ANNALES
POLONICI MATHEMATICI
XLIV (1984)

On solutions of a certain functional-integral equation

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Abstract. In this paper there are investigated solutions of equation (1), where f, g and h are given functions. Some of obtained results are applied to investigate integrable solutions of equation (2).

In the present paper we shall be concerned with solutions of the equation

$$\varphi(x) = \int_0^x g\,d\varphi\circ f + h(x), \tag{1}$$

where f, g and h are given real-valued functions of a real variable. In two final theorems we shall apply preceding results to consider integrable solutions of the equation

$$\varphi(x) = g(x)\,\varphi[f(x)] + h(x). \tag{2}$$

Let $I = [0, a]$ or $I = [0, a)$, where $0 < a \leqslant \infty$. Put $I_0 := I\setminus\{0\}$ and assume that:

(i) The function f is continuous and strictly increasing in I, $0 < f(x) < x$ for every $x \in I_0$.

(ii) The function g is continuous in I and $g(x) \neq 0$ for every $x \in I_0$.

(iii) The function h is of finite variation in I.

For an $x_0 \in I$ we put $x_n := f^n(x_0)$, $n \in N$. Moreover, we define the sequence of functions

$$G_0 := 1, \quad G_n := \prod_{k=0}^{n-1} g\circ f^k, \quad n\in N.$$

DEFINITION 1. Let hypotheses (i), (ii), (iii) be fulfilled. We say that a real-valued function φ is a *solution of equation* (1) in I if and only if φ is of finite variation in I and satisfies equality (1) for every $x \in I_0$.

At first we shall consider the homogeneous equation

$$\varphi(x) = \int_0^x g d\varphi \circ f. \tag{3}$$

LEMMA 1. *Suppose that hypotheses* (i) *and* (ii) *are fulfilled. If* φ *is a solution of equation* (3) *and for a certain* $x_0 \in I_0$, $\varphi|_{[x_1, x_0]} = c$, *then* $\varphi|_{(0, x_0]} = c$.

Proof. Put

$$\varphi^*(x) := \operatorname{Var} \varphi|_{[0,x]}, \qquad x \in I_0,$$

and note that for every $n \in N_0$ we can find a $\xi_n \in [x_{n+1}, x_n]$ such that

$$\varphi^*(x_n) - \varphi^*(x_{n+1}) = \int_{x_{n+1}}^{x_n} |g|\, d\varphi^* \circ f = |g(\xi_n)| [\varphi^*(x_{n+1}) - \varphi^*(x_{n+2})]. \tag{4}$$

Assumptions of lemma imply

$$\varphi^*(x_1) = \varphi^*(x_0).$$

Hence and by (ii) and (4) we get

$$\varphi^*(x_n) = \varphi^*(x_0) \quad \text{for } n \in N.$$

Therefore φ^* is a constant function in I_0. Since $\varphi|_{[x_1, x_0]} = c$, finally we get $\varphi|_{(0, x_0]} = c$.

Let hypotheses (i), (ii), (iii) be fulfilled. Let us fix an $x_0 \in I_0$. For a given function $\varphi\colon I \to \boldsymbol{R}$ of finite variation we define the sequence of functions $\varphi_n\colon [x_1, x_0] \to \boldsymbol{R}$ of finite variation by the formula

$$\begin{aligned} &\varphi_0(x) := \varphi(x), && x \in [x_1, x_0], \\ &\varphi_n(x) := \varphi_0(x) - \sum_{k=0}^{n-1} \int_{x_1}^{x} G_k dh \circ f^k, && x \in (x_1, x_0], \quad n \in N, \\ &\varphi_n(x_1) := \varphi_0(x_1), && n \in N. \end{aligned} \tag{5}$$

Moreover, we put

$$\begin{aligned} &\varphi_n^*(x) := \operatorname{Var} \varphi_n|[x_1, x], && x \in (x_1, x_0], \ n \in N_0, \\ &\varphi_n^*(x_1) := \varphi_n^*(x_1+) - |\varphi_n(x_1+) - \varphi_n(x_1)|, && n \in N_0. \end{aligned} \tag{6}$$

THEOREM 1. *Suppose that hypotheses* (i), (ii), (iii) *are fulfilled. If* φ *is a solution of equation* (1), *then for every* $x_0 \in I_0$

$$\sum_{n=0}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_n^*}{|G_n|} < \infty, \tag{7}$$

where functions φ_n^* *are given by* (5) *and* (6).

Moreover,

(8) $$\mu = g(0)[\mu-\varphi(0)]+h(0+),$$

where

(9) $$\mu := \varphi(x_0)-\sum_{n=0}^{\infty}\int_{x_1}^{x_0}\frac{d\varphi_n}{G_n}.$$

Proof. We shall show by induction that

(10) $$\varphi_n(x) = \varphi_0(x_1)+\int_{x_1}^{x} G_n d\varphi\circ f^n, \qquad x\in(x_1, x_0],$$

for every $n\in N_0$.

For $n=0$ we get (10) immediately by (5). Assume that (10) holds for a certain $n\in N_0$. Then by (5), (1) and (10)

$$\varphi_{n+1}(x) = \varphi_n(x)-\int_{x_1}^{x} G_n dh\circ f^n = \varphi_0(x_1)+\int_{x_1}^{x} G_n d\varphi\circ f^n-\int_{x_1}^{x} G_n dh\circ f^n$$
$$= \varphi_0(x_1)+\int_{x_1}^{x} G_n d(\varphi\circ f^n-h\circ f^n) = \varphi_0(x_1)+\int_{x_1}^{x} G_n(y)\, d\Big(\int_0^{y} g\circ f^n d\varphi\circ f^{n+1}\Big)$$
$$= \varphi_0(x_1)+\int_{x_1}^{x} G_n g\circ f^n d\varphi\circ f^{n+1} = \varphi_0(x_1)+\int_{x_1}^{x} G_{n+1} d\varphi\circ f^{n+1}.$$

Now by (10) and (5) we obtain

(11) $$\varphi_n(x_1+)-\varphi_n(x_1) = G_n(x_1)[\varphi(x_{n+1}+)-\varphi(x_{n+1})], \qquad n\in N_0.$$

Further by (6), (10) and (11) we have

(12) $$\varphi_n^*(x) = |\varphi_n(x_1+)-\varphi_n(x_1)|+\lim_{c\to x_1+}\operatorname{Var}\varphi_n|[c, x]$$
$$= |\varphi_n(x_1+)-\varphi_n(x_1)|+\lim_{c\to x_1+}\int_c^{x}|G_n|\, d\varphi^*\circ f^n$$
$$= |\varphi_n(x_1+)-\varphi_n(x_1)|+\int_{x_1}^{x}|G_n|\, d\varphi^*\circ f^n-$$
$$-|G_n(x_1)|\,|\varphi(x_{n+1}+)-\varphi(x_{n+1})|$$
$$= \int_{x_1}^{x}|G_n|\, d\varphi^*\circ f^n, \qquad x\in(x_1, x_0],\ n\in N_0.$$

Hence and again by (6) and (11)

$$\int_{x_1}^{x_0}\frac{d\varphi_n^*}{|G_n|} = \frac{\varphi_n^*(x_1+)-\varphi_n^*(x_1)}{|G_n(x_1)|}+\lim_{c\to x_1+}\int_c^{x_0}\frac{1}{|G_n(x)|}\, d\Big(\int_{x_1}^{x}|G_n|\, d\varphi^*\circ f^n\Big)$$

$$= \left|\frac{\varphi_n(x_1+)-\varphi_n(x_1)}{G_n(x_1)}\right| + \lim_{c\to x_1+} \int_c^{x_0} d\varphi^* \circ f^n$$

$$= \left|\frac{\varphi_n(x_1+)-\varphi_n(x_1)}{G_n(x_1)}\right| + \varphi^*(x_n) - \varphi^*(x_{n+1}) - |\varphi(x_{n+1}+)-\varphi(x_{n+1})|$$

$$= \varphi^*(x_n) - \varphi^*(x_{n+1}).$$

Finally we have

$$\sum_{n=0}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_n^*}{|G_n|} = \sum_{n=0}^{\infty} [\varphi^*(x_n) - \varphi^*(x_{n+1})] = \varphi^*(x_0) - \varphi^*(0+) < \infty.$$

To prove the second part of this theorem note that according to (1) we have

$$\varphi(0+) = g(0)[\varphi(0+)-\varphi(0)] + h(0+). \tag{13}$$

In view of (10) and (11) we get

$$\int_{x_1}^{x_0} \frac{d\varphi_k}{G_k} = \frac{\varphi_k(x_1+)-\varphi_k(x_1)}{G_k(x_1)} + \lim_{c\to x_1+} \int_c^{x_0} \frac{1}{G_k(x)} d\left(\int_{x_1}^{x} G_k d\varphi\circ f^k\right)$$

$$= \varphi(x_{k+1}+) - \varphi(x_{k+1}) + \lim_{c\to x_1+} \int_c^{x_0} d\varphi\circ f^k = \varphi(x_k) - \varphi(x_{k+1}).$$

Hence and by (9)

$$\varphi(0+) = \varphi(x_0) - \sum_{k=0}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_k}{G_k} = \mu.$$

Consequently, in view of (13) we get (8), which was to be proved.

Now we shall show that solutions of equation (1) depend on an arbitrary function.

THEOREM 2. *Let hypotheses* (i), (ii), (iii) *be fulfilled and let* $x_0 \in I_0$. *Suppose that* φ_0: $[x_1, x_0] \to \boldsymbol{R}$ *is a function of finite variation and the sequence* $(\varphi_n^*)_{n=0}^{\infty}$ *given by* (5) *and* (6) *satisfies condition* (7). *Further we assume that* η *is a real number such that*

$$\mu = g(0)[\mu - \eta] + h(0+), \tag{14}$$

where

$$\mu := \varphi_0(x_0) - \sum_{n=0}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_n}{G_n}. \tag{15}$$

Then equation (1) *has exactly one solution* φ: $I \to \boldsymbol{R}$ *such that* $\varphi|_{[x_1, x_0]} = \varphi_0$ *and* $\varphi(0) = \eta$.

Proof. Let us put

$$(16)\qquad \varphi(x) := \int_{x_1}^{f^{-n}(x)} \frac{d\varphi_n}{G_n} + \varphi_0(x_0) - \sum_{k=0}^{n} \int_{x_1}^{x_0} \frac{d\varphi_k}{G_k}, \quad x \in (x_{n+1}, x_n],\ n \in N_0,$$

$$\varphi(0) := \eta.$$

Note that $\varphi|_{[x_1, x_0]} = \varphi_0$. Given an $\varepsilon > 0$, in view of (7), we can find $n_0 \in N$ such that for $n \geqslant n_0$

$$\sum_{k=n}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_k^*}{|G_k|} < \varepsilon.$$

Hence, in view of (15) and (16), we have for $x \in (x_{n+1}, x_n]$, $n \geqslant n_0$,

$$|\varphi(x) - \mu| = \left| \int_{x_1}^{f^{-n}(x)} \frac{d\varphi_n}{G_n} + \sum_{k=n+1}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_k}{G_k} \right| \leqslant \sum_{k=n}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_k^*}{|G_k|} < \varepsilon,$$

thus

$$(17)\qquad \varphi(0+) = \varphi_0(x_0) - \sum_{k=0}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_k}{G_k} = \mu.$$

On account of (16) we obtain for $k \in N_0$

$$(18)\qquad \begin{aligned} \varphi(x_{k+1}+) - \varphi(x_{k+1}) &= \frac{\varphi_k(x_1+) - \varphi_k(x_1)}{G_k(x_1)}, \\ \varphi(x_k) - \varphi(x_k-) &= \frac{\varphi_k(x_0) - \varphi_k(x_0-)}{G_k(x_0)}. \end{aligned}$$

Now we shall verify that φ is a function of finite variation. We have by (16) and (18)

$$\begin{aligned} \operatorname{Var} \varphi\,|[x_{k+1}, x_k] &= |\varphi(x_{k+1}+) - \varphi(x_{k+1})| + \lim_{c \to x_{k+1}+} \int_{f^{-k}(c)}^{x_0} \frac{d\varphi_k^*}{|G_k|} \\ &= |\varphi(x_{k+1}+) - \varphi(x_{k+1})| + \int_{x_1}^{x_0} \frac{d\varphi_k^*}{|G_k|} - \left| \frac{\varphi_k(x_1+) - \varphi_k(x_1)}{G_k(x_1)} \right| \\ &= \int_{x_1}^{x_0} \frac{d\varphi_k^*}{|G_k|}. \end{aligned}$$

Hence and according to (7) we get

$$\operatorname{Var}\varphi|[0, x_0] = |\varphi(0+)-\varphi(0)| + \sum_{k=0}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_k^*}{|G_k|} < \infty.$$

Now we shall prove that φ fulfils equation (1). Let us fix an $n \in N_0$ and an $x \in (x_{n+1}, x_n]$. Evidently we have

$$\int_0^x g\,d\varphi\circ f = g(0)[\varphi(0+)-\varphi(0)] + \lim_{c \to x_{n+1}+} \int_c^x g\,d\varphi\circ f + \sum_{k=n+1}^{\infty} \int_{(x_{k+1}, x_k]} g\,d\varphi\circ f. \tag{19}$$

We get by (16)

$$\int_p^q g\,d\varphi\circ f = \int_{f(p)}^{f(q)} g\circ f^{-1}d\varphi = \int_{f(p)}^{f(q)} g\circ f^{-1}(x)\,d\left(\int_{x_1}^{f^{-(k+1)}(x)} \frac{d\varphi_{k+1}}{G_{k+1}}\right) \tag{20}$$

$$= \int_{f(p)}^{f(q)} g\circ f^{-1} \frac{d\varphi_{k+1}\circ f^{-(k+1)}}{G_{k+1}\circ f^{-(k+1)}} = \int_p^q \frac{d\varphi_{k+1}\circ f^{-k}}{G_k\circ f^{-k}}$$

for every $p, q \in (x_{k+1}, x_k)$, $p < q$, $k \in N_0$, whence

$$\lim_{c \to x_{n+1}+} \int_c^x g\,d\varphi\circ f = \int_{x_{n+1}}^x \frac{d\varphi_{n+1}\circ f^{-n}}{G_n\circ f^{-n}} - \frac{\varphi_{n+1}(x_1+)-\varphi_{n+1}(x_1)}{G_n(x_1)} \tag{21}$$

$$= \int_{x_1}^{f^{-n}(x)} \frac{d\varphi_{n+1}}{G_n} - \frac{\varphi_{n+1}(x_1+)-\varphi_{n+1}(x_1)}{G_n(x_1)}.$$

The following equality we get just like (20):

$$\int_{(x_{k+1}, x_k)} g\,d\varphi\circ f = \int_{(x_{k+1}, x_k)} \frac{d\varphi_{k+1}\circ f^{-k}}{G_k\circ f^{-k}} = \int_{(x_1, x_0)} \frac{d\varphi_{k+1}}{G_k}, \quad k \in N_0. \tag{22}$$

Further, we have by (18)

$$g(x_k)[\varphi(x_{k+1}+)-\varphi(x_{k+1}-)] \tag{23}$$

$$= \frac{\varphi_k(x_1+)-\varphi_k(x_1)}{G_{k-1}(x_1)} + \frac{\varphi_{k+1}(x_0)-\varphi_{k+1}(x_0-)}{G_k(x_0)}, \quad k \in N.$$

Now on account of (22), (23) and (5) we obtain

$$(24)\quad \int\limits_{(x_{k+1},x_k]} gd\varphi\circ f = \int\limits_{(x_{k+1},x_k)} gd\varphi\circ f + g(x_k)[\varphi(x_{k+1}+)-\varphi(x_{k+1}-)]$$

$$= \int\limits_{x_1}^{x_0} \frac{d\varphi_{k+1}}{G_k} - \frac{\varphi_{k+1}(x_0)-\varphi_{k+1}(x_0-)}{G_k(x_0)} - \frac{\varphi_{k+1}(x_1+)-\varphi_{k+1}(x_1)}{G_k(x_1)} + \frac{\varphi_{k+1}(x_0)-\varphi_{k+1}(x_0-)}{G_k(x_0)} + \frac{\varphi_k(x_1+)-\varphi_k(x_1)}{G_{k-1}(x_1)}$$

$$= \int\limits_{x_1}^{x_0} \frac{d\varphi_k}{G_k} - [h(x_k)-h(x_{k+1})] + \left[\frac{\varphi_k(x_1+)-\varphi_k(x_1)}{G_{k-1}(x_1)} - \frac{\varphi_{k+1}(x_1+)-\varphi_{k+1}(x_1)}{G_k(x_1)}\right], \quad k\in N.$$

Since $\lim\limits_{k\to\infty} \int\limits_{x_1}^{x_0} \frac{d\varphi_k^*}{|G_k|} = 0$, we have

$$\lim_{k\to\infty} \frac{\varphi_{k+1}(x_1+)-\varphi_{k+1}(x_1)}{G_k(x_1)} = 0.$$

Thus by (24) we get

$$(25)\quad \sum_{k=n+1}^{\infty} \int\limits_{(x_{k+1},x_k]} gd\varphi\circ f = \sum_{k=n+1}^{\infty} \int\limits_{x_1}^{x_0} \frac{d\varphi_k}{G_k} - h(x_{n+1}) + h(0+) + \frac{\varphi_{n+1}(x_1+)-\varphi_{n+1}(x_1)}{G_n(x_1)}.$$

Finally according to (14), (17), (19), (21), (25) and (5) we conclude that

$$\int\limits_0^x gd\varphi\circ f + h(x) = \varphi(0+) - h(0+) + \int\limits_{x_1}^{f^{-n}(x)} \frac{d\varphi_{n+1}}{G_n} - \frac{\varphi_{n+1}(x_1+)-\varphi_{n+1}(x_1)}{G_n(x_1)} + $$

$$+ \sum_{k=n+1}^{\infty} \int\limits_{x_1}^{x_0} \frac{d\varphi_k}{G_k} - h(x_{n+1}) + h(0+) + \frac{\varphi_{n+1}(x_1+)-\varphi_{n+1}(x_1)}{G_n(x_1)} + h(x)$$

$$= \varphi(0+) + \int_{x_1}^{f^{-n}(x)} \frac{d\varphi_n}{G_n} - [h(x) - h(x_{n+1})] + \sum_{k=n+1}^{\infty} \int_{x_1}^{x_0} \frac{d\varphi_k}{G_k} - h(x_{n+1}) + h(x)$$

$$= \int_{x_1}^{f^{-n}(x)} \frac{d\varphi_n}{G_n} + \varphi_0(x_0) - \sum_{k=0}^{n} \int_{x_1}^{x_0} \frac{d\varphi_k}{G_k} = \varphi(x).$$

So φ given by (16) fulfils equation (1) in $(0, x_0]$. The function φ may be uniquely extend on I_0 to a solution of equation (1). Uniqueness of this solution we can deduce from Lemma 1.

Remark 1. It follows from Theorems 1, 2 and Lemma 1 that any solution φ of equation (1) may be obtained as a unique extension of the function $\varphi_0 := \varphi|_{[x_1, x_0]}$, where x_0 is an arbitrary point of I_0.

Remark 2. Let us consider solutions of finite variation of the equation

$$\varphi(x) = s\varphi[f(x)] + h(x), \tag{26}$$

where s is a real number. Some results concerning uniqueness and dependence on an arbitrary function of these solutions may be deduced from Theorems 1 and 2, where $g = s$ in I. These results have been proved otherwise in Matkowski, Zdun [3] (cf. [3], Theorems 1, 2 and Corollaries 1, 2).

Now we assume that:

(iv) The function f is absolutely continuous and strictly increasing in I, $0 < f(x) < x$ for every $x \in I_0$.

(v) The function g is integrable in I and $g \neq 0$ a.e.

(vi) The function h is absolutely continuous in I.

LEMMA 2. *Suppose that hypotheses* (iv) *and* (v) *are fulfilled. If φ is an absolutely continuous solution of equation* (3) *and, for a certain $x_0 \in I_0$, $\varphi|_{[x_1, x_0]} = c$, then $\varphi = 0$.*

Proof. Put

$$\varphi^*(x) := \operatorname{Var} \varphi|_{[0,x]} \quad \text{for } x \in I_0.$$

We have by (3)

$$\varphi^*(x_n) - \varphi^*(x_{n+1}) = \int_{x_{n+1}}^{x_n} |g|\,|(\varphi \circ f)'| \quad \text{for } n \in N_0. \tag{27}$$

Since $\varphi^*(x_1) = \varphi^*(x_0)$ and $g \neq 0$ a.e., we obtain by (27) that $\varphi|_{(x_{n+1}, x_n]} = c_n$, $n \in N$, where $c_0 := c$. According to continuity of φ and (3) we conclude that $\varphi = 0$.

The following two results have proofs analogous to those presented for previous theorems.

THEOREM 3. *Suppose that hypotheses* (iv), (v), (vi) *are fulfilled. If* φ *is an absolutely continuous solution of equation* (1), *then for every* $x_0 \in I_0$

$$\sum_{n=0}^{\infty} \int_{x_1}^{x_0} \left|\frac{\varphi_n'}{G_n}\right| < \infty \tag{28}$$

and

$$\varphi(x_0) - \sum_{n=0}^{\infty} \int_{x_1}^{x_0} \frac{\varphi_n'}{G_n} = h(0),$$

where functions φ_n, $n \in N_0$, *are given by* (5).

THEOREM 4. *Let hypotheses* (iv), (v), (vi) *be fulfilled and let* $x_0 \in I_0$. *Suppose that* φ_0: $[x_1, x_0] \to \boldsymbol{R}$ *is an absolutely continuous function and the sequence* $(\varphi_n)_{n=0}^{\infty}$ *given by* (5) *satisfies condition* (28). *Further, we assume that*

$$\varphi_0(x_0) - \sum_{n=0}^{\infty} \int_{x_1}^{x_0} \frac{\varphi_n'}{G_n} = h(0). \tag{29}$$

Then equation (1) *has exactly one absolutely continuous solution* φ: $I \to \boldsymbol{R}$ *such that* $\varphi|_{[x_1, x_0]} = \varphi_0$.

Remark analogous to Remark 1 is also true:

Remark 3. Every absolutely continuous solution φ of equation (1) may be obtained as unique extension of the function $\varphi_0 := \varphi|_{[x_1, x_0]}$, where x_0 is an arbitrary point of I_0.

Remark 4. Let us consider absolutely continuous solutions of equation (26). Again note that some results concerning uniqueness and dependence on an arbitrary function of these solutions may be deduced from Theorems 3 and 4, where $g = s$ in I. These results have been proved otherwise in Matkowski [2] (cf. [2], Theorem 2.11).

From Theorems 3 and 4 we shall deduce two following corollaries.

COROLLARY 1. *Let hypotheses* (iv), (v), (vi) *be fulfilled and let* φ *be an absolutely continuous solution of equation* (1). *If for a certain* $x_0 \in I_0$

$$\sum_{n=0}^{\infty} \prod_{k=0}^{n-1} u_k(x_0) = \infty, \tag{30}$$

where

$$u_k(x_0) := \operatorname*{inf\,ess}_{[x_{k+1}, x_k]} |g|^{-1}, \qquad k \in N_0, \tag{31}$$

then there exists a subsequence $(k_n)_{n\in N}$ of the sequence of natural numbers such that

$$\lim_{n\to\infty} \sum_{k=0}^{k_n-1} \int_{x_1}^{x} G_k(h\circ f^k)' = \varphi(x)-\varphi(x_1) \tag{32}$$

uniformly in $[x_1, x_0]$ and

$$\lim_{n\to\infty} \sum_{k=0}^{k_n-1} G_k(h\circ f^k)' = \varphi' \quad \text{a.e. in } [x_1, x_0]. \tag{33}$$

Proof. According to Theorem 3 the sequence $(\varphi_n)_{n=0}^{\infty}$ defined by (5) satisfies condition (28). Note that

$$\sum_{n=0}^{\infty} \int_{x_1}^{x_0} \left|\frac{\varphi_n'}{G_n}\right| \geqslant \sum_{n=0}^{\infty} \operatorname{Var} \varphi_n|[x_1, x_0] \prod_{k=0}^{n-1} u_k(x_0).$$

Hence, in view of (28) and (29) we get

$$\liminf_{n\to\infty} \operatorname{Var} \varphi_n|[x_1, x_0] = 0.$$

Let $(m_n)_{n\in N}$ be a sequence of natural numbers such that

$$\lim_{n\to\infty} \operatorname{Var} \varphi_{m_n}|[x_1, x_0] = 0. \tag{34}$$

Since $\varphi_n(x_1) = \varphi_0(x_1)$ for $n\in N$, we have $\lim_{n\to\infty} \varphi_{m_n} = \varphi_0(x_1)$ uniformly in $[x_1, x_0]$. The equality

$$\varphi_n(x) = \varphi_0(x) - \sum_{k=0}^{n-1} \int_{x_1}^{x} G_k(h\circ f^k)', \qquad x\in[x_1, x_0],\ n\in N_0$$

implies

$$\lim_{n\to\infty} \sum_{k=0}^{m_n-1} \int_{x_1}^{\cdot} G_k(h\circ f^k)' = \varphi_0 - \varphi_0(x_1)$$

uniformly in $[x_1, x_0]$.

According to Fatou's lemma and (34) we conclude that

$$0 \leqslant \int_{x_1}^{x_0} \liminf_{n\to\infty} |\varphi_{m_n}'| \leqslant \liminf_{n\to\infty} \int_{x_1}^{x_0} |\varphi_{m_n}'| = \lim_{n\to\infty} \operatorname{Var} \varphi_{m_n}|[x_1, x_0] = 0.$$

Then for a certain subsequence $(k_n)_{n\in N}$ of the sequence $(m_n)_{n\in N}$ we have

$$\lim_{n\to\infty} \varphi_{k_n}' = 0 \quad \text{a.e. in } [x_1, x_0]$$

or

$$\lim_{n\to\infty}\sum_{k=0}^{k_n-1} G_k(h\circ f^k)' = \varphi' \quad \text{a.e. in } [x_1, x_0].$$

Remark 5. Let us replace condition (30) by a stronger one. Suppose that

$$\sum_{n=0}^{\infty}\prod_{k=0}^{m_n-1} u_k(x_0) = \infty, \tag{35}$$

for every subsequence $(m_n)_{n\in N}$ of the sequence of natural numbers, where $u_k(x_0)$, $k\in N_0$, are given by (31). Then the same proof shows that under hypotheses of Corollary 1 we have

$$\sum_{k=0}^{\infty}\int_{x_1}^{\cdot} G_k(h\circ f^k)' = \varphi - \varphi(x_1)$$

uniformly in $[x_1, x_0]$ and

$$\sum_{k=0}^{\infty} G_k(h\circ f^k)' = \varphi' \quad \text{in } [x_1, x_0]$$

in the norm of the space of all Lebesgue integrable functions (in particular, in Lebesgue measure).

Note that conditions (30) and (35) are fulfilled if $|g|\leqslant 1$ a.e. in a neighbourhood of zero.

Remark 6. Some conditions for the convergence or divergence of series

$$\sum_{n=0}^{\infty}\prod_{k=0}^{n-1} u_k(x_0)$$

have been given by Kuczma [1].

COROLLARY 2. *Suppose that hypotheses* (iv), (v), (vi) *are fulfilled. If for a certain* $x_0\in I_0$ *the function* $h'\sum_{k=0}^{\infty}\frac{1}{G_k}$ *is locally integrable and*

$$\sum_{n=0}^{\infty}\prod_{k=0}^{n-1} v_k(x_0) < \infty, \tag{36}$$

where

$$v_k(x_0) := \operatorname*{sup\,ess}_{[x_{k+1}, x_k]} |g|^{-1}, \quad k\in N_0, \tag{37}$$

then equation (1) *has an absolutely continuous solution depending on an arbitrary function. More precisely, for every absolutely continuous function* $\varphi_0\colon [x_1, x_0]\to \boldsymbol{R}$ *fulfilling condition* (29) *there exists exactly one absolutely continuous solution of equation* (1) *such that* $\varphi|_{[x_1,x_0]} = \varphi_0$.

Proof. Let $\varphi_0\colon [x_1, x_0] \to \boldsymbol{R}$ be an absolutely continuous function fulfilling (29). According to Theorem 4 it is sufficient to show that the sequence $(\varphi_n)_{n=0}^{\infty}$ given by (5) satisfies condition (28). Let us note that

$$(38)\qquad \sum_{n=0}^{\infty} \int_{x_1}^{x_0} \left|\frac{\varphi_n'}{G_n}\right| = \sum_{n=0}^{\infty} \int_{x_1}^{x_0} \left|\frac{\varphi_0' - \sum_{k=0}^{n-1} G_k (h\circ f^k)'}{G_n}\right| \leqslant \sum_{n=0}^{\infty} \int_{x_1}^{x_0} \left|\frac{\varphi_0'}{G_n}\right| + \sum_{n=0}^{\infty} \int_{x_1}^{x_0} \sum_{k=0}^{n-1} \left|\frac{G_k}{G_n}(h\circ f^k)'\right|.$$

In view of (36)

$$(39)\qquad \sum_{n=0}^{\infty} \int_{x_1}^{x_0} \left|\frac{\varphi_0'}{G_n}\right| \leqslant \operatorname{Var} \varphi_0 | [x_1, x_0] \sum_{n=0}^{\infty} \prod_{k=0}^{n-1} v_k(x_0) < \infty.$$

Further, we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \left|\frac{G_k}{G_n}(h\circ f^k)'\right| = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \left|\frac{(h\circ f^k)'}{G_{n-k}\circ f^k}\right| = \sum_{n=0}^{\infty} \sum_{k=1}^{n} \left|\frac{(h\circ f^{n-k})'}{G_k\circ f^{n-k}}\right|$$
$$= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \left|\frac{(h\circ f^{n-k})'}{G_k\circ f^{n-k}}\right| = \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \left|\frac{(h\circ f^{n})'}{G_k\circ f^{n}}\right| = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \left|\frac{(h\circ f^{n})'}{G_k\circ f^{n}}\right|,$$

whence

$$(40)\qquad \sum_{n=0}^{\infty} \int_{x_1}^{x_0} \sum_{k=0}^{n-1} \left|\frac{G_k}{G_n}(h\circ f^k)'\right| = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \int_{x_1}^{x_0} \left|\frac{(h\circ f^{n})'}{G_k\circ f^{n}}\right| = \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} \int_{x_{n+1}}^{x_n} \left|\frac{h'}{G_k}\right| \leqslant \int_{0}^{x_0} \sum_{k=1}^{\infty} \left|\frac{h'}{G_k}\right| < \infty.$$

Finally, from (38), (39) and (40) we get that condition (28) holds.

Now we shall apply previous results to present two theorems concerning Lebesgue integrable solutions of equation (2).

Let us assume that:

(vii) The function f is absolutely continuous and strictly increasing in I, $0 < f(x) < x$ for every $x \in I_0$; $f \neq 0$ a.e.

(viii) The function g is measurable and $g \neq 0$ a.e. in I.

(ix) The function h is locally integrable in I.

THEOREM 5. *Let hypotheses* (vii), (viii), (ix) *be fulfilled and let* $\varphi\colon I_0 \to \boldsymbol{R}$

be a locally integrable solution of equation (2). *If* g/f' *is a locally integrable function and condition* (30) *holds for a certain* $x_0 \in I_0$, *where*

$$u_k(x_0) := \operatorname*{inf\,ess}_{[x_{k+1}, x_k]} \frac{f'}{|g|}, \qquad k \in N_0, \tag{41}$$

then there exists a subsequence $(k_n)_{n \in N}$ *of the sequence of natural numbers such that*

$$\lim_{n \to \infty} \sum_{k=0}^{k_n - 1} G_k h \circ f^k = \varphi \qquad \text{a.e. in } [x_1, x_0].$$

Moreover, if condition (35) *is fulfilled for every subsequence* $(m_n)_{n \in N}$ *of the sequence of natural numbers, where* $u_k(x_0)$, $k \in N_0$, *are given by* (41), *then*

$$\sum_{k=0}^{\infty} G_k h \circ f^k = \varphi \qquad \text{in } [x_1, x_0]$$

in the norm of the space of all Lebesgue integrable functions (*in particular in Lebesgue measure*).

Proof. Put

$$\Phi(x) := \int_0^x \varphi, \qquad x \in I$$

and

$$H(x) := \int_0^x h, \qquad x \in I. \tag{42}$$

Then the function Φ is an absolutely continuous solution of the equation

$$\Phi(x) = \int_0^x \frac{g}{f'} (\Phi \circ f)' + H(x). \tag{43}$$

Therefore, by Corollary 1 we get the first part of theorem. To prove the second part it is enough to apply Remark 5.

Problem of the uniqueness of integrable solutions of equation (2) have been investigated in Matkowski [2] (cf. [2], Theorem 2.7). Results, which have been proved there, are different from Theorem 5.

THEOREM 6. *Suppose that hypotheses* (vii), (viii), (ix) *are fulfilled and* g/f' *is a locally integrable function. If for a certain* $x_0 \in I_0$ *the function*

$$h \sum_{n=0}^{\infty} (f^n)'/|G_n|$$

is integrable in $(0, x_0)$ *and condition* (36) *holds, where*

$$v_k(x_0) := \operatorname*{sup\,ess}_{[x_{k+1}, x_k]} \frac{f'}{|g|}, \qquad k \in N_0,$$

then equation (2) *has in* $(0, x_0)$ *the integrable solution depending on an arbitrary function. More precisely, for every integrable function* $\varphi_0\colon (x_1, x_0) \to \boldsymbol{R}$ *there exists essentialy one integrable solution* $\varphi\colon (0, x_0) \to \boldsymbol{R}$ *of equation* (2) *such that* $\varphi|_{(x_1, x_0)} = \varphi_0$ *a.e.*

Proof. Let $\varphi_0\colon (x_1, x_0) \to \boldsymbol{R}$ be an integrable function. Note that

$$\sum_{n=1}^{\infty} \int_{x_1}^{x_0} \left| \frac{\varphi_0 - \sum_{k=0}^{n-1} G_k h\circ f^k}{G_n} \right| (f^n)' \leqslant \sum_{n=1}^{\infty} \int_{x_1}^{x_0} |\varphi_0| \frac{(f^n)'}{|G_n|} + \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \int_{x_1}^{x_0} \left| \frac{G_k h\circ f^k}{G_n} \right| (f^n)'. \tag{44}$$

In view of (36)

$$\sum_{n=1}^{\infty} \int_{x_1}^{x_0} |\varphi_0| \frac{(f^n)'}{|G_n|} \leqslant \int_{x_1}^{x_0} |\varphi_0| \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} v_k(x_0) < \infty. \tag{45}$$

Further, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \int_{x_1}^{x_0} \left| \frac{G_k h\circ f^k}{G_n} \right| (f^n)' &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \int_{x_1}^{x_0} \left| \frac{h\circ f^k}{G_{n-k}\circ f^k} \right| (f^n)' \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} \int_{x_{n+1}}^{x_n} \left| \frac{h\circ f^{k-n}}{G_{n-k}\circ f^{k-n}} \right| \\ &= \sum_{n=1}^{\infty} \sum_{k=1}^{n} \int_{x_{n+1}}^{x_n} \left| \frac{h\circ f^{-k}}{G_k\circ f^{-k}} \right| \\ &= \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \int_{x_{n+1}}^{x_n} \left| \frac{h\circ f^{-k}}{G_k\circ f^{-k}} \right| = \sum_{k=1}^{\infty} \int_{0}^{x_0} \left| \frac{h}{G_k} \right| (f^k)' < \infty. \end{aligned} \tag{46}$$

So we get by (44), (45) and (46)

$$\sum_{n=1}^{\infty} \int_{x_1}^{x_0} \left| \frac{\varphi_0 - \sum_{k=0}^{n-1} G_k h\circ f^k}{G_n} \right| (f^n)' < \infty$$

and we may put

$$c := \sum_{n=1}^{\infty} \int_{x_1}^{x_0} \frac{\varphi_0 - \sum_{k=0}^{n-1} G_k h\circ f^k}{G_n} (f^n)'.$$

Now let us set

$$\Phi_0(x) := \int_{x_1}^{x} \varphi_0 + c, \qquad x \in [x_1, x_0]$$

and define function H by (42). It is easy to show that

$$\Phi_0(x_0) - \sum_{n=0}^{x} \int_{x_1}^{x_0} \frac{\Phi_n'}{\left[\dfrac{G_n}{(f^n)'}\right]} = H(0),$$

where functions Φ_n, $n \in N$, are given by

$$\Phi_n(x) := \Phi_0(x) - \sum_{k=0}^{n-1} \int_{x_1}^{x_0} \frac{G_k}{(f^k)'} (H \circ f^k)'.$$

Now it is sufficient to apply Corollary 2 to equation (43) and function Φ_0.

If we confine ourselves to the integrable solutions of equation (2), then Theorem 6 generalizes Theorem 2.10 which has been proved in Matkowski [2].

References

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Reçu par la Rédaction le 16.08.1979