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Some remarks on the uniqueness of solutions of the Darboux problem with conditions of the Krasnosielski-Krein type

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The present paper deals with some sufficient conditions of uniqueness for the solutions of the Darboux problem for equations of the form

$$\frac{\partial^2 u}{\partial x \partial y} = f(x, y, u) \tag{1}$$

and

$$\frac{\partial^3 u}{\partial x \partial y \partial z} = f(x, y, z, u) . \tag{2}$$

The criteria given below for equations of forms (1) and (2) are strictly connected with the results obtained by Krasnosielski and Krein [1], W. Walter [3] and M. Kwapisz, B. Palczewski, W. Pawelski [2].

In paper [1] it was shown for the ordinary differential equation

$$y' = f(t, y) \tag{$*$}$$

with initial condition

$$y(t_0) = y_0 ,$$

where the function $f(t, y)$ is defined and continuous on the rectangle $R = \{(t, y)\colon |t-t_0| \leqslant a,\ |y-y_0| \leqslant b\}$, that the supplementary conditions

$$\begin{cases} |f(t, y)-f(t, \bar{y})| \leqslant k|t-t_0|^{-1}|y-\bar{y}| , \\ |f(t, y)-f(t, \bar{y})| \leqslant A|y-\bar{y}|^{a} \end{cases} \tag{K-K}$$

guarantee the uniqueness of the solutions of the initial value problem for equation ($*$) where k, a, A are constants satisfying the conditions $k > 0$, $A > 0$, $0 < a < 1$, $k(1-a) < 1$. We will state now that the conditions analogical to (K-K) may be put for equations (1) and (2).

Let D denote the rectangle: $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$ $(a, b > 0)$ and let $f(x, y, u)$ be a function defined and continuous on the set $E = D \times \{-\infty < u < +\infty\}$. Then every solution $u(x, y)$ of the partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = f(x, y, u) \tag{3}$$

satisfying the conditions

(I) $$u(x,0)=\sigma(x)\,,\qquad u(0,y)=\tau(y)\,,\qquad \sigma(0)=\tau(0)=u_0\,,$$

(4) $$u(x,y)=\sigma(x)+\tau(y)-u_0+\int\limits_0^x\int\limits_0^y f\big(t,\tau,u(t,\tau)\big)\,dt\,d\tau$$

and conversely.

Let $\sigma(x)$ and $\tau(y)$ be the functions of class C^1 defined respectively on $\langle 0,a\rangle$ and $\langle 0,b\rangle$. Now we have for equation (3) the following

THEOREM 1. *If $f(x,y,u)$ is defined, continuous and bounded on E and satisfies the conditions*

(5) $$|f(x,y,u)-f(x,y,\bar u)|\leqslant \frac{k}{xy}|u-\bar u|\,,$$
$$|f(x,y,u)-f(x,y,\bar u)|\leqslant C|u-\bar u|^a\,,\qquad 0<a<1\,,$$

where

(6) $$k(1-a)^2<1\,,$$

then there exists at most one solution $u(x,y)$ of the Darboux problem for equation (3).

Proof. Indeed, putting $M=\sup\limits_E|f(x,y,u)|$ and denoting by $u(x,y)$ and $\bar u(x,y)$ two solutions of the Darboux problem fulfilling conditions (I), we obtain from (4)

$$|u(x,y)-\bar u(x,y)|\leqslant 2Mxy\,,\qquad (x,y)\in D$$

and from (5)

$$|u(x,y)-\bar u(x,y)|\leqslant\int\limits_0^x\int\limits_0^y\Big|f\big(t,\tau,u(t,\tau)\big)-f\big(t,\tau,\bar u(t,\tau)\big)\Big|\,dt\,d\tau$$
$$\leqslant\int\limits_0^x\int\limits_0^y C|u(t,\tau)-\bar u(t,\tau)|^a\,dt\,d\tau\leqslant C(2M)^a(xy)^{1+a}$$

and in general

$$|u(x,y)-\bar u(x,y)|\leqslant C^{1+a+\ldots+a^m}(2M)^{a^{m+1}}(xy)^{1+a+\ldots+a^{m+1}}$$
$$\text{for}\quad m=1,2,\ldots$$

Hence we have the following estimation

(7) $$|u(x,y)-\bar u(x,y)|\leqslant C^{1/(1-a)}(xy)^{1/(1-a)}\,.$$

We put

$$Q(x,y)=(xy)^{-\sqrt{k}}|u(x,y)-\bar u(x,y)|\qquad\text{for}\quad xy>0\,.$$

Then it follows from (7) that

$$0\leqslant Q(x,y)=Q(s)\leqslant C^{\frac{1}{1-a}}(xy)^{\frac{1-\sqrt{k}(1-a)}{1-a}}\,,$$

and hence, by the fact that $\sqrt{k}(1-\alpha) < 1$, we have $\lim_{D \ni s \to s_0 \in \Gamma} Q(s) = 0$, where Γ is a line composed of two segments: $\langle 0, a \rangle$ on x-axis and $\langle 0, b \rangle$ on y-axis.

Basing ourselves on this fact we can put $Q(s) = 0$ for $s \in \Gamma$; hence we infer that the function $Q(s)$ is continuous on D. We shall show that $Q(s) = 0$ for $s \in D$. Suppose that it is not true. Then there exists a point $\bar{s} = (\bar{x}, \bar{y}) \in D/\Gamma$ such that

$$0 < r = Q(\bar{s}) = \sup_{s \in D} Q(s) . \tag{8}$$

On the other hand, basing ourselves on the first part of condition (5) we have successively

$$\begin{aligned} r = Q(\bar{s}) &\leqslant (\bar{x}\bar{y})^{-\sqrt{k}} \int_0^{\bar{x}}\int_0^{\bar{y}} \left|f\big(t, \tau, u(t, \tau)\big) - f\big(t, \tau, \bar{u}(t, \tau)\big)\right| dt\, d\tau \\ &\leqslant (\bar{x}\bar{y})^{-\sqrt{k}} \int_0^{\bar{x}}\int_0^{\bar{y}} k t^{\sqrt{k}-1} \tau^{\sqrt{k}-1} Q(t, \tau)\, dt\, d\tau \\ &< r(\bar{x}\bar{y})^{-\sqrt{k}} \int_0^{\bar{x}} \sqrt{k} t^{\sqrt{k}-1} dt \int_0^{\bar{y}} \sqrt{k} t^{\sqrt{k}-1} dt = r , \end{aligned}$$

which contradicts (8).

Remark 1. We are very much obliged to Mr. J. Kisyński for his remarks concerning theorem 1, and for reporting Mr. C. Kluczny's opinion in [5] that in the case of ordinary differential equations the Krasnosielski-Krein theorem on the base of the minimum of the two comparative functions appearing there is reduced to Kamke's uniqueness theorem.

Theorem 1 of this paper cannot be reduced in the same manner to W. Walter's theorem ([3], p. 312, theorem 4), but the question remains open as to its resulting from W. Walter's theorem ([4], p. 201, theorem 7).

Remark 2. Now let V denote the set: $0 \leqslant x_i \leqslant a_i$, $a_i > 0$, $i = 1, 2, 3$ and let $f(x_1, x_2, x_3, u) = f(\xi, u)$ be a defined and continuous function in the domain $W = V \times \{-\infty < u < +\infty\}$. Then every solution $u(\xi)$ of equation

$$\frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = f(\xi, u) \tag{9}$$

fulfilling conditions

$$\begin{gathered} u(0, x_2, x_3) = \varphi_1(x_2, x_3), \quad u(x_1, 0, x_3) = \varphi_2(x_1, x_3), \\ u(x_1, x_2, 0) = \varphi_3(x_1, x_2) \end{gathered} \tag{II}$$

has the form

$$u(\xi) = \psi_0(\xi) + \int_0^{x_1}\int_0^{x_2}\int_0^{x_3} f\big(t_1, t_2, t_3, u(t_1, t_2, t_3)\big)\, dt_1\, dt_2\, dt_3, \tag{10}$$

where the functions φ_i are defined respectively on the sets $\Pi_{jk} = \langle 0, a_j\rangle \times \times \langle 0, a_k\rangle$ and have there sufficient regularity. The functions fulfil also the usual conditions of compatibility (as in [2]). The function $\psi_0(\xi)$ is defined by φ_i as in [2]. And conversely—every continuous solution $u(\xi)$ of equation (10) fulfils equation (9) and conditions (II).

Now we have for the above case the following

THEOREM 2. *If $f(\xi, u)$ is defined, continuous and bounded in W and satisfies the inequalities*

$$|f(\xi, u)-f(\xi, \bar{u})| \leqslant \frac{k}{x_1x_2x_3}|u-\bar{u}|, \quad k>0, \quad x_1x_2x_3>0,$$

$$|f(\xi, u)-f(\xi, \bar{u})| \leqslant C|u-\bar{u}|^a, \quad C\text{—any constant} > 0, \quad 0<a<1,$$

where $k(1-a)^3<1$, then there exists at most one solution of the Darboux problem for equation (9).

The proof of this theorem is analogical to the proof of theorem 1.

From the above-mentioned theorem 2 one can see how the respective theorems for higher orders corresponding in type to equation (9) may be presented.

We can further generalize those theorems to systems of equations corresponding in type to equations (3) and (9), supposing that f and u are vector n-dimensional functions and that $|w|$ denotes the norm of such a vector in the form, for instance, of the sum of the absolute values of its components, just is shown in paper [5] with regard to equation (*).

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