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Note on the existence of continuous solutions of a functional equation of n-th order

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Abstract. A sufficient condition is given for the existence of continuous solutions of the functional equation

$$\varphi(x) = h(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]),$$

where φ is the unknown function.

The object of this note is to give a theorem about the existence of the continuous solutions of the functional equation

$$\varphi(x) = h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big), \tag{1}$$

under the hypothesis of the existence of semicontinuous solutions of the functional inequalities

$$\varphi(x) \leqslant h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) \tag{2}$$

and

$$h\big(x, \varphi[f_1(x)], \ldots, \varphi[f_n(x)]\big) \leqslant \varphi(x). \tag{3}$$

A similar method has been used in [1] in the case $n = 1$.

We shall assume the following hypotheses (R denotes the set of all real numbers).

(i) X is a metric space; $h: X \times R^n \to R$ and $f_i: X \to X$, $i = 1, \ldots, n$, are continuous functions.

(ii) There exist a function $a: X \to [0, \infty)$ and an increasing (with respect to each variable) function $\beta: [0, \infty)^n \to [0, \infty)$ such that

$$|h(x, y_1, \ldots, y_n) - h(x, \bar{y}_1, \ldots, \bar{y}_n)| \leqslant a(x)\beta(|y_1 - \bar{y}_1|, \ldots, |y_n - \bar{y}_n|), \quad (x, y_i), (x, \bar{y}_i) \in X \times R,\ i = 1, \ldots, n. \tag{4}$$

Moreover,

$$a\big(f_1^k[f_i(x)]\big) \leqslant a[f_1^{k+1}(x)] \ (^1), \quad x \in X,\ k = 0, 1, 2, \ldots, \quad i = 1, \ldots, n. \tag{5}$$

(1) The upper indexes denote the function iterates.

(iii) Inequality (2) has a lower semicontinuous solution $\varphi_1^*: X \to R$ and inequality (3) has an upper semicontinuous solution $\varphi_2^*: X \to R$ such that $\varphi_1^*(x) \leqslant \varphi_2^*(x)$ for all $x \in X$. Furthermore, the function σ defined by

$$\sigma(x) = |\varphi_1^*(x) - \varphi_2^*(x)|, \quad x \in X, \tag{6}$$

fulfils the inequality

$$\sigma\big(f_1^k[f_i(x)]\big) \leqslant \sigma[f_1^{k+1}(x)], \quad x \in X, \ k = 0, 1, 2, \ldots, \ i = 1, \ldots, n. \tag{7}$$

(iv) For every fixed $x \in X$ the function $h(x, \cdot, \ldots, \cdot)$ is an increasing (with respect to each variable) function in the set

$$\big[\varphi_1^*[f_1(x)], \varphi_2^*[f_1(x)]\big] \times \ldots \times \big[\varphi_1^*[f_n(x)], \varphi_2^*[f_n(x)]\big].$$

Write

$$\gamma(t) = \beta(t, \ldots, t), \quad t \in [0, \infty), \tag{8}$$

$$G_k(x) = \prod_{l=0}^{k-1} a[f_1^l(x)], \quad x \in X, \ k = 1, 2, \ldots, \tag{9}$$

$$H_k(x) = G_k(x)\gamma^k\big(\sigma[f_1^k(x)]\big), \quad x \in X, \ k = 1, 2, \ldots \tag{10}$$

We shall show that the following theorem is true.

THEOREM. *If hypotheses* (i)–(iv) *and the conditions*

$$\gamma\big(a(x)t\big) \leqslant a(x)\gamma(t), \quad x \in f_1(X), t \in [0, \infty), \tag{11}$$

$$\lim_{k\to\infty} H_k(x) = 0, \quad x \in X, \tag{12}$$

are fulfilled, then equation (1) *has at least one continuous solution $\varphi: X \to R$ fulfilling the condition*

$$\varphi_1^*(x) \leqslant \varphi(x) \leqslant \varphi_2^*(x), \quad x \in X. \tag{13}$$

Proof. Let Φ_1 be the class of all lower semicontinuous functions $\varphi: X \to R$ fulfilling (13), and similarly, let Φ_2 be the class of all upper semicontinuous functions $\varphi: X \to R$ fulfilling condition (13). Put

$$\varphi_{j,0}(x) = \varphi_j^*(x), \quad \varphi_{j,k+1}(x) = h\big(x, \varphi_{j,k}[f_1(x)], \ldots, \varphi_{j,k}[f_n(x)]\big), \tag{14}$$
$$x \in X, \ j = 1, 2, \ k = 0, 1, 2, \ldots$$

It follows from the continuity of the functions f_i, $i = 1, \ldots, n$, and h and from hypotheses (iii) and (iv) that

$$\varphi_{j,k} \in \Phi_j, \quad j = 1, 2, \ k = 0, 1, 2, \ldots \tag{15}$$

and

$$\varphi_{1,k}(x) \leqslant \varphi_{1,k+1}(x), \quad \varphi_{2,k+1}(x) \leqslant \varphi_{2,k}(x), \quad x \in X, \ k = 0, 1, 2, \ldots \tag{16}$$

Hence the sequences $\{\varphi_{j,k}\}$, $j=1,2$, are pointwise convergent. Let

(17) $$\varphi_j(x) = \lim_{k\to\infty}\varphi_{j,k}(x), \quad x\in X,\ j=1,2.$$

On account of (15), (16) and (17)

(18) $$\varphi_j \in \Phi_j, \quad j=1,2.$$

Moreover, by (14) and (17) we have that φ_j, $j=1,2$, fulfil equation (1).

Note that relations (10), (9), (8), (5), (7) and the monotonicity of γ imply the inequality

(19) $$H_k[f_i(x)] \leqslant H_k[f_1(x)], \quad x\in X,\ k=1,2,\ldots,\ i=1,\ldots,n,$$

whereas (10), (9) and (11) give

(20) $$\alpha(x)\gamma\big(H_k[f_1(x)]\big) \leqslant H_{k+1}(x), \quad x\in X,\ k=1,2,\ldots$$

We shall prove that

(21) $$|\varphi_{1,k}(x)-\varphi_{2,k}(x)| \leqslant H_k(x), \quad x\in X,\ k=1,2,\ldots$$

In view of (14), (4), (6), (7), of the monotonicity of β, and of (8), (9) and (10), we obtain

$$\begin{aligned}
&|\varphi_{1,1}(x)-\varphi_{2,1}(x)|\\
&\quad = \big|h\big(x,\varphi_1^*[f_1(x)],\ldots,\varphi_1^*[f_n(x)]\big)-h\big(x,\varphi_2^*[f_1(x)],\ldots,\varphi_2^*[f_n(x)]\big)\big|\\
&\quad \leqslant \alpha(x)\beta\big(\sigma[f_1(x)],\ldots,\sigma[f_n(x)]\big) \leqslant \alpha(x)\gamma\big(\sigma[f_1(x)]\big) = H_1(x).
\end{aligned}$$

Thus (21) holds for $k=1$. Suppose that inequality (21) is fulfilled for a $k\geqslant 1$. By (14), (4), (21) for k, the monotonicity of β, (19), (8) and (20), we have

$$\begin{aligned}
&|\varphi_{1,k+1}(x)-\varphi_{2,k+1}(x)|\\
&= \big|h\big(x,\varphi_{1,k}[f_1(x)],\ldots,\varphi_{1,k}[f_n(x)]\big)-h\big(x,\varphi_{2,k}[f_1(x)],\ldots,\varphi_{2,k}[f_n(x)]\big)\big|\\
&\leqslant \alpha(x)\beta\big(H_k[f_1(x)],\ldots,H_k[f_n(x)]\big) \leqslant \alpha(x)\gamma\big(H_k[f_1(x)]\big) \leqslant H_{k+1}(x),
\end{aligned}$$

i.e. (21) holds for $k+1$, and thus it is valid for every positive integer. Recalling (17), (21) and (12) we have $\varphi_1=\varphi_2=\varphi$, and applying (18) we obtain that $\varphi\in\Phi_1\cap\Phi_2$ which implies that φ is a continuous function. This completes the proof.

Remark. In the case where the function f_i, $i=1,\ldots,n$, is the i-th iterate of a continuous function $f\colon X\to X$, i.e. $f_i(x)=f^i(x)$, $x\in X$, $i=1,\ldots,n$, the inequalities

$$\alpha[f(x)] \leqslant \alpha(x), \quad x\in X,$$
$$\sigma[f(x)] \leqslant \sigma(x), \quad x\in X,$$

imply (5) and (7), respectively.

Now, we are going to show how our result can be applied in a more specific situation by considering the following

EXAMPLE. Let f, g and F be real-valued continuous functions defined on an interval I such that

$$f(I) \subset I,$$
$$0 \leqslant g(x), \quad 0 \leqslant F(x), \quad 1+g(x)+F(x) \leqslant \pi/2, \quad x \in I,$$
$$g[f(x)] \leqslant g(x), \quad \sum_{k=0}^{\infty} g[f^k(x)] < \infty, \quad x \in I,$$

and consider the functional equation

$$\varphi(x) = \frac{1+g(x)}{2}\big(\sin\varphi[f(x)]+\sin\varphi[f^2(x)]\big)+F(x).$$

Putting

$$a(x) = 1+g(x), \quad x \in I,$$
$$\beta(t_1, t_2) = \tfrac{1}{2}\big(\gamma(t_1)+\gamma(t_2)\big), \quad t_1, t_2 \in [0, \infty),$$

where

$$\gamma(t) = \begin{cases} 2\sin t/2, & 0 \leqslant t \leqslant \pi, \\ 2, & t > \pi, \end{cases}$$

and

$$\varphi_1^*(x) = 0, \quad \varphi_2^*(x) = \pi/2, \quad x \in I,$$

we get, by the above theorem and by the remark, the existence of at least one continuous solution $\varphi: I \to [0, \pi/2]$ of this equation. This, however, does not result from theorems contained in papers [2]-[4] (cf. also Chapter XII in [5]).

References

[1] K. Baron, *On the continuous solutions of a non-linear functional equation of the first order*, Ann. Polon. Math. 28 (1973), p. 201–205.

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[3] — *Note on continuous solutions of a functional equation*, ibidem (to appear).

[4] J. Kordylewski, *On continuous solutions of systems of functional equations*, Ann. Polon. Math. 25 (1971), p. 53-83.

[5] M. Kuczma, *Functional equations in a single variable*, Monografie Mat. 46, Warszawa 1968.

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