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On the radius of starlikeness and convexity of certain classes of analytic functions

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Abstract. Let $\mathscr{A}_n$ denote the class of normalized functions $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ regular in the unit disc. For any natural number n and for real constants A, B satisfying $-1 \leqslant A < B \leqslant 1$, let $P_n(A, B) = \{p/p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots,$ regular in E and subordinate to $(1+Az)/(1+Bz)$ in $E\}$. We consider the following three classes:

$$U_n(A, B) = \{f/f \in \mathscr{A}_n \text{ and } f(z)/z \in P_n(A, B)\},$$

$$V_n(A, B) = \{f/f \in \mathscr{A}_n \text{ and } f'(z) \in P_n(A, B)\},$$

$$W_n(A, B) = \{f/f \in \mathscr{A}_n \text{ and } zf'(z)/f(z) \in P_n(A, B)\}.$$

In this paper we obtain the radius of starlikeness of the family $U_n(A, B)$ and the radii of convexity of $V_n(A, B)$ and $W_n(A, B)$. We also obtain some distortion theorems and coefficient estimates.

The results obtained generalize those of Karunakaran [2], Livingston [3], Padmanabhan [5] an Nikolaeva and Repnina [4].

1. Introduction. Let $\mathscr{A}_n = \{f/f(z)$ be regular in the unit disc E and have the Taylor series about the origin given by $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$ for z in $E\}$.

Let $\mathscr{B} = \{w/w(z)$ be regular in E, $w(0) = 0 =$ and $|w(z)| < 1$ for z in $E\}$. For any natural number n and real constants A, B satisfying $-1 \leqslant A < B \leqslant 1$, let $P_n(A, B) = \{p/p(z) = 1 + \sum_{k=n}^{\infty} p_k z^k$ be regular in E and be subordinate to $(1+Az)/(1+Bz)$ for z in $E\}$.

The class $P_n(A, B)$ has been introduced by Stankiewicz and Waniurski [7], who took A and B as fixed complex numbers satisfying $|A| \leqslant 1$, $|B| \leqslant 1$. They showed that $p \in P_n(A, B)$ implies that there exists $w \in \mathscr{B}$, $w(z) = \sum_{k=n}^{\infty} w_k z^k$ such that $p(z) = (1 + Aw(z))/(1 + Bw(z))$ for z in E. We note that $P_1(A, B) = P(A, B)$, the class introduced by Janowski [1].

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Now we introduce three classes $U_n(A, B)$, $V_n(A, B)$ and $W_n(A, B)$.

DEFINITION. For any natural number n and $-1 \leqslant A < B \leqslant 1$, let $U_n(A, B) = \{f/f \in \mathscr{A}_n$ and $f(z)/z \in P_n(A, B)\}$, let $V_n(A, B) = \{f/f \in \mathscr{A}_n$ and $f'(z) \in P_n(A, B)\}$, let $W_n(A, B) = \{f/f \in \mathscr{A}_n$ and $zf'(z)/f(z) \in P_n(A, B)\}$.

In this paper we obtain the radius of starlikeness of the family $U_n(A, B)$ and the radii of convexity of $V_n(A, B)$ and $W_n(A, B)$. We also obtain some distortion theorems and coefficient estimates. The results obtained generalize those of Karunakaran [2], Livingston [3], Padmanabhan [5] and Nikolaeva and Repnina [4].

2. Lemmas.

LEMMA 2.1. *Generalized Dieudonné's lemma: If* $w(z) = w_n z^n + w_{n+1} z^{n+1} + \ldots$, *then for* $|z| < 1$,

$$|zw'(z) - nw(z)| \leqslant \frac{|z|^{2n} - |w|^2}{|z|^{n-1}(1 - |z|^2)}.$$

Proof. Write $w(z) = z^n \Phi(z)$, where $\Phi(z)$ is regular in E and $|\Phi(z)| < 1$ therein. By the well-known theorem of Carathéodory,

$$|\Phi'(z)| \leqslant \frac{1 - |\Phi(z)|^2}{1 - |z|^2}, \qquad z \in E,$$

and the assertion follows. The lemma is sharp for the choice

$$w(z) = z^n \frac{(z - q)}{(1 - qz)}, \qquad |q| \leqslant 1.$$

Especially note that $w(z) = z^n$ and $w(z) = -z^n$ give sharp results.

LEMMA 2.2. *For all* $x > 0$ *and any natural number* n,

$$1 + x^2 + x^4 + \ldots + x^{2n-2} - nx^{n-1} \geqslant 0.$$

Proof. It easily follows from induction on n.

LEMMA 2.3. *If* $p(z) \in P_n(A, B)$, $\alpha \geqslant 0$, $\beta \geqslant 0$, *then on* $|z| = r < 1$, *we have*

$$\operatorname{Re}\left[\alpha p(z) + \beta z \frac{p'(z)}{p(z)}\right]$$

$$\geqslant \frac{\alpha + [2\alpha A - n\beta(B - A)]r^n + A^2 r^{2n}}{(1 + Ar^n)(1 + Br^n)} \qquad \text{if } R_1 \leqslant R_2,$$

$$\geqslant -n\beta \frac{(A + B)}{(B - A)} + \frac{2}{(B - A)r^{n-1}(1 - r^2)}(L_1 K_1)^{1/2} - (1 - ABr^{2n}) \qquad \text{if } R_1 \geqslant R_2,$$

where

$$R_1 = (L_1/K_1)^{1/2}, \qquad R_2 = (1 + Ar^n)/(1 + Br^n),$$

$$L_1 = nAr^{n-1}(1 - r^2) + 1 - A^2 r^{2n},$$

$$K_1 = (B - A) + nBr^{n-1}(1 - r^2) + 1 - B^2 r^{2n}.$$

The bounds are sharp.

Proof. This is proved by the author in [6].

LEMMA 2.4. *If* $p \in P_n(A, B)$, *on* $|z| = r < 1$, *then*

$$(1+Ar^n)/(1+Br^n) \leqslant \operatorname{re} p(z) \leqslant (1-Ar^n)/(1-Br^n).$$

The bounds are sharp.

Proof. See [7].

LEMMA 2.5. *If* $p \in P_n(A, B)$ *and* $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots$, *then*

(i) $$|p_k| \leqslant B - A, \quad k = n, n+1, \ldots;$$

(ii) $$\sum_{k=n}^{N} |p_k|^2 \leqslant (B-A)^2 \frac{(1-B^{2N})}{(1-B^2)}, \quad N = n, n+1, \ldots,$$

the estimate being sharp only for $n = 1$;

(iii) $$\sum_{k=n}^{\infty} |p_k|^2 \leqslant \frac{(B-A)^2}{(1-B^2)},$$

the estimate being sharp for every natural number n.

Proof. Refer to [7].

3. Theorems.

THEOREM 3.1. *If* $f \in \mathscr{A}_n$ *belongs to* $U_n(A, B)$, *then*

$$\operatorname{Re} f'(z) \geqslant \begin{cases} L(a+d) & \text{for } R_1 > a+d, \\ L(R_1) & \text{for } a-d \leqslant R_1 \leqslant a+d, \\ L(a-d) & \text{for } a-d > R_1, \end{cases}$$

where

$$L(R) = \frac{nA}{B-A} + \frac{(1-A^2 r^{2n})}{r^{n-1}(1-r^2)(B-A)} - \frac{R\,[\{(n-1)B+(n+1)A\}\, r^{n-1}(1-r^2) + 2(1-ABr^{2n})]}{(B-A)\, r^{n-1}(1-r^2)} + \frac{R^2\,[nBr^{n-1}(1-r^2)+1-B^2 r^{2n}]}{(B-A)\, r^{n-1}(1-r^2)},$$

$$R_1 = \frac{[(n-1)B+(n+1)A]\, r^{n-1}(1-r^2) + 2(1-ABr^{2n})}{2\,[nBr^{n-1}(1-r^2)+1-B^2 r^{2n}]},$$

$$a = \frac{1-ABr^{2n}}{1-B^2 r^{2n}}, \quad d = \frac{(B-A)\, r^n}{(1-B^2 r^{2n})}.$$

The estimates are sharp.

Proof.

$$\frac{f(z)}{z}=\frac{1+Aw(z)}{1+Bw(z)}, \quad w(z)\in\mathscr{B}.$$

Hence

$$\frac{zf'(z)}{f(z)}=1-\frac{(B-A)zw'(z)}{(1+Aw(z))(1+Bw(z))},$$

(1)
$$f'(z)=\frac{1+Aw(z)}{1+Bw(z)}-\frac{(B-A)zw'(z)}{[1+Bw(z)]^2}.$$

Since

$$p(z)=\frac{f(z)}{z}\prec\frac{1+Az}{1+Bz} \quad \text{in } E,$$

we have $|z|\leqslant r$ transformed by $p(z)$ to the disc

$$|p(z)-a|\leqslant d, \quad a=\frac{1-ABr^{2n}}{1-B^2r^{2n}}, \quad d=\frac{(B-A)r^n}{1-B^2r^{2n}}.$$

By using Lemma 2.1, we have

(2)
$$\operatorname{Re}\frac{zw'(z)}{[1+Bw(z)]^2}\leqslant n\operatorname{Re}\frac{w(z)}{[1+Bw(z)]^2}+\frac{r^{2n}-|w(z)|^2}{r^{n-1}(1-r^2)|1+Bw(z)|^2},$$

$$p(z)=\frac{1+Aw(z)}{1+Bw(z)} \quad \text{yields} \quad w(z)=\frac{1-p(z)}{Bp(z)-A}$$

and

$$\frac{w(z)}{[1+Bw(z)]^2}=\frac{[1-p(z)][Bp(z)-A]}{(B-A)^2}.$$

Substituting in (1) and (2) gives

$$\operatorname{Re} f'(z)$$

$$\geqslant \operatorname{Re} p(z)-(B-A)\left[n\operatorname{Re}\frac{(1-p(z))(Bp(z)-A)}{(B-A)^2}+\frac{r^{2n}|Bp(z)-A|^2-|1-p(z)|^2}{r^{n-1}(1-r^2)(B-A)^2}\right]$$

$$=\frac{nA}{(B-A)}+\frac{1}{(B-A)}\left[-\{(n-1)B+(n+1)A\}\operatorname{Re} p(z)+nB\operatorname{Re}[p(z)]^2-\right.$$

$$\left.-\frac{r^{2n}|Bp(z)-A|^2-|1-p(z)|^2}{r^{n-1}(1-r^2)}\right].$$

Putting $p(z) = a+u+iv$, $R = |p(z)|$ and calling the right-hand side $S(u, v)$, we have

$$\text{(3)}\quad S(u, v) = \frac{nA}{(B-A)}+\frac{1}{(B-A)}\times$$
$$\times\Bigg[-\{(n-1)B+(n+1)A\}(a+u)+nB\{(a+u)^2-v^2\}-$$
$$-\frac{(1-B^2r^{2n})}{r^{n-1}(1-r^2)}(d^2-u^2-v^2)\Bigg],$$

$$\text{(4)}\qquad \frac{\partial S}{\partial v} = \frac{2v[(1-B^2r^{2n})-nBr^{n-1}(1-r^2)]}{(B-A)r^{n-1}(1-r^2)}.$$

When B is equal to $-ve$, the expression in the brackets on right-hand side is certainly positive, but when B is positive, it cannot be taken for granted as n may be large, We now prove that it is positive, even when B is positive.

CLAIM 1. *For* $-1 < B \leqslant 1$, $1-B^2r^{2n} \geqslant B(1-r^{2n})$.

Proof. This is equivalent to showing that $(1-B)(1-Br^{2n}) \geqslant 0$, which is true.

Applying Claim 1 to the expression in the brackets on right-hand side of (4) we obtain

$$1-B^2r^{2n}-nBr^{n-1}(1-r^2) > B(1-r^{2n})-nBr^{n-1}(1-r^2)$$
$$= B(1-r^2)(1+r^2+r^4+\ldots+r^{2n-2}-nr^{n-1}) = +ve,$$

since B is equal to $+ve$ and because of Lemma 2.2. Thus the minimum of $S(u, v)$ is obtained at $v = 0$. Putting $v = 0$, we infer

$$\text{(5)}\quad S(u, 0) = L(R) = \frac{nA}{(B-A)}+\frac{1}{(B-A)}\Bigg[-R\{(n-1)B+(n+1)A\}+$$
$$+nBR^2-\frac{1-B^2r^{2n}}{r^{n-1}(1-r^2)}\{d^2-(R-a)^2\}\Bigg]$$
$$= \frac{nA}{(B-A)}-\frac{[B(n-1)+(n+1)A]}{(B-A)}R+\frac{nB}{(B-A)}R^2+$$
$$+\frac{(1-A^2r^{2n})}{r^{n-1}(1-r^2)(B-A)}+\frac{(1-B^2r^{2n})R^2}{r^{n-1}(1-r^2)(B-A)}-\frac{2(1-ABr^{2n})R}{r^{n-1}(1-r^2)(B-A)},$$

$$L'(R) = 2R\frac{[nBr^{n-1}(1-r^2)+1-B^2r^{2n}]}{r^{n-1}(1-r^2)(B-A)}-$$
$$-\frac{[\{(n-1)B+(n+1)A\}r^{n-1}(1-r^2)+2(1-ABr^{2n})]}{r^{n-1}(1-r^2)(B-A)}.$$

This vanishes at $R = R_1$ given by

$$R_1 = \frac{\{(n-1)B+(n+1)A\}r^{n-1}(1-r^2)+2(1-ABr^{2n})}{2[nBr^{n-1}(1-r^2)+1-B^2r^{2n}]}.$$

The absolute minimum of $L(R)$ is attained at R_1 only if $a-d \leqslant R_1 \leqslant a+d$ and then it is $L(R_1)$. When $R_1 < a-d$, $L(R)$ increases, and hence the minimum is attained at $a-d$ and it is $L(a-d)$. When $R_1 > a+d$, $L(R)$ decreases, and hence the minimum is attained at $a+d$ and it is $L(a+d)$. Equality of the estimates is achieved

(i) if $R_1 > a+d$ for $f_1(z) = \dfrac{z(1-Az^n)}{(1-Bz^n)}$ at $z = r$,

(ii) if $a-d \leqslant R_1 \leqslant a+d$ for $f_2(z) = \dfrac{z[1+Aw_1(z)]}{[1+Bw_1(z)]}$

when $w_1(z) = z^n\dfrac{(z-q)}{(1-qz)}$, q determined from $\operatorname{Re}\left\{\dfrac{1+Rw_1(z)}{1+Bw_1(z)}\right\}_{z=r} = R_1$,

(iii) if $a-d > R_1$ for $f_3(z) = z\dfrac{(1+Az^n)}{(1+Bz^n)}$ at $z = r$.

Corollary. *Putting $n = 1$, we obtain the theorem of Karunakaran* [2].

Theorem 3.2. *If $f(z) \in \mathscr{A}_n$ belongs to $U_n(A, B)$, then*

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geqslant 1-\frac{(B-A)nr^n}{(1+Ar^n)(1+Br^n)}, \qquad \text{if } R_1 \leqslant R_2,$$

$$\geqslant 1-n\frac{(A+B)}{(B-A)}+\frac{2[(L_1K_1)^{1/2}-(1-ABr^{2n})]}{(B-A)r^{n-1}(1-r^2)}, \qquad \text{if } R_1 \geqslant R_2,$$

where R_1 and R_2 are given by Lemma 2.3. The bounds are sharp.

Proof.

$$p(z) = \frac{f(z)}{z} \quad \text{yields} \quad \frac{zp'(z)}{p(z)} = \frac{zf'(z)}{f(z)}-1.$$

Putting $\alpha = 0$, $\beta = 1$ in Lemma 2.3, we obtain the theorem. The bounds are sharp.

Theorem 3.3. *If $f \in \mathscr{A}_n$ belongs to $U_n(A, B)$, the radius of starlikeness of f is given by the least positive root of equations* (i) *or* (ii), *according as $R_1 \leqslant R_2$ or $R_1 \geqslant R_2$, the equations being*

(i) $$ABr^{2n}+[A+B-n(B-A)]r^n+1 = 0,$$

(ii) $$4ABr^{2n+2}-4ABr^{2n}-[(1+n^2)(B-A)-2n(B+A)]r^{n+3}+$$
$$+2[(n^2-1)(B-A)-2n(B+A)]r^{n+1}-[(1+n^2)(B-A)-2n(B+A)]r^{n-1}-$$
$$-4r^2+4=0.$$

The results are sharp.

Proof. It follows from the previous theorem and the estimates are clearly sharp.

Remark. The theorem above is interesting for the following reason. If $f\in U_n(A, B)$ and if $g\in \mathcal{A}_n$ is defined by $g(z)=\lambda z+(1-\lambda)f(z)$ for some λ in $(0, 1]$, then $g(z)\in U_n(\lambda B+(1-\lambda)A, B)$. The radius of starlikeness of such families also follows from the theorem.

THEOREM 3.4. *If $f\in \mathcal{A}_n$ belongs to $U_n(A, B)$ we have on $|z|=r<1$:*

(i) $r\dfrac{(1+Ar^n)}{(1+Br^n)}\leqslant |f(z)|\leqslant r\dfrac{(1-Ar^n)}{(1-Br^n)}$. *The bounds are sharp.*

(ii) *If* $f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k$, *then:*

(a) $|a_k|\leqslant (B-A)$, $k=n+1, n+2, \ldots$ *The result is sharp.*

(b) *For* $-1<B<1$,

(i′) $$\sum_{k=n+1}^{N+1} |a_k|^2\leqslant (B-A)^2(1-B^{2N})/(1-B^2),\qquad N=n,\ n+1, \ldots,$$

(ii′) $$\sum_{k=n+1}^{\infty} |a_k|^2\leqslant (B-A)^2/(1-B^2).$$

The result in (a) *is sharp for $n=1$ only and the second result is sharp for every natural number n. The extremal function is of the form*

$$f(z)=z\frac{(1+AKz^n)}{(1+BKz^n)},\qquad |K|=1.$$

Proof. (i) follows by putting $p(z)=f(z)/z$ and using Lemma 2.4. Sharpness is attained for $f(z)=z(1+Az^n)/(1+Bz^n)$ at $z=r$ for left-hand side, etc.

(ii) $$\frac{f(z)}{z}=1+a_{n+1}z^n+a_{n+2}z^{n+1}+\ldots,$$
$$\frac{f(z)}{z}=p(z)=1+p_n z^n+p_{n+1}z^{n+1}+\ldots$$

Result (a) follows from Lemma 2.5 (i). We have sharpness for $f(z)=z(1+Az^n)/(1+Bz^n)$.

(b) follows from Lemma 2.5 (ii) and (iii).

Remark. Stankiewicz and Waniurski in [7] have extensively dealt with the class $V_n(A, B)$, and reference can be had from them. We only remark that the radius of convexity of the family $V_n(A, B)$ is the same as the radius of starlikeness of the family $U_n(A, B)$, because f is convex iff zf' is starlike. Thus the radius of convexity of the family $V_n(A, B)$ is given by Theorem 3.3.

THEOREM 3.5. *If* $f \in \mathscr{A}_n$ *belongs to* $W_n(A, B)$, *then*

(i) $$\frac{r}{(1+Br^n)^{(B-A)/nB}} \leqslant |f(z)| \leqslant \frac{r}{(1-Br^n)^{(B-A)/nB}} \quad \textit{if } B \neq 0,$$

(ii) $$r \exp\frac{Ar^n}{n} \leqslant |f(z)| \leqslant r \exp\frac{-Ar^n}{n} \quad \textit{if } B = 0.$$

The result is sharp.

Proof.

$$f(z) \in W_n(A, B) \quad \text{implies} \quad \frac{zf'(z)}{f(z)} = \frac{1+Az^n \Phi(z)}{1+Bz^n \Phi(z)},$$

where Φ is regular in E, $\Phi(0) = 0$ and $|\Phi(z)| < 1$ in E; thus

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = \frac{-(B-A)z^{n-1}\Phi(z)}{1+Bz^n\Phi(z)}.$$

We have, on integrating from 0 to z on both sides,

$$\log\left|\frac{f(z)}{z}\right| = \text{Re} \log\frac{f(z)}{z} = \text{Re} \int_0^z \left\{\frac{f'(t)}{f(t)} - \frac{1}{t}\right\} dt$$

$$= Re \int_0^{|z|} \frac{-(B-A)t^{n-1} e^{i(n-1)\theta} \Phi(te^{i\theta}) e^{i\theta} dt}{1+Bt^n e^{in\theta} \Phi(te^{i\theta})}$$

$$\leqslant \int_0^{|z|} \frac{(B-A)|\Phi(te^{i\theta})| t^{n-1} dt}{|1+Bt^n e^{in\theta} \Phi(te^{i\theta})|}$$

$$\leqslant (B-A) \int_0^r \frac{t^{n-1} dt}{(1-Bt^n)} = -\frac{(B-A)}{nB} \log(1-Br^n) \quad \text{if } B \neq 0$$

$$= -A \int_0^r t^{n-1} dt = \frac{-A}{n} r^n \quad \text{if } B = 0.$$

This gives the right-hand side of the inequality. We also have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} \geqslant \frac{1+Ar^n}{1+Br^n} \quad \text{on } |z| = r < 1.$$

$$r \operatorname{Re}\frac{\partial}{\partial r}\left\{\log\frac{f(z)}{z}\right\} = \operatorname{Re}\left\{\frac{zf'(z)}{f(z)}-1\right\} \geqslant \frac{(1+Ar^n)}{(1+Br^n)}-1 = \frac{-(B-A)r^n}{(1+Br^n)}.$$

Moreover,

$$\log\left|\frac{f(z)}{z}\right| - \operatorname{Re}\log\frac{f(z)}{z} \geqslant \int_0^r -(B-A)\frac{t^{n-1}}{(1+Bt^n)}\,dt$$

$$= \frac{-(B-A)}{nB}\log(1+Br^n) \quad \text{if } B \neq 0.$$

If $B = 0$, $\log|f(z)/z| \geqslant A\int_0^r t^{n-1}\,dt = Ar^n/n$. This settles the left-hand side of the inequality. Sharpness can be indicated easily.

THEOREM 3.6. *If $f \in W_n(A, B)$, then on $|z| = r < 1$ we have*

$$\frac{1+Ar^n}{1+Br^n} \leqslant \operatorname{Re}\frac{zf'(z)}{f(z)} \leqslant \left|\frac{zf'(z)}{f(z)}\right| \leqslant \frac{1-Ar^n}{1-Br^n}.$$

The result is sharp.

Proof. Put $p(z) = zf'(z)/f(z)$ and apply Lemma 2.4.

Remark. The radius of convexity of the class $W_n(A, B)$ will be obtained from the next section which deals with a more general class.

DEFINITION 3.1. For each $\lambda \in [0, 1]$, define the family

$Q_n(\lambda, A, B) = \{g(z) = \lambda f(z) + (1-\lambda)zf'(z)$ where $f \in \mathscr{A}_n$ belongs to the family $W_n(A, B)\}$.

THEOREM 3.7. *Let $g(z) \in Q_n(\lambda, A, B)$ and $|z| = r < 1$. Then*

$$\operatorname{Re}\frac{zg'(z)}{g(z)} \geqslant (1-\alpha) + \frac{\alpha + [2\alpha D - n(B-D)]r^n + \alpha D^2 r^{2n}}{(1+Dr^n)(1+Br^n)} \qquad \textit{if } R_1 \leqslant R_2,$$

$$\geqslant (1-\alpha) - n\frac{(D+B)}{(B-D)} + \frac{2[(L_1K_1)^{1/2} - (1-BDr^{2n})]}{(B-D)r^{n-1}(1-r^2)} \qquad \textit{if } R_1 \geqslant R_2,$$

where

$$\alpha = 1/(1-\lambda), \quad D = \lambda B + (1-\lambda)A, \quad R_1 = [L_1/K_1]^{1/2},$$

$$R_2 = \frac{1+Dr^n}{1+Br^n}, \quad L_1 = nDr^{n-1}(1-r^2)+1-D^2r^{2n},$$

$$K_1 = [\alpha(B-D)+nB]\,r^{n-1}(1-r^2)+(1-B^2r^{2n}).$$

The estimates are sharp.

Proof. We have $g(z) = \lambda f(z)+(1-\lambda)zf'(z)$, where

$$\frac{zf'(z)}{f(z)} = \frac{1+Aw(z)}{1+Bw(z)}, \quad w(z) \in \mathscr{B},$$

$$\frac{zg'(z)}{g(z)} = \frac{zf'(z)}{f(z)} \cdot \frac{1+(1-\lambda)\cdot[zf''(z)/f'(z)]}{\lambda+(1-\lambda)\cdot[zf'(z)/f(z)]}$$

$$= \frac{1+Aw(z)}{1+Bw(z)} - \frac{(1-\lambda)(B-A)zw'(z)}{[1+Bw(z)][1+Dw(z)]}$$

$$= -\frac{\lambda}{1-\lambda}+\frac{1}{(1-\lambda)}p(z)-\frac{(B-D)zw'(z)}{[1+Bw(z)][1+Dw(z)]},$$

where $p(z) = [1+Dw(z)]/[1+Bw(z)]$. Thus

$$\frac{zg'(z)}{g(z)} = (1-\alpha)+\alpha p(z)+\frac{zp'(z)}{p(z)}.$$

Putting $\beta = 1$, keeping α and changing A to D in Lemma 2.3, we obtain the result. Sharpness is easy to check and follows from the same lemma.

Remark. Putting $\lambda = 0$, the radius of convexity of the class $W_n(A, B)$ is obtained. In this, if we put $n = 1$, several results due to many authors will be derived at once. Putting $n = 1$, $\lambda = \frac{1}{2}$ gives special results due to Livingston [3], Padmanabhan [5], Nikolaeva and Repnina [4] etc.

THEOREM 3.8. *The radius of convexity of the class $W_n(A, B)$ is given by the smallest positive root of the following equations according to $R_1 \leqslant R_2$ or $R_1 \geqslant R_2$:*

(i) $$1+[2A-n(B-A)]\,r^n+A^2r^{2n} = 0,$$

(ii) $$4A^2r^{2n+2}-4A^2r^{2n}+n[4A-n(B-A)]\,r^{n+3}+$$
$$+2[(n^2-2)(B-A)-4nA]\,r^{n+1}+n[4A-n(B-A)]\,r^{n-1}-4r^2+4 = 0.$$

Proof. Put $\lambda = 0$ in Theorem 3.7 and simplify. Sharpness is immediate.

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