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Asymptotic behaviour of harmonic polynomials bounded on a compact set

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1. Introduction. Trying to solve the Dirichlet problem by interpolating harmonic polynomials one has to solve the problem of choosing nodes suited to the interpolation. The problem was treated by J. L. Walsh [14] and mainly by J. H. Curtiss [1]-[4]. The latter showed that points of the Fekete type defined through an extremal property give good results. He also investigated the role of Fejér's points in the harmonic interpolation [4].

On the basis of known results it is evident that, just as in the case of approximation and interpolation by polynomials in z, also in the theory of interpolation and approximation by harmonic polynomials the Green function $G(z, E)$ of the unbounded component of the complement of a compact set E plays a very important role (cf. [12]-[15]). Therefore it seems natural that one should try to elucidate the mutual correlation between the Green function and interpolating harmonic polynomials. The problem has already been treated in [10].

This paper is intended as a supplement to paper [10]. When a review of [10] appeared in Mathematical Reviews 31 (3) (1966) # 2415 the author found what follows: 1° In accordance with the suspicion of the reviewer (J. H. Curtiss) theorems 5.1 and 5.3 in [10] cannot be proved by a straightforward application of the reasoning valid for polynomials in z. Moreover, it remains an unsolved problem whether these theorems are true or not; 2° Theorem 5.2 and the Corollary on p. 403 in [10] are false, as we shall show in § 3 of the present paper; 3° Theorem 6.1 in [10] is fortunately true and may be generalized to the case of a compact set E consisting of a finite number of mutually exterior closed simply connected domains. The generalization will be given by Theorem 1 in § 2.

In § 4 we discuss a sufficient condition on E under which Theorem 1 remains true. The condition concerns an asymptotic behaviour of harmonic polynomials in the vicinity of E and it is analogous to the "polynomial condition" of F. Leja ([5], [6], [7]) concerning polynomials in z.

2. Harmonic polynomials bounded in modulus by 1 on a compact set E and the Green function $G(z, E)$. Let $z^{(n)} = \{z_0, \ldots, z_{2n}\}$ be an arbitrary system of $2n+1$ points of the complex plane C and let $A(z^{(n)})$ denote the determinant whose jth row reads

$$[1, z_j, \ldots, z_j^n, \bar{z}_j, \ldots, \bar{z}_j^n] \quad (j = 0, \ldots, 2n).$$

Given any $z^{(n)}$ such that $A(z^{(n)}) \neq 0$, we put

$$H^{(j)}(z, z^{(n)}) = A(\{z_0, \ldots, z_{j-1}, z, z_{j+1}, \ldots, z_{2n}\})/A(z^{(n)}), \tag{1}$$

$j = 0, \ldots, 2n$. Then $H^{(j)}$ is a real harmonic polynomial of degree $\leqslant n$ such that $H^{(j)}(z_k, z^{(n)}) = \delta_{jk}$. If U_n is an arbitrary harmonic polynomial (real or complex) of degree $\leqslant n$, then the following interpolation formula holds [3]

$$U_n(z) = \sum_{j=0}^{2n} U_n(z_j) H^{(j)}(z, z^{(n)}), \quad z \in C. \tag{2}$$

Let E be a compact set in C. Let $q^{(n)} = \{q_0, \ldots, q_{2n}\}$ be an *n-th extremal system of E* with respect to the determinant A, i.e., the points of $q^{(n)}$ belong to E and

$$|A(q^{(n)})| \geqslant |A(z^{(n)})| \quad \text{for every } z^{(n)} \subset E. \tag{3}$$

We say that E is *unisolvent* if for every $n = 1, 2, \ldots$ there is a $z^{(n)} \subset E$ such that $A(z^{(n)}) \neq 0$. If E contains the boundary of a bounded domain, then E is unisolvent. If E is unisolvent, then $A(q^{(n)}) \neq 0$, where $q^{(n)}$ is an arbitrary nth extremal system of E with respect to A. In the sequel we shall consider only unisolvent compact sets E.

Denote by $H_n(z) = H_n(z, E)$, $H_n^{(k)}(z) = H_n^{(k)}(z, E)$, $k = 1, 2, 3, 4$, $n = 1, 2, \ldots$ the functions defined for every $z \in C$ by

$$H_n(z) = \sup |U_n(z)|, \tag{a$_0$}$$

the sup being taken over all harmonic polynomials U_n (complex or real) of degree $\leqslant n$ such that $\|U_n\|_E = \sup_{z \in E} |U_n(z)| \leqslant 1$;

$$H_n^{(1)}(z) = \max_{(j)} |H^{(j)}(z, q^{(n)})|, \tag{a$_1$}$$

$$H_n^{(2)}(z) = \sum_{j=0}^{2n} |H^{(j)}(z, q^{(n)})|, \tag{a$_2$}$$

$$H_n^{(3)}(z) = \inf_{z^{(n)} \subset E} [\max_{(j)} |H^{(j)}(z, z^{(n)})|], \tag{a$_3$}$$

$$H_n^{(4)}(z) = \inf_{z^{(n)} \subset E} \sum_{j=0}^{2n} |H^{(j)}(z, z^{(n)})|. \tag{a$_4$}$$

THEOREM 1. *If the compact set E consists of a finite number of mutually exterior closed simply connected domains, then the sequences*

$$\{\sqrt[n]{H_n(z)}\}\,,\quad \{\sqrt[n]{H_n^{(k)}(z)}\}\,,\quad k=1,2,3,4$$

are convergent at every point $z\in C$ to the same limit $F(z)=F(z,E)$, where $F(z)=\exp G(z)$ for $z\in C-E$, $F(z)=1$ for $z\in E$ and $G(z)=G(z,E)$ denotes the Green function of $C-E$ with the pole at infinity.

This theorem generalizes theorem 6.1 of [10], where E is assumed to be a closed simply connected domain (or equivalently, in virtue of the maximum principle, a boundary of a simply connected domain).

Proof. First we shall prove the inequalities

$$(*)\qquad H_n\leqslant(2n+1)H_n^{(1)}\leqslant(2n+1)H_n^{(2)}\leqslant(2n+1)^3H_n^{(3)}\leqslant(2n+1)^3H_n^{(4)}\leqslant(2n+1)^4H_n\,,$$

valid for every $z\in C$ and $n=1,2,\ldots$ They imply that a necessary and sufficient condition that all the sequences of Theorem 1 be convergent at a fixed point z is that at least one of the sequences be convergent at the point z.

The first inequality of $(*)$ follows from the interpolation formula

$$U_n(z)=\sum_{j=0}^{2n}U_n(q_j)H^{(j)}(z,q^{(n)})\,,$$

where U_n is an arbitrary harmonic polynomial of degree $\leqslant n$ such that $\|U_n\|_E\leqslant 1$ and $q^{(n)}$ denotes a fixed nth extremal system of E with respect to A. The second and the fourth inequalities are direct consequences of the definitions (a_1)-(a_4). To prove the third inequality it is enough to observe that by the interpolation formula

$$H^{(j)}(z,q^{(n)})=\sum_{k=0}^{2n}H^{(j)}(z_k,q^{(n)})H^{(k)}(z,z^{(n)})\,,\qquad z\in C\,,\ z^{(n)}\subset E\,,\ A(z^{(n)})\neq 0\,,$$

whence

$$|H^{(j)}(z,q^{(n)})|\leqslant(2n+1)H_n^{(3)}(z,E)\,,$$

because by the extremum property (3) of $q^{(n)}$ we have

$$(4)\qquad |H^{(j)}(z,q^{(n)})|\leqslant 1\quad\text{for}\quad z\in E\,.$$

The last inequality in $(*)$ follows from

$$H_n^{(4)}(z)\leqslant H_n^{(2)}(z)=\sum_{j=0}^{2n}|H^{(j)}(z,q^{(n)})|\leqslant(2n+1)H_n(z)\,.$$

We shall need the following two theorems, already known. Given a compact set E of positive transfinite diameter denote by $F(z) = F(z, E)$ the function defined for $z \in C$ by $F(z) = \exp G(z)$ for z in the unbounded component D of $C-E$ and $F(z) = 1$ for z in $C-D$. Let $E_\varrho = \{z \in C: F(z, E) = \varrho\}$.

THEOREM I ([12], [13]). *Let E satisfy the assumptions of Theorem 1. Let U be a function defined on E and let harmonic polynomials U_n of respective degrees $\leqslant n$ satisfy*

$$\limsup (\|U - U_n\|_E)^{1/n} \leqslant 1/\varrho < 1;$$

then

1° *the sequence $\{U_n\}$ converges uniformly on closed subsets of the interior of E_ϱ (consequently the function U can be extended harmonically from E to the interior of E_ϱ);*

2° $\limsup (\|U - U_n\|_{E_\sigma})^{1/n} \leqslant \sigma/\varrho$ $(1 < \sigma < \varrho)$.

THEOREM II ([9]). *If E is a compact set of positive transfinite diameter, then*

$$F(z, E) = \sup_{n\geqslant 1} \sqrt[n]{F_n(z, E)} = \lim \sqrt[n]{F_n(z, E)}, \qquad \text{where } F_n(z, E) = \sup |p_n(z)|,$$

the sup *being taken over all polynomials $p_n(z)$ in z of degree $\leqslant n$ such that $\|p_n\|_E \leqslant 1$.*

We are now ready to prove Theorem 1. Let $p_n(z)$ be an arbitrary polynomial in z of degree $\leqslant n$ such that $\|p_n\|_E \leqslant 1$. Since p_n is also a harmonic polynomial of dergee $\leqslant n$, we have

$$|p_n(z)| \leqslant H_n(z, E), \quad z \in C, \ n \geqslant 1,$$

whence by Theorem II

$$F(z) = \lim \sqrt[n]{F_n(z)} \leqslant \liminf \sqrt[n]{H_n(z)}, \quad z \in C. \tag{5}$$

To end the proof we have to show that $\limsup \sqrt[n]{H_n(z)} \leqslant F(z)$, $z \in C$. To this aim let $1 < \sigma < \varrho$, σ, ϱ being arbitrary fixed numbers. Given $z_0 \in E_\sigma = \{z: F(z, E) = \sigma\}$, let

$$H_n^{(1)}(z_0, E) = \max_{(j)} |H^{(j)}(z_0, q^{(n)})| = |H^{(j_n)}(z_0, q^{(n)})|.$$

Put

$$U_k(z) = \sum_{n=1}^{k} \frac{1}{\varrho^n} H^{(j_n)}(z, q^{(n)}), \quad U(z) = \sum_{n=1}^{\infty} \frac{1}{\varrho^n} H^{(j_n)}(z, q^{(n)}).$$

Then by (4)

$$\|U - U_k\|_E \leqslant \sum_{n=k+1}^{\infty} \frac{1}{\varrho^n} = \frac{1}{\varrho^k} \cdot \frac{1}{\varrho - 1}.$$

Therefore in view of Theorem I

$$\limsup(\|U-U_k\|_{E_\sigma})^{1/k}\leqslant\sigma/\varrho\,.$$

But

$$\frac{1}{\varrho^n}|H^{(jn)}(z_0,q^{(n)})|\leqslant\|U_n-U_{n-1}\|_{E_\sigma}\leqslant\|U_n-U\|_{E_\sigma}+\|U_{n-1}-U\|_{E_\sigma}\,.$$

Therefore, in virtue of (*),

$$\limsup\sqrt[n]{H_n(z_0,E)}=\limsup\sqrt[n]{H_n^{(1)}(z_0,E)}$$
$$\leqslant\varrho\limsup[\|U_n-U\|_{E_\sigma}+\|U_{n-1}-U\|_{E_\sigma}]^{1/n}\leqslant\sigma=F(z_0)\,,$$

since for every $\varepsilon>0$ we have

$$\|U_n-U\|_{E_\sigma}\leqslant\left(\frac{e^\varepsilon\sigma}{\varrho}\right)^n,\qquad n\geqslant n_0(\varepsilon)\,.$$

So

$$F(z)=\lim\sqrt[n]{H_n(z)}\qquad\text{for}\qquad z\in C-E\,.$$

Since for $z\in E$ we have

$$H_n(z)=1=F(z)\,,\qquad\text{so}\qquad F=\lim\sqrt[n]{H_n}$$

for every $z\in C$. The proof is completed.

Remark 1. Theorem I (and consequently Theorem 1) remains true also under weaker assumptions on E (cf. [12], p. 344); e.g. the theorem holds if E is a compact set such that $C-E$ is connected, the interior $\mathring{E}$ of E consists of a finite number of components and $\partial\mathring{E}=\partial(C-E)$.

Remark 2. (Cf. the remarks on p. 172 in [15]). Let $p^{(n)}=\{p_0,\ldots,p_{2n}\}$ be an arbitrary fixed system of $2n+1$ points of E chosen in such a way that either

$$\max_{z\in E}\sum_{j=0}^{2n}|H^{(j)}(z,p^{(n)})|=\inf_{z^{(n)}\subset E}\Big[\max_{z\in E}\sum_{j=0}^{2n}|H^{(j)}(z,z^{(n)})|\Big]$$

or

$$\max_{(j)}[\max_{z\in E}|H^{(j)}(z,p^{(n)})|]=\inf_{z^{(n)}\subset E}\{\max_{(j)}[\max_{z\in E}|H^{(j)}(z,z^{(n)})|]\}\,.$$

One may easily check that if in Theorem 1 the functions $H_n^{(1)}$ and $H_n^{(2)}$ are defined by taking $p^{(n)}$ instead of $q^{(n)}$, then the sequences $\{\sqrt[n]{H_n^{(1)}}\}$ and $\{\sqrt[n]{H_n^{(2)}}\}$ are also convergent to F in C.

Similarly one may verify that in the theorems of Curtiss (e.g. Theorems 3.4 and 3.5 in [3]) concerning the solution of the Dirichlet problem by interpolating harmonic polynomials or the maximal convergence of harmonic polynomials found by interpolation the points $q^{(n)}$ may be replaced by $p^{(n)}$.

3. An example. As a direct consequence of Theorem II we get the following Bernstein-Walsh inequality

$$|p_n(z)| \leqslant \|p_n\|_E F^n(z, E), \quad z \in \boldsymbol{C}, \tag{6}$$

valid for every polynomial in z of degree $\leqslant n$, $n = 1, 2, \ldots$

One may ask whether the same or an analogous inequality holds if p_n is replaced by a harmonic polynomial and F is replaced by $H(z, E) = \limsup \sqrt[n]{H_n(z, E)}$. G. Szegö proved in [11] (see also [8]) that (6) holds for harmonic polynomials if E is a circle. We shall show that (6) does not hold for harmonic polynomials if E is an ellipse with foci ± 1 given by

$$E = \{z\colon |z+\sqrt{z^2-1}| = (r+\sqrt{r^2+1})\} \quad (r > 0).$$

Indeed, in view of Theorem 1,

$$H(z) = F(z) = \max\big(1, |z+\sqrt{z^2+1}|/(r+\sqrt{r^2+1})\big).$$

For the harmonic polynomial $U_n(z) = \operatorname{Im}(z/r)$ we have $|U_n(z)| \leqslant 1$ on E. Given any fixed $n = 1, 2, \ldots$, let $0 < r < 1/\sqrt{n^2-1}$. If $z = iy$, $y > r$, we have

$$F(iy) = (y+\sqrt{y^2+1})/(r+\sqrt{r^2+1}),$$

$$U_n(iy) = y/r \quad \text{and} \quad |U_n(iy)| > F^n(iy),$$

if $r < y < r+\varepsilon$, $\varepsilon > 0$ being sufficiently small.

In the next section we shall give a modification of (6) valid for harmonic polynomials.

4. Asymptotic behaviour of harmonic polynomials near a compact set. Given a compact set E denote by (H) the following condition

(H) *For every $\varepsilon > 0$ there exist two positive numbers δ and M such that if U_n is an arbitrary harmonic polynomial of degree $\leqslant n$ with $\|U_n\|_E \leqslant 1$, then $|U_n(z)| \leqslant Me^{n\varepsilon}$ for all z with* $\operatorname{dist}(z, E) < \delta$.

Denote by (L) an analogous condition where U_n is replaced by an algebraic polynomial in z. F. Leja [6] proved that a compact set E satisfies condition (L) if and only if the function $F(z, E)$ is continuous in $\boldsymbol{C}$. His polynomial lemma saying that every continuum satisfies condition (L) proved very useful in the complex analysis (cf. [5], [6], [7]).

It is obvious that every compact set E satisfying condition (H) satisfies also condition (L). But the opposite is not true, as can be easily shown for E being a line segment.

It seems to be an interesting problem to find what sets satisfy condition (H). A partial answer to this problem is given by the following theorems.

THEOREM 2. *If E is a compact set satisfying condition* (H), *then*

$$\lim \sqrt[n]{H_n(z, E)} = F(z, E), \quad z \in \boldsymbol{C}.$$

Proof. Given $\varepsilon > 0$ let $\varrho > 0$ be so small that $\varrho < \varepsilon$ and $E_\varrho = \{z\colon F(z) \leqslant e^\varrho\}$ is contained in the set $\{z\colon \operatorname{dist}(z, E) < \delta\}$, where $\delta = \delta(\varepsilon)$ is chosen in accordance with (H). Moreover, we choose ϱ in such a way that E_ϱ consists of a finite number of mutually disjoint closed Jordan domains. For every fixed $n = 1, \ldots$ denote by $q^{(n)} = \{q_0, \ldots, q_{2n}\}$ an nth extremal system of E with respect to the determinant A. For every harmonic polynomial U_n such that $\|U_n\|_E \leqslant 1$ we have, in view of (H) and by the interpolation formula,

$$|U_n(z)| \leqslant Me^{n\varepsilon} H_n^{(2)}(z, E_\varrho), \quad z \in \boldsymbol{C},$$

whence

$$H_n(z, E) \leqslant Me^{n\varepsilon} H_n^{(2)}(z, E_\varrho), \quad z \in \boldsymbol{C}.$$

By Theorem 1 we get

$$\limsup \sqrt[n]{H_n(z, E)} \leqslant e^\varepsilon F(z, E_\varrho) = e^\varepsilon \max[1, e^{-\varrho} F(z, E)], \quad z \in \boldsymbol{C}.$$

Since $\varepsilon > 0$ and $\varrho > 0$ may be chosen arbitrarily small, we have

$$\limsup \sqrt[n]{H_n(z, E)} \leqslant F(z, E), \quad z \in \boldsymbol{C}.$$

This and (5) give the required result.

THEOREM 3. *If E is a compact set consisting of a finite number of mutually disjoint closed simply connected domains, then E satisfies condition* (H).

Proof. It is enough to prove the theorem under the assumption that E is a bounded closed simply connected domain. Let E_n, $n = 1, \ldots$ be a closed domain contained in the interior of E and such that $E_n \subset E_{n+1}$ and $\bigcup_{n=1}^{\infty} E_n = \mathring{E}$, $\mathring{E}$ denoting the interior of E. By Lemma 1.2 of [13] for every $m = 1, \ldots$ there exists a positive constant K_m such that for every function U harmonic in E and in modulus not greater than 1 there exists a conjugate function $\overset{*}{U}$ in E such that

$$|\overset{*}{U}(z)| \leqslant K_m \quad \text{for} \quad z \in E_m, \quad m = 1, \ldots$$

Let $q^{(n)} = \{q_0, \ldots, q_{2n}\}$ be an nth extremal system of E with respect to A. Then by (4) and by the lemma there exists a polynomial $\overset{*}{H}{}^{(j)}$ conjugate with $H^{(j)}(z, q^{(n)})$ such that

$$|H^{(j)}(z, q^{(n)})| \leqslant |H^{(j)} + i\overset{*}{H}{}^{(j)}| \leqslant 1 + K_m \quad \text{for} \quad z \in E_m.$$

Since $H^{(j)}+i\overset{*}{H}^{(j)}$ is a polynomial in z, we may apply the Bernstein-Walsh inequality and get

(7) $$|H^{(j)}(z, q^{(n)})| \leqslant (1+K_m)F^n(z, E_m), \quad z \in \boldsymbol{C}, \quad j = 0, \dots, 2n.$$

It is easy to see that $F(z, E_m) \geqslant F(z, E_{m+1}) \geqslant F(z, E)$ in $\boldsymbol{C}$. The function $f_m(z) = \mathrm{Log}\big(F(z, E_m)/F(z, E)\big)$ is continuous in the Riemann sphere $S = \boldsymbol{C}+\{\infty\}$, harmonic in $C-E_m$, $f_{m+1} \leqslant f_m$ in S, $f_m(z) = 0$ for $z \in E_m$ and $f_m(\infty) \to 0$ ([9]). So by the Harnack principle $f_n(z) \to 0$ in S and by Dini's theorem the convergence is uniform in S. Therefore, given any $\varepsilon > 0$, we have

$$F(z, E_m) \leqslant e^{\varepsilon/4}F(z, E), \quad z \in S, \quad m \geqslant m_0 = m_0(\varepsilon).$$

Therefore, in virtue of (7),

(8) $$H_n^{(1)}(z, E) \leqslant (1+K)e^{n\varepsilon/4}F^n(z, E), \quad K = K_{m_0}.$$

Let $\delta > 0$ be so small that $F(z, E) < e^{\varepsilon/4}$, if $\operatorname{dist}(z, E) < \delta$; then

(9) $$H_n^{(1)}(z, E) \leqslant (1+K)e^{n\varepsilon/2}, \quad \operatorname{dist}(z, E) < \delta.$$

Given any harmonic polynomial U_n of degree $\leqslant n$ such that $\|U_n\|_E \leqslant 1$ we see by the interpolation formula that

$$|U_n(z)| \leqslant (2n+1)H_n^{(1)}(z, E), \quad z \in \boldsymbol{C},$$

whence by (9)

$$|U_n(z)| \leqslant e^{n\varepsilon}, \quad \operatorname{dist}(z, E) < \delta, \quad n \geqslant N = N(\varepsilon),$$

N being chosen in such a way that $(2n+1)(1+K) < e^{n\varepsilon/2}$ for $n \geqslant N$. The application of the interpolation formula shows that there exists an $M > 0$ such that $|U_n(z)| < M$, if $\operatorname{dist}(z, E) < \delta$ and if U_n is an arbitrary harmonic polynomial of degree $\leqslant n \leqslant N$ such that $\|U_n\|_E \leqslant 1$. The proof is completed.

THEOREM 4. *If E is a continuum satisfying the condition* (H) *(e.g. if E is a bounded closed simply connected domain), then for every $\varepsilon > 0$ there exists a positive number $M = M(\varepsilon)$ such that for every harmonic polynomial U_n of degree $\leqslant n$ we have*

(10) $$|U_n(z)| \leqslant \|U_n\|_E Me^{n\varepsilon}F^n(z, E), \quad z \in \boldsymbol{C}, \quad n = 1, 2, \dots$$

This theorem may be considered as a weaker version of the Corollary on p. 403 in [10]. This weaker version, however, is sufficient for deriving some results on the maximal convergence of sequences of harmonic polynomials (e.g. Theorems 6.2 and 6.3 in [10]).

Proof. Given $\varepsilon > 0$ let $\varrho > 0$ be so small that $\varrho < \varepsilon$ and $E_\varrho = \{z\colon F(z, E) \leqslant e^\varrho\}$ is contained in the set $\{z\colon \operatorname{dist}(z, E) < \delta\}$, where

$\delta = \delta(\varepsilon/2)$ is chosen in accordance with condition (H). For every $\varrho > 0$ $F(z, E_\varrho) < F(z, E)$ in C and E_ϱ is a closure of a Jordan domain. By Lemma 1.2 of [13] there exists a constant K_ϱ such that for every function U harmonic in E_ϱ with $\|U\|_{E_\varrho} \leqslant 1$ there exists a conjugate function $\overset{*}{U}$ in E_ϱ such that $|\overset{*}{U}(z)| \leqslant K_\varrho$ in E. Application of this lemma, inequality (4) and condition (H) give

$$|H^{(j)}(z, q^{(n)}) + i\overset{*}{H}{}^{(j)}(z)| \leqslant 1 + MK_\varrho e^{n\varepsilon/2} \quad \text{for} \quad z \in E\,,$$

whence by (6) we get

$$|H^{(j)}(z, q^{(n)})| \leqslant B_\varrho e^{n\varepsilon/2} F^n(z, E)\,, \qquad B_\varrho = (1 + MK_\varrho)\,.$$

Therefore, by the interpolation formula,

$$|U_n(z)| \leqslant \|U_n\|_E B_\varrho e^{n\varepsilon/2}(2n+1) F^n(z, E)\,,$$

$\varrho > 0$ being sufficiently small. Hence by a standard reasoning we get Theorem 4.

THEOREM 5. *If E is a continuum satisfying the condition* (H), *then the sequences*

$$\{\sqrt[n]{H_n}\}\,, \qquad \{\sqrt[n]{H_n^{(k)}}\} \qquad (k = 1, 2, 3, 4) \tag{11}$$

considered in Theorem 1 *are convergent uniformly on compact subsets of the complex plane* C.

Proof. In virtue of the inequalities (*) it is enough to prove that for every $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ such that

$$F(z, E) e^{-\varepsilon} \leqslant \sqrt[n]{H_n(z)} \leqslant F(z, E) e^{\varepsilon}\,, \quad z \in C\,, \quad n \geqslant n_0\,.$$

Since E satisfies condition (H), it satisfies also condition (L) and therefore the functions $F_n(z, E)$ and $F(z, E)$ are continuous in C. Moreover, the function $f_n(z)$ defined by $f_n(z) = F(z, E)/\sqrt[n]{F_n(z, E)}$ for $z \in C$ and $f_n(\infty) = \lim_{z\to\infty} F(z, E)/\sqrt[n]{F_n(z, E)}$ is continuous in $C + \{\infty\}$. Further, $f_n(z) \geqslant 1$ and $1 = \lim_{n\to\infty} f_n(z)$ for z in $C + \{\infty\}$. Therefore by the Dini theorem the convergence of $\{f_n\}$ is uniform in $C + \infty$, whence

$$F(z, E) e^{-\varepsilon} \leqslant \sqrt[n]{F_n(z, E)} \leqslant F(z, E) e^{\varepsilon}\,, \quad z \in C\,, \quad n \geqslant n_0(\varepsilon)\,.$$

This and (10) give the required result.

We shall need the following

POLYNOMIAL LEMMA. *Let $\mathcal{F}$ be a family of polynomials in k complex variables $z = (z_1, \ldots, z_k)$ bounded at every point z of a Cartesian product $E = E_1 \times \ldots \times E_k$, where E_j $(j = 1, \ldots, k)$ is a continuum not reduced to* *a single point. The bound may depend on the point z. Then for every* $\varepsilon > 0$ *there exist two positive numbers* δ *and* M *such that*

$$|P_n(z)| \leqslant Me^{n\varepsilon}, \quad \operatorname{dist}(z, E) < \delta,$$

where P_n *is an arbitrary polynomial of degree at most n belonging to the family* $\mathcal{F}$.

Proof. For $k = 1$ the lemma is due to Leja [5]. If $k > 1$ the lemma follows by induction. Indeed, put $z' = (z_1, \ldots, z_{k-1})$. Then

$$|P(z)| = |P(z', z_k)| \leqslant M(z', z_k) \quad \text{for} \quad z' \in E_1 \times \ldots \times E_{k-1},\ z_k \in E_k,\ P \in \mathcal{F}.$$

By the validity of the lemma for $k = 1$ we have

$$|P_n(z', z_k)| \leqslant M(z')e^{n\varepsilon_1}, \quad z' \in E_1 \times \ldots \times E_{k-1}, \quad \operatorname{dist}(z_k, E_k) < \delta_1,$$

δ_1 depending only on $\varepsilon_1 > 0$ and not on z' or P_n. By the induction assumption

$$|P_n(z', z_k)|/e^{n\varepsilon_1} \leqslant Me^{n\varepsilon_2}, \quad \operatorname{dist}(z', E_1 \times \ldots \times E_{k-1}) < \delta_2,\ \operatorname{dist}(z_k, E_k) < \delta_1,$$

$M = \text{const} > 0$. So

$$|P_n(z)| \leqslant Me^{n\varepsilon}, \quad \operatorname{dist}(z, E) < \delta, \quad P_n \in \mathcal{F},$$

δ being sufficiently small depends only on ε.

We are now able to give a generalization of Theorem I.

THEOREM 6. *Let E be a compact set satisfying the condition* (H). *Let U be a function defined on E and let harmonic polynomials* U_n *of respective degrees at most n satisfy*

$$\limsup(\|U - U_n\|_E)^{1/n} \leqslant 1/\varrho < 1. \tag{12}$$

Then

1° *the sequence* $\{U_n\}$ *converges uniformly on closed subsets of the interior of* E_ϱ;

2° $\limsup(\|U - U_n\|_{E_\sigma})^{1/n} \leqslant \sigma/\varrho$ $(1 < \sigma < \varrho)$.

Proof. Let ε, σ and σ_1 be arbitrary positive numbers such that $1 < \sigma < \sigma_1 < \varrho$, $e^{2\varepsilon}\sigma_1/\varrho < 1$. By (12)

$$\|U_n - U_{n-1}\|_E \leqslant \|U_n - U\|_E + \|U_{n-1} - U\|_E \leqslant M_1(e^\varepsilon/\varrho)^n, \qquad n \geqslant 1,\ M_1 = \text{const}.$$

By the interpolation formula and by (4) and (a_0)

$$|U_n(z) - U_{n-1}(z)| \leqslant M_1(e^\varepsilon/\varrho)^n \sum_{j=0}^{2n} |H^{(j)}(z, q^{(n)})| \leqslant M_1(e^\varepsilon/\varrho)^n(2n+1)H_n(z)$$

for $n \geqslant 1$ and $z \in C$, whence

$$\limsup \left(|U_n(z) - U_{n-1}(z)|\right)^{1/n} \leqslant e^{\varepsilon}\sigma_1/\varrho < 1\,, \quad z \in E_{\sigma_1}\,.$$

Thus the sequence $\{U_n\}$ converges for every $z \in E_{\sigma_1}$.

The sequence of polynomials of two variables x and y defined by

$$P_n(x, y) = [U_n(z) - U_{n-1}(z)]/(e^{\varepsilon}\sigma_1/\varrho)^n\,, \quad n = 1, \ldots$$

is bounded at every $z = x+iy \in E$. The set E_σ is a compact subset of the interior of E_{σ_1}. By the Heine-Borel theorem there is a finite number of squares

$$Q_j = \{(x, y)\colon |x-x_j| \leqslant r_j\,, \ |y-y_j| \leqslant r_j\}\,, \quad j = 1, \ldots, l$$

such that

$$E_\sigma \subset \bigcup_{j=1}^{l} Q_j \subset E_{\sigma_1}\,.$$

By the polynomial lemma applied to the sequence $\{P_n(x, y)\}$ and to every Q_j there exists a constant M such that for every $n \geqslant 1$

$$|U_n(z) - U_{n-1}(z)| \leqslant M(e^{2\varepsilon}\sigma_1/\varrho)^n\,, \tag{13}$$

z in a neighbourhood of E_σ. This implies that $\{U_n\}$ converges uniformly in E_σ, whence in view of the arbitrariness of ε, σ and σ_1 it converges uniformly on compact subsets of E_ϱ.

By (13)

$$\|U - U_n\|_{E_\sigma} = \Big\| \sum_{k=n+1}^{\infty} (U_k - U_{k-1}) \Big\|_{E_\sigma} \leqslant M(e^{2\varepsilon}\sigma_1/\varrho)^{n+1}/\left(1 - (e^{2\varepsilon}\sigma_1/\varrho)\right)\,,$$

so

$$\limsup (\|U - U_n\|_{E_\sigma})^{1/n} \leqslant e^{2\varepsilon}\sigma_1/\varrho\,,$$

whence by the arbitrariness of ε, σ and σ_1 we get 2°. The proof is completed.

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