ANNALES
POLONICI MATHEMATICI
XXXVIII (1980)

A difference method for a non-linear system of elliptic equations with mixed derivatives

by Z. KOWALSKI (Kraków)

Abstract. In this paper we consider the non-linear system of partial differential equations (1.1). We assume that system (1.1) is of the elliptic type in the sense of Definition 1 (cf. Section 4) and we prove the convergence of the difference method for the boundary problem (5.3), (5.4). The error estimate is also given. The proofs are based on the method of difference inequalities.

1. This paper contains many formulas; nevertheless, the entire reasoning is distinguished by a clear geometric thought.

I consider a difference method for a non-linear system of partial differential equations

$$f^l(x, u_{lx}, u_{lxx}) = 0 \qquad (l = 1, \ldots, p) \tag{1.1}$$

in the n-dimensional cube Q, $x = (x_1, \ldots, x_n) \in Q$, cf. (2.2), where the l-th equation contains all the functions $u_1, u_2, \ldots, u_p$ and the derivatives of the l-th function $u_l(x)$ only. I was forced to choose such a system since I wanted to apply the geometrical methods worked out previously in papers [1] and [3], which are concerned with a single non-linear differential equation of the elliptic type (the methods of papers [1] and [3] fail for system (1.1) if there are the derivatives of all functions $u_1, \ldots, u_p$ in the l-th equation).

I assume that system (1.1) is of elliptic type in the sense of Definition 1, cf. Section 4. I was led to that definition also from geometrical considerations in the following way: First, I wrote the difference equation, cf. (5.7), associated with system (1.1), and I defined the error $r_l^M = u_l^M - v_l^M$ $(l = 1, \ldots, p)$, where u_l^M $(l = 1, \ldots, p)$ is the solution of system (1.1) and v_l^M $(l = 1, \ldots, p)$ — the solution of the difference equation.

Then I considered the point $R(h)$ in the p-dimensional space:

$$R(h) = (r_1^{A_1}, \ldots, r_p^{A_p}), \tag{1.2}$$

with coordinates

$$0 \leqslant r_l^{A_l} = \max_{x^M \in Q} r_l^M \qquad (l = 1, \ldots, p), \tag{1.3}$$

the maximum being taken over all the nodal points x^M in the set Q (in a similar way I defined the minimum).

Obviously, the point $R(h)$ depends on the mesh size h imposed on Q. A single look at Fig. 1 is now sufficient to catch the idea of the paper.

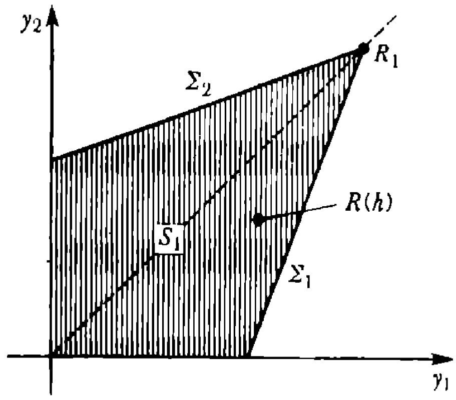

Fig. 1. The set S_1 in the two dimensional case ($p = 2$), the point $R(h)$ and R_1, cf. paper [2]

I found that the point $R(h)$ is in the domain S_1 bounded by the hyperplanes Σ_l ($l = 1, \ldots, p$) in the p-dimensional space. Those hyperplanes Σ_l possess a single intersection point R_1 if and only if (rather unexpected) condition (10.7) in Definition 1 of ellipticity is satisfied. In addition, the domain S_1 shrinks toward the origin as $h \to 0$, the point $R(h)$ being inside S_1 for all $h > 0$.

In consequence, all coordinates $r_l^{A_l}$ of the point $R(h)$ tend toward zero as $h \to 0$, which means that the difference method in question is convergent.

Thus I was forced to accept condition (10.7), since in the opposite case (where R_1 does not exist) the point $R(h)$ could escape into infinity as $h \to 0$, and the difference method could be divergent.

The calculations form the remainder of the paper, but each formula possesses a geometrical significance and I would like the reader could see it.

There is no possibility to hide own research program before the men who work in the same research center. M. Malec has undertaken the problems connected with difference methods for elliptic equations and has written the following series of papers: [4] and [5] on difference inequalities and a single non-linear differential equation; [6] and [7] on systems of difference inequalities and systems of non-linear differential equations; [8] on the Neumann problem for a single non-linear differential equation.

His method of checking the convergence differs from that of mine. Writing the difference equation out, he disrupts the difference quotients and collects corresponding elements so as to obtain terms of constant sign (i.e. always positive for $h > 0$ or always negative for $h > 0$). Then he omits these terms in order to achieve simple difference inequalities and estimates for errors r_l^M. Thus, his method could be termed as the method of disrupting the difference quotients. This is a powerful algebraic method, since it enables one to obtain the convergence and error estimate in a short and almost painless way.

The calculations as performed by Malec in his method of disrupting the difference quotients have led him to a different definition of elliptic equation.

The non-linear system (1.1) is of the elliptic type in the sense of Malec if, among others, the matrix $(\partial f^l/\partial w_{ij})$ $(i, j = 1, \ldots, n)$ has the dominating diagonal line. In consequence, the class of elliptic equations in the sense of Malec is contained in the class of elliptic equations in my sense, cf. Definition 1, Section 4, papers [4]–[8], and [9], p. 106 (we have not investigated, for the time being, whether the opposite inclusion actually holds).

There is still one detail which differs our papers. I replace the mixed derivatives $u_{x_i x_j}$ of the second order by "large" difference quotients u^{Mij}, cf. (3.3), and Malec uses for that purpose the arithmetic means of "small" difference quotients (3.5), the choice being dependent from the sign of the derivative $\partial f^l/\partial w_{ij}$.

The last way may be better in practical computations, cf. Samarskiĭ [11], p. 264.

In the papers of Malec and in my papers there remains open the difficult problem of the existence of solutions, for a differential and difference equations of elliptic type, respectively.

2. Let us consider the nodal points x^M

$$x^M = (x_1^M, \ldots, x_n^M), \tag{2.1}$$

in the set Q:

$$Q\colon\ 0 \leqslant x_j \leqslant \sigma \quad (j = 1, \ldots, n), \qquad 0 < \sigma = \text{const}, \tag{2.2}$$

the coordinates x_j^M $(j = 1, \ldots, n)$ of the point x^M being defined by

$$x_j^M = m_j \cdot h \quad (j = 1, \ldots, n), \qquad 0 < h = \sigma/N, \tag{2.3}$$

where M denotes a multi-index

$$M = (m_1, m_2, \ldots, m_n), \qquad 0 \leqslant m_j \leqslant N \quad (j = 1, \ldots, n), \tag{2.4}$$

and N is a natural number.

It will be convenient to introduce the nodal points

$$\begin{gathered} x^{j(M)},\ x^{-j(M)} \quad \text{for } j = 1, \ldots, n, \quad \text{and} \quad x^{ij(M)},\ x^{-ij(M)}, \\ x^{-i-j(M)},\ x^{i-j(M)} \quad (i \neq j;\ i, j = 1, \ldots, n), \end{gathered} \tag{2.5}$$

which belong to the h-neighbourhood of the nodal point x^M and are characterized by the following multi-indices:

$$(2.6)\quad \begin{aligned} j(M) &= (m_1', \ldots, m_n'), & m_j' &= m_j+1, & m_i' &= m_i \quad \text{for } i \neq j\\ -j(M) &= (m_1', \ldots, m_n'), & m_j' &= m_j-1, & m_i' &= m_i \quad \text{for } i \neq j\\ &&&& (i, j &= 1, \ldots, n), \end{aligned}$$

and, for $i \neq j$, by

$$(2.7)\quad \begin{aligned} ij(M) &= (m_1', \ldots, m_n'), & m_i' &= m_i+1, & m_j' &= m_j+1,\\ -ij(M) &= (m_1', \ldots, m_n'), & m_i' &= m_i-1, & m_j' &= m_j+1,\\ -i-j(M) &= (m_1', \ldots, m_n'), & m_i' &= m_i-1, & m_j' &= m_j-1,\\ i-j(M) &= (m_1', \ldots, m_n'), & m_i' &= m_i+1, & m_j' &= m_j-1, \end{aligned}$$

where $m_s' = m_s$ in formulas (2.7) for $s \neq i, s \neq j$ $(s = 1, \ldots, n)$, cf. Fig. 2.

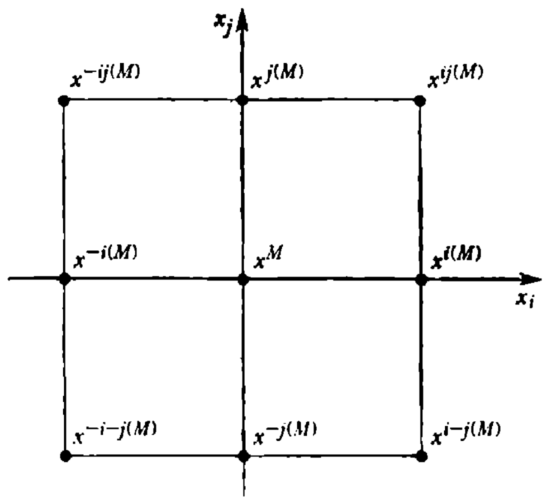

Fig. 2. The nodal points x^M, $x^{i(M)}$, $x^{ij(M)}$, ... For the sake of simplicity the nodal point x^M has been located at the origin

The nodal point $x^{ij(M)}$ will be denoted also by $x^{ji(M)}$, since we assume

$$(2.8)\quad \begin{aligned} ij(M) = ji(M), \quad -ij(M) = j-i(M), \quad -i-j(M) = -j-i(M),\\ i-j(M) = -ji(M) \quad \text{for } i \neq j \ (i, j = 1, \ldots, n). \end{aligned}$$

We denote by $\operatorname{int} Q$ the set of nodal points (2.1) which belong to the interior of Q, cf. (2.2), and by $\operatorname{sym} A$ the set of nodal points x^M such that $x^M \in \operatorname{int} Q$ and $x^{M^*} \in \operatorname{int} Q$ simultaneously, x^{M^*} and x^M being symmetric with respect to the nodal point x^A.

3. Let us denote by u_l^M $(l = 1, \ldots, p)$ the value of the function $u_l(x)$ $(l = 1, \ldots, p)$ at the nodal point x^M. We shall use the forward, backward and symmetric difference quotients, respectively:

$$(3.1)\qquad u_{l+}^{Mj} = \frac{1}{h}\cdot(u_l^{j(M)} - u_l^M), \qquad u_{l-}^{Mj} = \frac{1}{h}\cdot(u_l^M - u_l^{-j(M)}),$$
$$(l = 1, \ldots, p;\ j = 1, \ldots, n),$$

$$(3.2)\qquad u_l^{Mj} = \frac{1}{2h}\cdot(u_l^{j(M)} - u_l^{-j(M)}) \qquad (l = 1, \ldots, p;\ j = 1, \ldots, n)$$

for the first partial derivatives, and the difference quotients

$$(3.3)\qquad \begin{aligned} u_l^{Mjj} &= h^{-2}(u_l^{j(M)} - 2\cdot u_l^M + u_l^{-j(M)}),\\ u_l^{Mij} &= \tfrac{1}{4}\cdot h^{-2}\cdot(u_l^{ij(M)} - u_l^{-ij(M)} - u_l^{i-j(M)} + u_l^{-i-j(M)}),\end{aligned}$$
$$(i \neq j)\ (l = 1, \ldots, p;\ i = 1, \ldots, n;\ j = 1, \ldots, n),$$

for the second derivatives. In particular, u_l^{Mij} will be called the ***large difference quotient*** for the mixed derivative of the second order, since it is based on the large square formed by the nodal points $x^{ij(M)}$, $x^{-ij(M)}$, $x^{i-j(M)}$, $x^{-i-j(M)}$, cf. Fig. 2.

From definitions (3.1), (3.2) and (3.3) we have

$$(3.4)\qquad u_l^{Mj} = \tfrac{1}{2}\cdot(u_{l+}^{Mj} + u_{l-}^{Mj}), \qquad u_l^{Mjj} = \frac{1}{h}\cdot(u_{l+}^{Mj} - u_{l-}^{Mj})$$

for $l = 1, \ldots, p$; $j = 1, \ldots, n$. We shall use also the small difference quotients for mixed derivatives of the second order, cf. Fig. 2:

$$(3.5)\qquad \begin{aligned} u_{l++}^{Mij} &= h^{-2}\cdot(u_l^{ij(M)} - u_l^{j(M)} - u_l^{i(M)} + u_l^M),\\ u_{l-+}^{Mij} &= h^{-2}\cdot(u_l^{j(M)} - u_l^{-ij(M)} - u_l^M + u_l^{-i(M)}),\\ u_{l--}^{Mij} &= h^{-2}\cdot(u_l^M - u_l^{-i(M)} - u_l^{-j(M)} + u_l^{-i-j(M)}),\\ u_{l+-}^{Mij} &= h^{-2}\cdot(u_l^{i(M)} - u_l^M - u_l^{i-j(M)} + u_l^{-j(M)})\end{aligned}$$

for $i \neq j$; $i = 1, \ldots, n$; $j = 1, \ldots, n$; $l = 1, \ldots, p$.

From (3.5) and the definition of u_l^{Mij}, cf. (3.3), we have

$$(3.6)\qquad u_l^{Mij} = \tfrac{1}{4}\cdot(u_{l++}^{Mij} + u_{l-+}^{Mij} + u_{l--}^{Mij} + u_{l+-}^{Mij})$$

for $i \neq j$; $i = 1, \ldots, n$; $j = 1, \ldots, n$; $l = 1, \ldots, p$.

We introduce also the vectors $u_l^{M\triangle}$ with coordinates

$$(3.7)\qquad u_l^{M\triangle} = (u_l^{M1}, u_l^{M2}, \ldots, u_l^{Mp}) \qquad (l = 1, \ldots, p),$$

and the $n\times n$ matrices $u_l^{M\square}$ $(l = 1, \ldots, p)$:

$$(3.8)\qquad u_l^{M\square} = (u_l^{Mij}) \qquad (i = 1, \ldots, n;\ j = 1, \ldots, n)\ (l = 1, \ldots, p).$$

4. The following three conditions W_1, W_2 and W_3 provide a means for the definition of system (1.1) to be of the elliptic type:

CONDITION W_1. The functions $f^l(x, u, q, w)$ are of class C^1 in the set Q_1:

$$Q_1 = Q \times R^p \times R^n \times R^{n^2}; \tag{4.1}$$

the quadratic forms

$$\sum_{i,j=1}^{n} f^l_{w_{ij}} \cdot \lambda_i \cdot \lambda_j \quad (l = 1, \ldots, p) \; (f^l_{w_{ij}} = f^l_{w_{ji}}), \tag{4.2}$$

are positive defined at every point of the set Q_1, and the characteristic roots s_{lj} $(l = 1, \ldots, p)$ $(j = 1, \ldots, n)$ of the form (4.2) are bounded:

$$0 < \delta_1 \leqslant s_{lj} \leqslant \delta_2, \tag{4.3}$$

the constants δ_1, δ_2 being independent from the point (x, u, q, w) in the set Q_1.

CONDITION W_2. The elements of the $p \times p$ matrix $(f^l_{u_k})$ $(l = 1, \ldots, p;$ $k = 1, \ldots, p)$ satisfy the relations

$$f^l_{u_l} \leqslant \eta < 0 \quad (l = 1, \ldots, p; \eta = \text{const}), \tag{4.4}$$

$$0 \leqslant f^l_{u_k} < \delta \quad (\delta = \text{const}; l \neq k; l = 1, \ldots, p; k = 1, \ldots, p) \tag{4.5}$$

and

$$-\frac{1}{p-1} < \gamma < 0 \quad (p \geqslant 2), \tag{4.6}$$

the constant γ being defined by

$$\gamma = \eta^{-1} \cdot \delta \quad (\gamma < 0). \tag{4.7}$$

CONDITION W_3. The derivatives $f^l_{w_{ij}}$ and $f^l_{q_j}$ are bounded in the set Q_1:

$$|f^l_{w_{ij}}| \leqslant \zeta, \quad |f^l_{q_j}| \leqslant \beta. \tag{4.7}$$

DEFINITION 1. The non-linear system (1.1) is called to be of the *elliptic type* if and only if conditions W_1, W_2 and W_3 are satisfied.

5. Before proceeding to a specific discussion of the difference method, it will be useful to introduce the following

ASSUMPTIONS H. (i) System (1.1) is of the elliptic type in the sense of Definition 1.

(ii) The functions $u_l = u_l(x)$ $(l = 1, \ldots, p)$ satisfy the following conditions in the set Q:

1° $u_l(x)$ is of class C^2 for $x \in Q$,

2° the derivatives $u^{\prime}_{l,x_ix_j}$ fulfil the Lipschitz condition

$$|u_{l,x_ix_j}(x) - u_{l,x_ix_j}(x')| \leqslant \tfrac{1}{2} \cdot L \cdot |x_s - x'_s| \quad (i, j = 1, \ldots, n), \tag{5.1}$$

the points x and x' being on the x_s-axis $(s = 1, \ldots, n)$, $x = (x_1, \ldots, x_n)$

$\in Q$, $x' = (x'_1, \ldots, x'_n) \in Q$, $x_s \neq x'_s$, $x_k = x'_k$ $(k \neq s;\ k = 1, \ldots, n)$, $0 < L = \text{const}$.

3° The inequalities

$$|u_{l,x_ix_j}| \leqslant \tfrac{1}{2}\cdot\Lambda \qquad (i, j = 1, \ldots, n) \tag{5.2}$$

hold for $x \in Q$, the constant Λ being independent from x.

4° The function $u_l(x)$ $(l = 1, \ldots, p)$ $(x \in Q)$ is the solution of system (1.1), i.e. we have the identity

$$f^l(x, u, u_{lx}, u_{lxx}) \equiv 0 \quad \text{for } x \in Q \ (l = 1, \ldots, p), \tag{5.3}$$

where $u = (u_1, \ldots, u_p)$, $u_{lx} = (u_{lx_1}, \ldots, u_{lx_n})$, $u_{lxx} = (u_{l,x_ix_j})$, and takes on prescribed values $\varphi_l(x)$ at the boundary ∂Q of the set Q:

$$u_l(x) = \varphi_l(x) \quad \text{for } x \in \partial Q \ (l = 1, \ldots, p), \tag{5.4}$$

the function $\varphi_l(x)$ being continuous on ∂Q.

(iii) We assume that the discrete function v_l^M $(l = 1, \ldots, p)$ satisfies the following conditions:

1° v_l^M is defined at the nodal points x^M $(x^M \in Q)$.

2° The difference quotients v_l^{Mij} of the second order satisfy the inequalities:

$$\begin{gathered} |v_l^{Mjj} - v_l^{Pjj}| \leqslant \tfrac{1}{2}\cdot hL, \qquad |v_{l++}^{Mij} - v_{l++}^{Pij}| \leqslant \tfrac{1}{2}\cdot hL, \\ |v_{l-+}^{Mij} - v_{l-+}^{Pij}| \leqslant \tfrac{1}{2}\cdot hL, \qquad |v_{l--}^{Mij} - v_{l--}^{Pij}| \leqslant \tfrac{1}{2}\cdot hL, \\ |v_{l+-}^{Mij} - v_{l+-}^{Pij}| \leqslant \tfrac{1}{2}\cdot hL \ \text{for } h > 0,\ 0 < L = \text{const} \ (i \neq j;\ l = 1, \ldots, p), \end{gathered} \tag{5.5}$$

at the nodal points x^M and x^P, $P = s(M)$ $(s = \pm 1, \pm 2, \ldots, \pm n)$, the distance between x^M and x^P being h in the direction of the x_s-axis.

3° The inequalities

$$|v_l^{Mij}| \leqslant \tfrac{1}{2}\cdot\Lambda \qquad (i, j = 1, \ldots, n;\ l = 1, \ldots, p), \tag{5.6}$$

hold for $x^M \in \operatorname{int} Q$.

4° The function v_l^M $(l = 1, \ldots, p)$ is the solution of the difference equation

$$f^l(x^M, v^M, v_l^{M\triangle}, v_l^{M\square}) = 0 \qquad (l = 1, \ldots, p) \text{ for } x^M \in \operatorname{int} Q, \tag{5.7}$$

where $v^M = (v_1^M, \ldots, v_p^M)$, cf. (3.7), (3.8), and takes on prescribed values at the boundary ∂Q of the set Q:

$$v_l^M = \varphi_l(x^M) \quad \text{for } x^M \in \partial Q \ (l = 1, \ldots, p). \tag{5.8}$$

(iv) We suppose that the functions

$$r_l^M = u_l^M - v_l^M \qquad (l = 1, \ldots, p) \ (x^M \in Q), \tag{5.9}$$

satisfy the relations

$$(5.10)\qquad \begin{aligned} &|r_{l+}^{Mj}| \leqslant h\cdot\vartheta \quad \text{for } x^M \in \partial Q,\ x^{j(M)} \in \operatorname{int} Q \ (j = 1, \ldots, n),\\ &|r_{l-}^{Mj}| \leqslant h\cdot\vartheta \quad \text{for } x^M \in \partial Q,\ x^{-j(M)} \in \operatorname{int} Q \ (j = 1, \ldots, n),\end{aligned}$$

for $l = 1, \ldots, p$, where $0 < \vartheta = \text{const}$.

6. In the preceding section we have summarized the properties of the solutions $u_l(x)$ and v_l^M of the system of differential equations (5.3), and the system of difference equations (5.7), respectively. Unfortunately, at the time being we are forced to assume the existence of the solutions $u_l(x)$ and v_l^M.

Conditions (5.10) are imposed along the boundary ∂Q of the set Q, only, and can be viewed as a compatibility conditions for the difference quotients of the first order v_l^{Mj} and u_l^{Mj}. It is not quite excluded that they may help to gain the uniqueness of the solution v_l^M of system (5.7), cf. Pliś [10], when for any two solutions v_l^M and V_l^M of (5.7) the conditions

$$(6.1)\qquad \begin{aligned} &|v_{l+}^{Mj} - V_{l+}^{Mj}| \leqslant h\cdot\vartheta \quad \text{for } x^M \in \partial Q,\ x^{j(M)} \in \operatorname{int} Q,\\ &|v_{l-}^{Mj} - V_{l-}^{Mj}| \leqslant h\cdot\vartheta \quad \text{for } x^M \in \partial Q,\ x^{-j(M)} \in \operatorname{int} Q,\end{aligned}$$

are assumed along the boundary ∂Q of the set Q.

7. The main idea of the remainder of this paper is the following. First, we shall prove that the errors r_l^M satisfy two systems of linear difference inequalities I and II, cf. (5.9) and Theorem 1, Section 14.

We have investigated those systems I and II in detail in our previous paper [2]. The chief point of that paper was to indicate that, under the suitable assumptions, the solutions r_l^M of the systems I and II converge to zero, as the mesh size h tends to zero. This result is of particular importance for our systems (5.3) and (5.7), since it guarantees the convergence of the difference method in question, the error estimate being taken from [2] as an additional result.

8. The following Remark 1 and Remark 2 will be used in order to verify that the errors r_l^M, cf. (5.9), satisfy the assumptions imposed upon them in paper [2] on systems of linear difference inequalities:

Remark 1. From assumptions (5.5) and (5.1) it follows that the errors r_l^M, cf. (5.9), satisfy the inequalities

$$(8.1)\qquad \begin{aligned} &|r_l^{Mjj} - r_l^{Pjj}| \leqslant h\cdot L, &&|r_{l++}^{Mij} - r_{l++}^{Pij}| \leqslant h\cdot L,\\ &|r_{l-+}^{Mij} - r_{l-+}^{Pij}| \leqslant h\cdot L, &&|r_{l--}^{Mij} - r_{l--}^{Pij}| \leqslant h\cdot L,\\ &|r_{l+-}^{Mij} - r_{l+-}^{Pij}| \leqslant h\cdot L &&(i \neq j)\ (l = 1, \ldots, p),\end{aligned}$$

for $h > 0$, $0 < L = \text{const}$, at the nodal points x^M and x^P, $P = s(M)$ $(s = \pm 1, \pm 2, \ldots, \pm n)$, the distance between x^M and x^P being h in the direction of the x_s-axis.

From (5.6) and (5.2) we obtain also

(8.2) $|r_l^{Mij}| \leqslant A$ for $x^M \in \mathrm{int}Q$ $(i, j = 1, \ldots, n;\ l = 1, \ldots, p)$.

9. Remark 2. The solution $u_l(x)$ $(l = 1, \ldots, p)$ of the system of differential equations (5.3) satisfies at the point x^M the equation

(9.1) $$f^l(x^M, u^M, u_l^{M\triangle}, u_l^{M\square}) = \varepsilon_l^M \text{ for } x^M \in \mathrm{int}Q \ (l = 1, \ldots, p),$$

ε_l^M depending on x^M.

Let us denote

(9.2) $$\varepsilon(h) = \max|\varepsilon_l^M|,$$

the maximum being taken for $l = 1, \ldots, p$ and $x^M \in \mathrm{int}Q$.

It can be seen that

(9.3) $$\varepsilon(h) \to 0, \quad \text{as } h \to 0,$$

because f^l is of class C^1 and $u_l(x)$ is of class C^2, cf. Assumptions H, Section 5.

10. We will return to the problem of convergence of the difference method (5.7) in Section 14, where suitable assumptions and theorems from paper [2] permit us to prove Theorem 1, cf. Section 14.

Let us now consider two systems of difference linear inequalities for the function z_l^M $(l = 1, \ldots, p)$ defined at the nodal points x^M:

(10.1) System I:

$$\sum_{i,j=1}^{n} a_{lij}^M \cdot z_l^{Mij} + \sum_{j=1}^{n} b_{lj}^M \cdot z_l^{Mj} + \sum_{k=1}^{p} c_{lk}^M \cdot z_k^M \geqslant -\varepsilon(h),$$

(10.2) System II:

$$\sum_{i,j=1}^{n} a_{lij}^M \cdot z_l^{Mij} + \sum_{j=1}^{n} b_{lj}^M \cdot z_l^{Mj} + \sum_{k=1}^{p} c_{lk}^M \cdot z_k^M \leqslant +\varepsilon(h),$$

where $0 < \varepsilon(h) = \mathrm{const}$ $(l = 1, \ldots, p)$.

Let us also consider the following conditions W_1', W_2' and W_3':

CONDITION W_1'. The quadratic forms

(10.3) $$\sum_{i,j=1}^{n} a_{lij}^M \cdot \lambda_i \cdot \lambda_j \quad (l = 1, \ldots, p) \ (x^M \in \mathrm{int}Q),$$

are positive defined and the characteristic roots s_{lj}^M $(l = 1, \ldots, p; j = 1, \ldots, n)$, are bounded:

(10.4) $$0 < \delta_1 \leqslant s_{lj}^M \leqslant \delta_2 \quad (l = 1, \ldots, p;\ j = 1, \ldots, n),$$

the constants δ_1 and δ_2 being independent of the mesh size h.

CONDITION W_2'. The elements of the matrix (c_{lk}^M) $(l, k = 1, \ldots, p)$ $(x^M \in \operatorname{int} Q)$ satisfy the inequalities:

$$c_{ll}^M \leqslant \eta < 0 \quad (l = 1, \ldots, p;\ \eta = \text{const}), \tag{10.5}$$

$$0 \leqslant c_{lk}^M < \delta \quad (l \neq k)\ (l, k = 1, \ldots, p;\ \delta = \text{const}), \tag{10.6}$$

where the constants η and δ do not depend on the mesh size h, and

$$-\frac{1}{p-1} < \gamma < 0 \quad (p \geqslant 2), \tag{10.7}$$

the coefficient γ being defined by

$$\gamma = +\eta^{-1} \cdot \delta \quad (\gamma < 0). \tag{10.8}$$

CONDITION W_3'. The coefficients a_{lij}^M, b_{lj}^M are bounded:

$$|a_{lij}^M| \leqslant \zeta, \quad |b_{lj}^M| \leqslant \beta \tag{10.9}$$

for $l = 1, \ldots, p$; $i, j = 1, \ldots, n$ $(x^M \in \operatorname{int} Q)$, the constants ζ and β being independent of the mesh size h.

We shall use the following

DEFINITION 2. The linear systems I and II of difference inequalities are of the *elliptic type* if and only if conditions W_1', W_2' and W_3' are fulfilled.

11. We shall use the following assumptions H(LS) on linear systems of inequalities:

ASSUMPTIONS H(LS). 1° We suppose that the functions z_l^M $(l = 1, \ldots, p)$ are defined at the nodal points of the set Q $(x^M \in Q)$, cf. (2.2).

2° There exists a positive constant $\vartheta > 0$ (independent of the mesh size h) such that the first order difference quotients satisfy the conditions

$$\begin{aligned} |z_{l+}^{Mj}| &\leqslant h \cdot \vartheta \quad \text{for } x^M \in Q,\ x^{j(M)} \in \operatorname{int} Q\ (j = 1, \ldots, n), \\ |z_{l-}^{Mj}| &\leqslant h \cdot \vartheta \quad \text{for } x^M \in Q,\ x^{-j(M)} \in \operatorname{int} Q\ (j = 1, \ldots, n) \end{aligned} \tag{11.1}$$

for $l = 1, \ldots, p$ at the nodal points x^M on the boundary ∂Q of the set Q.

3° There exists a positive constant $L > 0$ (independent of the mesh size h) such that the second order difference quotients satisfy the relations:

$$\begin{aligned} &|z_l^{Mjj} - z_l^{Pjj}| \leqslant h \cdot L, \quad |z_{l++}^{Mij} - z_{l++}^{Pij}| \leqslant h \cdot L, \\ &|z_{l-+}^{Mij} - z_{l-+}^{Pij}| \leqslant h \cdot L, \quad |z_{l--}^{Mij} - z_{l--}^{Pij}| \leqslant h \cdot L, \\ &|z_{l+-}^{Mij} - z_{l+-}^{Pij}| \leqslant h \cdot L \quad (i \neq j)\ (l = 1, \ldots, p), \end{aligned} \tag{11.2}$$

at the nodal points x^M and x^P, $P = s(M)$ $(s = \pm 1, \pm 2, \ldots, \pm n)$, the distance between x^M and x^P being h in the direction of the x_s-axis.

4° Suppose that

$$(11.3)\qquad |z_l^{Mij}| \leqslant \Lambda \quad (i, j = 1, \ldots, n;\ l = 1, \ldots, p)\ (x^M \in \operatorname{int} Q),$$

where the constant Λ is independent of the mesh size h.

5° We also suppose that z_l^M takes on prescribed values

$$(11.4)\qquad z_l^M = 0 \quad \text{for } x^M \in \partial Q\ (l = 1, \ldots, p),$$

at the nodal points x^M on the boundary ∂Q of the set Q.

6° Finally, we suppose that the linear systems (10.1) (10.2) of difference inequalities are of the elliptic type in the sense of Definition 2, cf. Section 10.

Let us denote

$$(11.5)\qquad \varkappa = \tfrac{1}{2}\cdot(\Lambda+\vartheta)^{-1/2}, \qquad \tfrac{1}{2} \leqslant a < 1,$$

where ϑ occurs in (11.1), Λ in (11.3), and a is an arbitrary fixed number satisfying (11.5).

Let us also define

$$(11.6)\qquad z_l^{A_l} = \max z_l^M, \quad x^{A_l} \in \operatorname{int} Q; \quad z_l^{B_l} = \min z_l^M, \quad x^{B_l} \in \operatorname{int} Q,$$

the maximum and minimum being taken for $x^M \in Q$.

12. Assumptions H(LS) of Section 11 permit us to recall the basic Theorem A, cf. Theorem 6 in [2]:

THEOREM A. *Let us suppose that the linear systems* I *and* II *of difference inequalities, cf.* (10.1) *and* (10.2), *are of the elliptic type in the sense of Definition* 2, *cf. Section* 10.

Let us also suppose that the functions z_l^M $(l = 1, \ldots, p)$ $(x^M \in Q)$, *satisfy assumptions* H(LS), *cf. Section* 11, *and the relation*

$$(12.1)\qquad 0 < \varepsilon(h) \to 0 \quad \text{as } h \to 0,$$

where $\varepsilon(h)$ *stands on the right-hand member of* (10.1), (10.2).

Under these assumptions:

1° *We have the convergence*

$$(12.2)\qquad z_l^M \to 0, \quad \text{as } h \to 0\ (l = 1, \ldots, p)\ (x^M \in Q).$$

2° *If h is a sufficiently small positiqe number, i.e., h satisfies the condition*

$$(12.3)\qquad [2\cdot(\varkappa h^a + h)]^2 + (n-1)\cdot(2h)^2 < \delta_1, \qquad \tfrac{1}{2} \leqslant a < 1,\ 0 < h < 1,$$

cf. notations (11.5), *where δ_1 stands for a lower bound of the characteristic values, cf.* (10.4), *then we have the following estimate:*

$$(12.4)\qquad -\Omega^{B_l}(h) \leqslant z_l^M \leqslant +\Omega^{A_l}(h) \quad \text{for } x^M \in Q\ (l = 1, \ldots, p).$$

In formula (12.4) we have $A_l = A_l(h), B_l = B_l(h)$ $(l = 1, \ldots, p)$ and

$$\Omega^{A_l}(h) = \begin{cases} h, & \text{if } z_l^{A_l} < h, \\ \dfrac{\Omega(h)}{1+(p-1)\cdot\gamma}, & \text{if } z_l^{A_l} \geqslant h, \end{cases} \tag{12.5}$$

$$\Omega^{B_l}(h) = \begin{cases} h, & \text{if } z_l^{B_l} > -h, \\ \dfrac{\Omega(h)}{1+(p-1)\cdot\gamma}, & \text{if } z_l^{B_l} \leqslant -h, \end{cases} \tag{12.6}$$

cf. notations (11.6).

The function $\Omega(h)$ in (12.5), (12.6) is defined by

$$\Omega(h) = -\eta^{-1}\cdot[\omega(h)+D(h)+\varepsilon(h)], \tag{12.7}$$

$$\omega(h) = n^2\cdot\delta_2\cdot[2\cdot L\varkappa h^a + n\cdot 2\Lambda\cdot\varkappa^{-1}\cdot h^{1-a} + n^2\cdot\Lambda\cdot\varkappa^{-2}\cdot h^{2(1-a)}], \tag{12.8}$$

$$D(h) = n\beta h\Lambda, \tag{12.9}$$

cf. (10.9), where δ_2 stands for an upper bound of the characteristic values cf. (10.4).

13. With Theorem A at hand we can return to the problem of convergence of the difference method (5.7).

The idea of what should now be done is the following: We shall first define the functions

$$z_l^M = r_l^M \quad (l = 1, \ldots, p) \ (x^M \in Q), \tag{13.1}$$

the errors $r_l^M = u_l^M - v_l^M$ being defined by (5.9).

Then we shall prove that the functions z_l^M defined by (13.1) fulfil two linear system of difference inequalities I and II, cf. (10.1), (10.2), as well as the remaining Assumptions H(LS), cf. Section 11, which form a basis for Theorem A.

As a result we shall obtain from Theorem A the desired convergence of the difference method (5.7) and the corresponding estimate for r_l^M.

14. THEOREM 1. *Let us suppose that Assumptions* H *are fulfilled cf. Section* 5. *Suppose, in addition, that the errors* r_l^M $(l = 1, \ldots, p)$ *are defined by formula* (5.9) *and the quantity* $\varepsilon(h)$ *by* (9.2) *and* (9.1).

Under these assumptions the functions r_l^M *satisfy the following linear systems of difference inequalities of elliptic type*:

$$\sum_{i,j=1}^{n} a_{lij}^M\cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^M\cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^M\cdot r_k^M \geqslant -\varepsilon(h), \tag{14.1}$$

$$\sum_{i,j=1}^{n} a_{lij}^M\cdot r_l^{Mij} + \sum_{j=1}^{n} b_{lj}^M\cdot r_l^{Mj} + \sum_{k=1}^{p} c_{lk}^M\cdot r_k^M \leqslant +\varepsilon(h) \tag{14.2}$$

for $l = 1, \ldots, p$ *and* $x^M \in \operatorname{int} Q$.

In formula (14.1) *and* (14.2), $\varepsilon(h)$ *satisfies relation* (9.3), *and the coefficients* a^M_{lij}, b^M_{lj}, c^M_{lk} *are defined by*

$$(14.3)\qquad a^M_{lij} = f^l_{w_{ij}}(\sim),\qquad b^M_{lj} = f^l_{q_j}(\sim),\qquad c^M_{lk} = f^l_{u_k}(\sim)$$

for $l = 1, \ldots, p$; $i = 1, \ldots, n$; $j = 1, \ldots, n$, *the derivatives being taken at a suitable point* $(\sim)$.

Proof. From (9.1) and (5.7) we obtain

$$(14.4)\qquad f^l(x^M, u^M, u_l^{M\triangle}, u_l^{M\square}) - f^l(x^M, v^M, v_l^{M\triangle}, v_l^{M\square}) = \varepsilon_l^M$$

for $l = 1, \ldots, p$ $(x^M \in \operatorname{int} Q)$. Now we can apply the mean value theorem to the left-hand member of (14.4) and we get by (3.7) and (3.8):

$$(14.5)\qquad \sum_{k=1}^{p} f^l_{u_k}(\sim)\cdot r_k^M + \sum_{j=1}^{n} f^l_{q_j}(\sim)\cdot r_l^{Mj} + \sum_{i,j=1}^{n} f^l_{w_{ij}}(\sim)\cdot r_l^{Mij} = \varepsilon_l^M,$$

the derivatives being taken at the suitable point $(\sim)$.

(14.5) can be written in the form

$$(14.6)\qquad \sum_{i,j=1}^{n} a^M_{lij}\cdot r_l^{Mij} + \sum_{j=1}^{n} b^M_{lj}\cdot r_l^{Mj} + \sum_{k=1}^{p} c^M_{lk}\cdot r_k^M = \varepsilon_l^M,$$

because of (14.3); hence from equality (14.6) and the definition of $\varepsilon(h)$, cf. (9.2), we obtain two systems of inequalities:

$$(14.7)\qquad \sum_{i,j=1}^{n} a^M_{lij}\cdot r_l^{Mij} + \sum_{j=1}^{n} b^M_{lj}\cdot r_l^{Mj} + \sum_{k=1}^{p} c^M_{lk}\cdot r_k^M \geqslant -\varepsilon(h),$$

and

$$(14.8)\qquad \sum_{i,j=1}^{n} a^M_{lij}\cdot r_l^{Mij} + \sum_{j=1}^{n} b^M_{lj}\cdot r_l^{Mj} + \sum_{k=1}^{p} c^M_{lk}\cdot r_k^M \leqslant +\varepsilon(h)$$

for $l = 1, \ldots, p$ $(x^M \in \operatorname{int} Q)$.

This ends the proof of Theorem 1.

15. THEOREM 2. *Let us suppose that Assumptions* H *are satisfied, cf. Section* 5, *and* $r_l^M = u_l^M - v_l^M$ $(l = 1, \ldots, p)$, *cf.* (5.9), *where* $u_l(x)$ $(l = 1, \ldots, p)$ *is the solution of the non-linear system of differential equations* (5.3) *of the elliptic type in the sense of Definition* 1, *cf. Section* 4, *and* v_l^M $(l = 1, \ldots, p)$ *is the solution of the difference equation* (5.7).

Under these assumptions:

1° *the difference method is convergent, i.e.*

$$(15.1)\qquad r_l^M \to 0,\quad \text{as } h \to 0\ (l = 1, \ldots, p)\ (x^M \in Q),$$

2° *If* h *is a sufficiently small positive number, i.e.* h *satisfies the condition*

$$(15.2)\qquad [2\cdot(\varkappa h^a + h)]^2 + (n-1)\cdot(2h)^2 < \delta_1,\qquad \tfrac{1}{2} \leqslant a < 1,\ 0 < h < 1,$$

Thus the functions $z_l^M = r_l^M$ $(l = 1, \ldots, p)$ satisfy all the assumptions of Theorem A, cf. Section 12; hence we get from Theorem A the convergence of the difference method, cf. (15.1), as well as the desired error estimate (15.3).

This ends the proof of Theorem 2.

References

[1] Z. Kowalski, *A difference method for a non-linear elliptic equation with mixed derivatives*, Ann. Polon. Math. (in preparation),

[2] — *Systems of difference inequalities of the elliptic type*, ibidem (in preparation).

[3] — and A. Pliś, *Difference inequalities of the elliptic type*, ibidem 26 (1972), p. 239–251.

[4] M. Malec, *Inégalités aux différences finies du type elliptique*, Bull. Acad. Polon. Sci., Sér. sci. math., astr., phys. 23 (1975), p. 859–863.

[5] — *Schéma des différences finies pour une équation non linéaire particlle du type elliptique avec dérivées mixtes*, Ann. Polon. Math. 34 (1977), p. 119–123.

[6] — *Système d'inégalités différences finies du type elliptique*, ibidem 33 (1977), p. 235–239.

[7] — *Schéma de différences finies pour un système d'équations non linéaires particlles elliptiques aux dérivées mixtes et avec les conditions aux limites du type de Dirichlet*, ibidem 33 (1977), p. 241–246.

[8] — *Sur une méthode des différences finies pour l'équation différentielle non linéaire elliptique avec dérivées mixtes et la condition aux limites du type de Neumann*, Zeszyty Nauk. Uniw. Jagiell. 475 (1977), p. 159–167.

[9] A. Mostowski i M. Stark, *Elementy algebry wyższej*, Warszawa 1950.

[10] A. Pliś, *Loss of uniqueness property in difference approximation of a Dirichlet problem*, Comm. Math. 14 (1970), p. 97–99.

[11] А. А. Самарский, *Введение в теорию разностных схем*, Москва 1971.

Reçu par la Rédaction le 6. 12. 1974