ANNALES
POLONICI MATHEMATICI
XLVIII (1988)

On a generalization of a theorem of S. Bernstein

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Abstract. In this paper we obtain "weak solutions" via Topological Transversality to nonlinear boundary value problems of the form $y'' = f(t, y, y')$, $t \in [0, 1]$, with y satisfying appropriate boundary conditions, where f: $[0, 1] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ satisfies the Carathéodory Conditions. Our analysis is based on the notions of an essential map and on a priori bounds on solutions.

1. Introduction. In this paper we study the existence of solutions to second order boundary value problems of the form

$$y'' = f\big(t, y(t), y'(t)\big), \quad y \in \boldsymbol{B}, \quad t \in [0, 1], \tag{1.1}$$

where in fact f: $[0, 1] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ may be discontinuous. Here of course B denotes suitable boundary conditions. We examine in this paper the case where f satisfies the *Carathéordory Conditions*, i.e.,

(a) For fixed $(u, v) \in \boldsymbol{R}^2$, $f(\blacksquare, u, v)$ is Lebesgue measurable on $[0, 1]$.

(b) For all $t \in [0, 1]$, $f(t, \blacksquare, \blacksquare)$ is continuous on $\boldsymbol{R}^2$.

For notational purposes let $L^2(0, 1)$ denote the space of Lebesgue measurable functions g on $(0, 1)$ with $\int_0^1 |g(t)|^2\,dt < \infty$. $L^2(0, 1)$ with norm $\|g\|_{L^2} = \left(\int_0^1 |g(t)|^2\,dt\right)^{1/2}$ is a Banach Space.

By a weak solution to (1.1) we mean a function $y \in \boldsymbol{B}$ which together with its derivative y' is absolutely continuous on $[0, 1]$ with $y'' \in L^2(0, 1)$ and $y'' = f(t, y, y')$ almost everywhere on $[0, 1]$. This paper in fact extends results of Granas, Guenther and Lee [10] which deals with the case where f is continuous. We shall establish, with f satisfying the same physical assumptions as in [10], that (1.1) has bounded weak solutions. Our analysis is based on the Topological Transversality Theorem and known results on Sobolev Spaces.

2. Preliminary notation and results. Let $H^2(0, 1)$ denote the space of all functions u on the interval $[0, 1]$ which are absolutely continuous on $[0, 1]$ together with their derivative u' and whose derivative u'' (which exists almost everywhere) is an element of $L^2(0, 1)$. $H^2(0, 1)$ with norm

$$\|u\|_{H^2} = \|u\|_{L^2} + \|u'\|_{L^2} + \|u''\|_{L^2}$$

is a Banach Space. Also we let

$$H_B^2(0, 1) = \{u \in H^2(0, 1): u \in B\}.$$

Finally we state (without proof) some standard theorems which will be used in this paper:

THEOREM 2.1. *Let g be a monotone increasing absolutely continuous function on $[a, b]$ with $g(a) = c$, $g(b) = d$. If f is a nonnegative measurable function on $[c, d]$, then*

$$\int_c^d f(y)\,dy = \int_a^b f(g(x))g'(x)\,dx.$$

THEOREM 2.2 (Sobolev Imbedding Theorem). *$H^2(0, 1)$ is compactly imbedded into $C^1[0, 1]$, i.e., the imbedding operator j: $H^2(0, 1) \to C^1[0, 1]$ is continuous and completely continuous.*

3. Homogeneous boundary value problems. In this section we examine problems of the form

$$(3.1) \qquad \begin{aligned} &y'' = f(t, y, y'), \quad t \in [0, 1], \\ &y \in B, \end{aligned}$$

where f is defined in $[0, 1] \times \mathbf{R}^2$. Here B denotes either the boundary conditions

(i) $y(0) = 0$, $y(1) = 0$

or

(ii) $-\alpha y(0) + \beta y'(0) = 0$; $\alpha, \beta > 0$; $ay(1) + by'(1) = 0$; $a, b > 0$.

Now suppose that f satisfies the following hypothesis:

(3.2) f satisfies the Carathéordory Conditions;

(3.3) There is a constant $M \geqslant 0$ such that

$$yf(t, y, 0) > 0 \quad \text{for } |y| > M,$$

$$(3.4) \qquad |f(t, u, p)| \leqslant A(t, u)p^2 + B(t, u),$$

where $A(t, u)$, $B(t, u) > 0$ are functions bounded on bounded (t, u) sets;

(3.5) $yf(t, y, p)$ is lower semicontinuous at all points of the form $(t, y, 0)$.

PROPOSITION 3.1. *Suppose that f: $[0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ satisfies* (3.2) *and* (3.4).

Then $F\colon C^1[0,1]\to L^2(0,1)$ *given by* $(Fu)(t)=f(t,u(t),u'(t))$ *is defined and continuous.*

Proof. Let $\varepsilon>0$ be given and suppose $u_0\in C[0,1]$. Consider

$$G_{vm}=\{t\in[0,1]\colon \|v-(u_0(t),u_0'(t))\|<1/m \Rightarrow |f(t,v_1,v_2)-f(t,u_0(t),u_0'(t))|<\varepsilon/L\},$$

where $v=(v_1,v_2)\in \mathbf{R}^2$, $\|v\|=\max\{|v_1|,|v_2|\}$ and L is a predetermined constant which will be described below. G_{vm} is measurable since f satisfies (3.2). Let $E_{m\varepsilon}=\bigcap_{v\in\mathbf{R}^2}G_{vm}$. Now $E_{m\varepsilon}$ is measurable and $E_{1\varepsilon}\subset E_{2\varepsilon}\subset\ldots$ Also $\bigcup_{m=1}^{\infty}E_{m\varepsilon}=(0,1)$ for if $t_0\in(0,1)$, then there exists m such that $\|v-(u_0(t_0),u_0'(t_0))\|<1/m$, and hence $|f(t_0,v_1,v_2)-f(t_0,u_0(t_0),u_0'(t_0))|<\varepsilon/L$ since f satisfies (3.2). Hence there exists $m_0\in N$ such that $\operatorname{mes}(E_{m_0\varepsilon})>1-\varepsilon/L$. Let A, B be constants such that $|A(t,v_1)|\leqslant A$ and $|B(t,v_1)|\leqslant B$ for $|v_1|\leqslant 1+\|u_0\|_1$, where $\|u_0\|_1=\max\{\|u_0\|_\infty,\|u_0'\|_\infty\}$ and $\|u_0\|_\infty=\sup_{t\in[0,1]}|u(t)|$. Now for all $\varepsilon>0$ there exists $\eta=\eta(\varepsilon)>0$ such that

$$\operatorname{mes}(S)<\eta\Rightarrow\int_S\left[9A^2(u_0'(t))^4+2B^2\right]dt<\tfrac{1}{3}(\tfrac{1}{2}\varepsilon)^2.$$

Put

$$0<\delta<\min\left\{\frac{1}{m_0},\ \frac{1}{2}\left\{\frac{\varepsilon}{A(6)^{1/2}}\right\}^{1/2}\right\}\quad\text{and}\quad\max\left\{\frac{\varepsilon}{\eta},(3)^{1/2}\right\}<L.$$

Let $u\in C^1[0,1]$ such that $\|u-u_0\|_1<\delta$. We will now show that $\|Fu-Fu_0\|_{L^2}<\varepsilon$. If $t\in E_{m_0\varepsilon}$, then

$$|f(t,u(t),u'(t))-f(t,u_0(t),u_0'(t))|<\varepsilon/L$$

and so

$$\int_{E_{m_0\varepsilon}}|f(t,u(t),u'(t))-f(t,u_0(t),u_0'(t))|^2\,dt<\varepsilon^2/L^2<\varepsilon^2/3.$$

However, $\operatorname{mes}(E^c_{m_0\varepsilon})<\varepsilon/L<\varepsilon\eta/\varepsilon=\eta$, and so

$$\int_{E^c_{m_0\varepsilon}}|f(t,u(t),u'(t))-f(t,u_0(t),u_0'(t))|^2\,dt$$

$$\leqslant 2\int_{E^c_{m_0\varepsilon}}\{|f(t,u(t),u'(t))|^2+|f(t,u_0(t),u'(t))|^2\}\,dt$$

$$\leqslant 2\int_{E^c_{m_0\varepsilon}}\{|A(t,u(t))(u'(t))^2+B(t,u(t))|^2+$$

$$+|A(t,u_0(t))(u_0'(t))^2+B(t,u_0(t))|^2\}\,dt$$

$$\leqslant 4 \int_{E^c_{m_0\varepsilon}} \left(A^2 (u'(t))^4 + B^2 + A^2 (u_0'(t))^4 + B^2\right) dt$$

$$\leqslant 4 \int_{E^c_{m_0\varepsilon}} \{8A^2 (|u'(t) - u_0'(t)|^4 + |u_0'(t)|^4) + A^2 |u_0'(t)|^4 + 2B^2\} dt$$

$$\leqslant 4 \int_{E^c_{m_0\varepsilon}} \{9A^2 |u_0'(t)|^4 + 2B^2 + 8A^2 |u_0'(t) - u'(t)|^4\} dt$$

$$< 4 \{\tfrac{1}{3} (\tfrac{1}{2}\varepsilon)^2 + 8A^2 \delta^4\} < \tfrac{1}{3}\varepsilon^2 + \tfrac{1}{3}\varepsilon^2 = \tfrac{2}{3}\varepsilon^2.$$

Hence $\|Fu - Fu_0\|_{L^2} < \varepsilon$, so F: $C^1[0, 1] \to L^2(0, 1)$ is continuous.

The Sobolev Imbedding Theorem together with Proposition 3.1 are now used to extend Theorem 2.1 of [11] for the new class of problems (3.1).

THEOREM 3.2 *Let* f: $[0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ *satisfy* (3.2), (3.4) *and* $0 \leqslant \lambda \leqslant 1$. *Suppose there is a constant* K *independent of* λ *such that* $\|y\|_{H^2} \leqslant K$ *for each solution* $y(t)$ *to*

$$(3.1)_\lambda \qquad y'' = \lambda f(t, y, y'), \quad t \in [0, 1], \quad y \in B.$$

Then the boundary value (3.1) *has a solution* y *in* $H^2(0, 1)$.

Proof. Let $\bar{V} = \{u \in H_B^2(0, 1)\colon \|u\|_{H^2} \leqslant K + 1\}$ and define F_λ: $C^1[0, 1] \to L^2(0, 1)$ by $(Fv)(t) = \lambda f(t, v(t), v'(t))$. Now F_λ is continuous by Proposition 3.1. We have the imbedding j: $H_B^2(0, 1) \to C^1[0, 1]$ completely continuous by Theorem 2.2. Finally we define N: $H_B^2(0, 1) \to L^2(0, 1)$ by $Ny = y''$. It is easy to check N is linear, onto and continuous. To show N is one-to-one we observe that the boundary conditions (i) or (ii) imply that y' vanishes at least once in $[0, 1]$. So if $Ny = 0$ the absolute continuity of y and y' with the above observation implies $y = 0$. Thus N^{-1} is a bounded linear operator by Theorem 5.10 of [15]. Now $H_\lambda = N^{-1} F_\lambda j$: $\bar{V} \to H_B^2(0, 1)$ defines a homotopy. It is clear that the fixed points of H_λ are precisely the solutions to $(3.1)_\lambda$. Now H_λ is fixed point free on ∂V. Moreover, the complete continuity of j together with the continuity of N^{-1} and F_λ imply that the homotopy H_λ is compact. Now H_0 is essential so Theorem 1.5 of [12] implies that H_1 is essential. Thus (3.1) has a solution.

Next sufficient conditions of f are given which imply a priori bounds for solutions to (3.1). Let $y \in H_B^2(0, 1)$ be a solution to (3.1). Suppose $[y(t)]^2$ has a maximum at $t_0 \in (0, 1)$. Then from elementary calculus $y'(t_0) = 0$.

THEOREM 3.3. *Suppose* f: $[0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ *satisfies* (3.2), (3.5) *and* (3.3). *Then any solution* y *to* (3.1) *satisfies*

$$|y(t)| \leqslant M, \quad t \in [0, 1].$$

Proof. We first show that $|y|$ cannot have a nonzero maximum at 0 or 1. This is true automatically if y satisfies (i). On the other hand suppose y satisfies (ii) and that $|y|$ has a nonzero maximum at 0. Then $y(0)y'(0) \leqslant 0$. However,

$$y(0)y'(0) = \frac{\beta}{\alpha}\{y(0)\}^2 > 0,$$

a contradiction. A similiar argument works for the case $t = 1$. We conclude that $|y|$ can only have a nonzero maximum at $t_0 \in (0, 1)$. Now assume the maximum of $|y|$ is at $t_0 \in (0, 1)$, so $y'(t_0) = 0$. Suppose $|y(t_0)| > M$. Then from (3.3), $y(t_0)f(t_0, y(t_0), 0) > 0$. The continuity of y and y' together with (3.5) implies there exists a neighborhood N_{t_0} of $(t_0, y(t_0), 0)$ such that

$$(*) \qquad y(t)f(t, y(t), y'(t)) > 0 \quad \text{for } (t, y(t), y'(t)) \in N_{t_0}.$$

On the other hand $y'(t) = \int_{t_0}^{t} y''(s)ds$ and so Fubini's Theorem implies

$$y(t) = y(t_0) + \int_{t_0}^{t} (t-u)y''(u)du.$$

Thus,

$$y^2(t) = y^2(t_0) + 2\int_{t_0}^{t} (t-u)[y(u)f(u, y(u), y'(u))] + [y'(u)]^2 du.$$

Since $|y|$ has a maximum at t_0, then for t near t_0

$$\int_{t_0}^{t} (t-u)[y(u)f(u, y(u), y'(u))] + [y'(u)]^2 du \leqslant 0$$

which contradicts $(*)$. Thus $|y(t_0)| \leqslant M$.

We now prove our basic existence theorem for second order boundary value problems.

THEOREM 3.4. *Suppose that* $f\colon [0, 1] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ *satisfies* (3.2), (3.3), (3.4) *and* (3.5). *Then the boundary value problem* (3.1) *has at least one solution in* $H^2(0, 1)$.

Proof. To prove existence of a solution in $H^2(0, 1)$ we apply Theorem 3.2. To establish a priori bounds for $(3.1)_\lambda$, let $y(t)$ be a solution to $(3.1)_\lambda$. If $\lambda = 0$, we have the unique solution $y \equiv 0$. Otherwise, for $0 < \lambda \leqslant 1$, $yf(t, y, 0) > 0$ for $|y| > M$ implies $\lambda yf(t, y, 0) > 0$ for $|y| > M$. Thus Theorem 3.3 implies $|y| \leqslant M$ for any solution y to $(3.1)_\lambda$ and for each $\lambda \in [0, 1]$. Hence $(\int_0^1 |y(s)|^2 ds)^{1/2} \leqslant M$. Finally we obtain a priori bounds on derivatives of y. It is easy to observe that boundary conditions (i) or (ii) imply that y' vanishes at least once on [0, 1], so each point $t \in [0, 1]$ for which $y'(t) \neq 0$ belongs to an interval $[\mu, v]$ such that y' maintains a fixed sign on $[\mu, v]$ and $y'(\mu)$ and/or $y'(v)$ is zero. Assume that $y'(\mu) = 0$ and $y' \geqslant 0$ on $[\mu, v]$. Thus, with A_0, B_0 denoting upper bounds of $A(t, u)$, $B(t, u)$ respectively for $(t, u) \in [0, 1] \times [-M, M]$ and since

$$|\lambda f(t, y, y')| \leqslant A_0 (y')^2 + B_0,$$

we have

$$\int_\mu^t \frac{y'(u)\,|y''(u)|}{A_0 [y'(u)]^2 + B_0}\, du \leqslant 2M.$$

For $\mu \leqslant u \leqslant t$

$$[y'(u)]^2 = |[y'(u)]^2| = 2\left|\int_\mu^u y'(s)\, y''(s)\, ds\right| \leqslant 2 \int_\mu^u y'(s)\,|y''(s)|\, ds, \tag{3.6}$$

so

$$A_0 [y'(u)]^2 + B_0 \leqslant 2A_0 \int_\mu^u y'(s)\,|y''(s)|\, ds + B_0.$$

Thus the previous inequality implies

$$\int_\mu^t \left\{ \frac{2A_0\, y'(u)\,|y''(u)|}{2A_0 \int_\mu^u y'(s)\,|y''(s)|\, ds + B_0} \right\} du \leqslant 4A_0 M.$$

Theorem 2.1 with $g(u) = 2A_0 \int_\mu^u y'(s)\,|y''(s)|\, ds$ yields

$$\int_0^{g(t)} \frac{du}{u + B_0} \leqslant 4A_0 M,$$

and so $g(t) \leqslant B_0 (e^{4A_0 M} - 1)$. Moreover, (3.6) yields

$$[y'(t)]^2 \leqslant 2 \int_\mu^t y'(s)\,|y''(s)|\, ds \leqslant \frac{B_0}{A_0} (e^{4A_0 M} - 1)$$

and so

$$|y'(t)| \leqslant \left\{ \frac{B_0}{A_0} (e^{4A_0 M} - 1) \right\}^{1/2} \equiv M_1.$$

The other cases are treated similarly and the same bound is obtained. Thus

$|y'| \leqslant M_1$ and, in particular, $(\int_0^1 |y'(s)|^2 ds)^{1/2} \leqslant M_1$ for each solution y to $(3.1)_\lambda$ and for each $\lambda \in [0, 1]$. Also (3.3) and the differential equation yields

$$(\int_0^1 |y''(t)|^2 dt)^{1/2} \leqslant (\int_0^1 [A_0(y'(t))^2 + B_0]^2 dt)^{1/2}$$
$$\leqslant A_0 M_1^2 + B_0 \equiv M_2.$$

So $\|y\|_{H^2} \leqslant K = M_0 + M_1 + M_2$ and the existence of a solution to (3.1) is established.

Remark. A priori bounds, independent of λ, for y' and y'' (assuming we have such a bound on solutions) can be obtained as in Theorem 3.4 if instead of the fact that y' vanishes at least once on $[0, 1]$, we have

$$|y'(\mu)| \leqslant K,$$

$K \geqslant 0$ a fixed constant independent of λ, for some $\mu \in [0, 1]$. This result will be used in our analysis of the inhomogeneous problem.

4. Inhomogeneous boundary value problems. We have analogue results for the inhomogeneous problem

$$(4.1) \qquad \begin{aligned} &y'' = f(t, y, y'), \quad t \in [0, 1], \\ &y \in B, \end{aligned}$$

where B denotes either the boundary conditions

(iii) $y(0) = r$, $y(1) = s$ or

(iv) $-\alpha y(0) + \beta y'(0) = r$; $\alpha, \beta > 0$; $ay(1) + by'(1) = s$; $a, b > 0$.

THEOREM 4.1 *Suppose that f satisfies* (3.2), (3.3), (3.4) *and* (3.5). *Then the boundary value problem* (4.1) *has at least one solution in* $H^2(0, 1)$.

Proof. Consider the family of problems:

$$(4.1)_\lambda \qquad \begin{aligned} &y'' = \lambda f(t, y, y'), \quad 0 \leqslant \lambda \leqslant 1, \\ &y \in B, \end{aligned}$$

The existence of a solution in $H^2(0, 1)$ follows from a slight modification of the proof of Theorem 5.1 of [13] once a priori bounds independent of λ are established for solutions y to $(4.1)_\lambda$. To establish a priori bounds for $(4.1)_\lambda$, let $y(t)$ be a solution to $(4.1)_\lambda$. Now if $\lambda = 0$ we have a unique solution and thus $|y(t)| \leqslant L$ for some constant $L < \infty$. Otherwise for $0 < \lambda \leqslant 1$, $yf(t, y, 0) > 0$ for $|y| > M$ implies $\lambda yf(t, y, 0) > 0$ for $|y| > M$. If y satisfies (iii) it follows immediately from Theorem 3.3 that

$$|y| \leqslant M_0 = \max\{M, |r|, |s|\}.$$

On the other hand, if y satisfies (iv) we have

$$|y| \leqslant M_1 = \max\{M, |r/\alpha|, |s/a|\}.$$

To see this suppose that $|y(t)|$ assumes its maximum at $t = 0$. Then $y(0)\,y'(0) \leqslant 0$. So

$$0 \geqslant y(0)\,\beta y'(0) = \alpha\big(y(0)\big)^2\left\{\frac{r}{\alpha y(0)} + 1\right\}$$

and consequently $|y(0)| \leqslant |r/\alpha|$. Likewise $|y(1)| \leqslant |s/a|$ if $|y|$ achieves its maximum at $t = 1$. Thus $|y| \leqslant M_2 = \max\{M_0, M_1, L\}$ for any solution y to $(4.1)_\lambda$. A priori bounds independent of λ, for y' and y'' follow from the remark after Theorem 3.4 since it is easy to observe that

$$|y'(\mu)| \leqslant K,$$

$K \geqslant 0$ a fixed constant independent of λ, for some point $\mu \in [0, 1]$. Thus existence of a solution to (4.1) follows from Theorem 5.1 of [13].

THEOREM 4.2 (Granas, Guenther and Lee). *Suppose* $f\colon [0, 1] \times \boldsymbol{R}^2 \to \boldsymbol{R}$ *is continuous and satisfies* (3.3) *and* (3.4). *Then the solution* u *of* (4.1) *is a classical solution.*

Proof. Let u be a solution (guaranteed by Theorem 4.1) of (4.1), i.e., $u''(t) = f(t, u(t), u'(t))$ almost everywhere for $t \in [0, 1]$. Now with $g(t) = f(t, u(t), u'(t)) \in C[0, 1]$ and $u''(t) = g(t)$ almost everywhere for $t \in [0, 1]$ we have $u \in C^2[0, 1]$ by the uniqueness of generalized derivatives.

EXAMPLE (Heat Conduction). Suppose V is an isotropic heat conducting medium with S denoting the surface and $\vec{n}$ the outer normal. We define $u = u(x, t)$ to be the temperature at location $x \in V$ and $t > 0$. Also $c = c(x, u)$ denotes the specific heat, $p = p(x, u)$ the density and $k = k(x, u)$ the thermal conductivity. Now the Divergence Theorem and Fourier's Law together with conservation of energy yields the heat equation

$$\frac{\partial}{\partial t}(cpu) = \operatorname{div}(k\,\vec{\nabla} u) + h; \qquad x \in V,\ t > 0,$$

where $h = h(x, u)$ represents the rate of heat generation by internal sources.

We now set up boundary conditions which describe the heat transfer across S. Suppose the surroundings of V are kept at a time independent temperature and that heat radiates into the surroundings (according to Newton's Law of Cooling) at a rate proportional to the temperature difference between S and its surrounding environment. The energy balance of heat flow across S together with Fourier's Law yields

$$z(x, u)u(x, t)+\sigma(x, u)\frac{\partial u(x, t)}{\partial n} = g(x); \qquad x\in S,\ t > 0,$$

where $z \geqslant 0$, $\sigma \geqslant 0$ and $\sigma+z > 0$.

We wish to find a steady state solution (temperature distribution) $y = y(x)$. It will satisfy

$$\Delta y = -\frac{1}{k}\{\vec{\nabla} k\cdot\vec{\nabla} y+h\}, \qquad x\in V,$$

$$z(x, y)y(x)+\sigma(x, y)\frac{\partial u(x)}{\partial n} = g(x), \qquad x\in S.$$

Now if V is a rod of unit length and insulated laternal surfaces then the steady state problem is

$$y'' = -\frac{1}{k(x, y)}[k_x(x, y)y'+k_y(x, y)(y')^2+h(x, y)],$$

$$z(0, y(0))y(0)-\sigma(0, y(0))y'(0) = g(0),$$

$$z(1, y(1))y(1)+\sigma(1, y(1))y'(1) = g(1).$$

We will assume the case where z, σ are independent of temperature and set $\alpha = z(0) > 0$, $\beta = \sigma(0) > 0$, $a = z(1) > 0$, $b = \sigma(1) > 0$, $r = g(0)$ and $s = g(1)$.

So our problem reduces to

$$y'' = -\frac{1}{k(x, y)}\{k_x y'+k_y(y')^2+h(x, y)\} \equiv f(x, y, y'),$$

(4.2)
$$\alpha y(0)-\beta y'(0) = r; \qquad \alpha, \beta > 0,$$

$$ay(1)+by'(1) = s; \qquad a, b > 0.$$

Now we make the following assumptions on h and k:

(4.3) $k_x(x, y)$, $k_y(x, y)$ are continuous for $(x, y)\in[0, 1]\times\boldsymbol{R}$. Also suppose for $(x, y)\in[0, 1]\times\boldsymbol{R}$, $k(x, y)$ is continuous and $k(x, y) \geqslant m > 0$, where m is a constant.

(4.4) Suppose $h(x, y)$ is bounded for bounded (x, y) sets. Suppose also h satisfies the Carathéordory Conditions.

(4.5) $$yh(x, y) < 0 \quad \text{for large} \quad |y|.$$

(4.6) $\dfrac{y(x)}{k(x, y)}h(x, y)$ is lower semicontinuous on $[0, 1]\times\boldsymbol{R}$.

The assumption (4.5) that $yh(x, y) < 0$ for large $|y|$ means that the internal heat generation $h(x, y)$ opposes large temperature extremes, i.e., if $y > 0$ and $|y|$ large, then $h(x, y) < 0$ so heat is removed from the rod by internal sources and the temperature tends to drop.

Now assumptions (4.3)–(4.6) together with Theorem 4.1 implies that (4.2) has at least one solution in $H^2(0, 1)$.

Acknowledgements. We would like to thank Professor Andrzej Granas and John W. Lee for their invaluable assistance during the research and writing of this paper.

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Reçu par la Rédaction le 1986.04.14