/2b} \quad \text{for any } |\alpha| \geqslant b. \tag{18}$$

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