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On the existence and uniqueness of solutions of the Darboux problem for partial differential-functional equations in a Banach space

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Abstract. In this paper we consider the Darboux problem for partial differential-functional equations of the form

$$(1)\qquad u_{xy}(x,y) = F\big(x,y,u(\cdot,\cdot),u_x(\cdot,\cdot),u_y(\cdot,\cdot),u_{xy}(\cdot,\cdot),\\ u_{xy}(\alpha_1(x,y),\beta_1(x,y)),\ldots,u_{xy}(\alpha_\nu(x,y),\beta_\nu(x,y))\big),$$

with boundary conditions

$$(2)\qquad \begin{aligned} u(x,0) &= \sigma(x) \quad \text{for } 0\leqslant x\leqslant a,\\ u(0,y) &= \tau(y) \quad \text{for } 0\leqslant y\leqslant b, \end{aligned}$$

where $F\colon \Delta\times[C(\Delta,E)]^4\times E^\nu\to E$, $\sigma\colon [0,a]\to E$, $\tau\colon [0,b]\to E$, and E denotes a Banach space.

In this paper we prove the existence and uniqueness of solutions of the Darboux problem (1)–(2) by the method of successive approximations, imposing on the operator F certain general regularity conditions.

We also give the error estimations and a theorem on the continuous dependence of solutions on the right-hand side of equation (1).

The main part of the paper constitutes considerations connected with the above-mentioned problems in the case where the suitable function appearing in the estimation of the norm of increment of operator F is linear. We formulate in this case sufficient conditions for the existence of solutions of equation (1), exploiting the specific features of the dependence of the operator F on the last $\nu+4$ variables; in particular, in these conditions there appear connections between the estimations imposed on the operator F, the functions α_i and β_i, $i=1,\ldots,\nu$.

All our results have been obtained by using the general idea of Ważewski [9] (see also [1], [4], [5]).

Let E be a Banach space with norm $\|\cdot\|$, $\Delta=[0,a]\times[0,b]$, and $u\colon\Delta\to E$ (we shall continue to denote the function u of the variables x and y, $(x,y)\in\Delta$, also by the symbol $u(\cdot,\cdot)$ or $(u(\xi,\eta))_\Delta$).

We shall consider the partial differential-functional equation

$$(1)\qquad u_{xy}(x,y) = F\big(x,y,u(\cdot,\cdot),u_x(\cdot,\cdot),u_y(\cdot,\cdot),u_{xy}(\cdot,\cdot),u_{xy}(\alpha_1(x,y),\\ \beta_1(x,y)),\ldots,u_{xy}(\alpha_\nu(x,y),\beta_\nu(x,y))\big),$$

with boundary conditions

$$(2)\qquad \begin{aligned} u(x, 0) &= \sigma(x) \quad \text{for } 0 \leqslant x \leqslant a, \\ u(0, y) &= \tau(y) \quad \text{for } 0 \leqslant y \leqslant b, \end{aligned}$$

where $\sigma\colon [0, a] \to E$ and $\tau\colon [0, b] \to E$ are functions of the class C^1 satisfying the conditions $\sigma(0) = \tau(0)$, the operator F is defined on a suitable set (more exact assumptions will be given further on) and the known functions $a_i, \beta_i, a_i\colon \Delta \to [0, a]$, $\beta_i\colon \Delta \to [0, b]$, $i = 1, \ldots, \nu$, are continuous.

We shall be interested only in solutions u which are continuous on Δ together with their partial derivatives u_x, u_y, u_{xy}. The set of all such functions will be denoted by $C^*(\Delta, E)$. The problem consisting in finding a solution of equation (1) fulfilling conditions (2) will be called the *Darboux problem*.

In this paper we prove the existence and uniqueness of the Darboux problem (1)–(2) by the method of successive approximations. We also give the error estimations and a theorem on the continuous dependence of solutions on the right-hand side of equation (1). Our results are generalizations of the result of paper [5]. All our results are obtained by using the general idea of Ważewski [9] (see also [1], [4], [5]).

The Darboux problem (1)–(2) is equivalent to the problem of the solution of the equation

$$(3)\qquad z(x, y) = F\Big(x, y, \Big(\sigma(\xi) + \tau(\eta) - \sigma(0) + \int_0^\xi \int_0^\eta z(s, t)\,ds\,dt\Big)_\Delta, \Big(\sigma'(\xi) + \int_0^\eta z(\xi, t)\,dt\Big)_\Delta, \Big(\tau'(\eta) + \int_0^\xi z(s, \eta)\,ds\Big)_\Delta, \big(z(\xi, \eta)\big)_\Delta, z\big(a_1(x, y), \beta_1(x, y)\big), \ldots, z\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big),$$

where

$$u(x, y) = \sigma(x) + \tau(y) - \sigma(0) + \int_0^x \int_0^y z(s, t)\,ds\,dt.$$

Putting in equation (3)

$$\begin{aligned} & f\big(x, y, (z(\xi, \eta))_\Delta, (p(\xi, \eta))_\Delta, (q(\xi, \eta))_\Delta, (s(\xi, \eta))_\Delta, r_1, \ldots, r_\nu\big) \\ &\quad = F\big(x, y, (\sigma(\xi) + \tau(\eta) - \sigma(0) + z(\xi, \eta))_\Delta, (\sigma'(\xi) + p(\xi, \eta))_\Delta, (\tau'(\eta) + \\ &\qquad + q(\xi, \eta))_\Delta, (s(\xi, \eta))_\Delta, r_1, \ldots, r_\nu\big), \end{aligned}$$

we get an equation of the form

$$(4)\qquad z(x, y) = f\Big(x, y, \Big(\int_0^\xi \int_0^\eta z(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta z(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi z(s, \eta)\,ds\Big)_\Delta, \big(z(\xi, \eta)\big)_\Delta, z\big(a_1(x, y), \beta_1(x, y)\big), \ldots, z\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big),$$

with which we shall occupy oneselves.

In this equation the dependence of the operator f on the last $\nu+4$ variables has been distinguished with regard to the fact that further on sufficient conditions will be formulated for the existence of the solution of the equation exploiting the specific features of the above-mentioned dependence; in particular, in these conditions there appear connections between the estimations imposed on the operator f, the functions a_i and β_i (see Theorem 8). The existence and uniqueness of the Darboux problem and its generalization to equations of a more special form than (1) was considered by many authors; for more detailed information and references see [3], [7], [2]. Especially paper [3] gives good information about the problems in question.

The main difference between our case and that considered by other authors is thus: the right-hand side of equation (1) depends on u_{xy} and our conditions for the existence and uniqueness of solution of problem (1)–(2) involve, as was pointed above, some relations between the estimations imposed on the operator F and on the functions a_i, β_i. In our consideration also we pay more attention to the error estimations of the approximate solutions of problem (1)–(2).

1. Assumptions and lemmas. We introduce

ASSUMPTION H_1. *Suppose that*

1° *the operator* $f\colon\ \Delta\times[C(\Delta,E)]^4\times E^\nu\to E$ *is continuous* ($C(\Delta,E)$ *will denote the class of all* E*-valued functions continuous on* Δ);

2° *if* $z\in C(\Delta,E)$ *and*

$$v(x,y)=f\Big(x,y,\Big(\int_0^\xi\int_0^\eta z(s,t)\,ds\,dt\Big)_\Delta,\Big(\int_0^\eta z(\xi,t)\,dt\Big)_\Delta,\Big(\int_0^\xi z(s,\eta)\,ds\Big)_\Delta,$$
$$\big(z(\xi,\eta)\big)_\Delta, z\big(a_1(x,y),\beta_1(x,y)\big),\ldots,z\big(a_\nu(x,y),\beta_\nu(x,y)\big)\Big),$$

then $v\in C(\Delta,E)$;

3° *the functions* $a_i\colon\ \Delta\to[0,a]$, $\beta_i\colon\ \Delta\to[0,b]$, $i=1,\ldots,\nu$, *are continuous on* Δ;

4° *there exists a functional* $\Omega\colon\ \Delta\times[C_0(\Delta,R^1_+)]^4\times R^\nu_+\to R^1_+$ ($C_0(\Delta,R^1_+)$ *denote the class of all* R^1_+*-valued functions upper semicontinuous on* Δ) *which fulfils the condition* $\Omega(x,y,0,\ldots,0)\equiv 0$, *and* Ω *has the following properties*:

(a) *if* $g\in C(\Delta,R^1_+)$ *and*

$$v(x,y)=\Omega\Big(x,y,\Big(\int_0^\xi\int_0^\eta g(s,t)\,ds\,dt\Big)_\Delta,\Big(\int_0^\eta g(\xi,t)\,dt\Big)_\Delta,\Big(\int_0^\xi g(s,\eta)\,ds\Big)_\Delta,$$
$$\big(g(\xi,\eta)\big)_\Delta, g\big(a_1(x,y),\beta_1(x,y)\big),\ldots,g\big(a_\nu(x,y),\beta_\nu(x,y)\big)\Big),$$

then $v\in C(\Delta,R^1_+)$,

(b) *if* $\bar{g}, \bar{\bar{g}} \in C(\Delta, R^1_+)$ *and* $\bar{g}(x, y) \leqslant \bar{\bar{g}}(x, y)$ *for* $(x, y) \in \Delta$, *then*

$$\Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta \bar{g}(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta \bar{g}(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi \bar{g}(s, \eta)\,ds\Big)_\Delta, \big(\bar{g}(\xi, \eta)\big)_\Delta,$$
$$\bar{g}\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, \bar{g}\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big)$$
$$\leqslant \Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta \bar{\bar{g}}(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta \bar{\bar{g}}(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi \bar{\bar{g}}(s, \eta)\,ds\Big)_\Delta, \big(\bar{\bar{g}}(\xi, \eta)\big)_\Delta,$$
$$\bar{\bar{g}}\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, \bar{\bar{g}}\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big)$$

for $(x, y) \in \Delta$,

(c) *if* $g_n \in C(\Delta, R^1_+)$, $g_{n+1} \leqslant g_n$, $n = 0, 1, \ldots$, *and* $\lim_{n\to\infty} g_n(x, y) = g(x, y)$, *then*

$$\lim_{n\to\infty}\Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta g_n(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g_n(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi g_n(s, \eta)\,ds\Big)_\Delta, \big(g_n(\xi, \eta)\big)_\Delta,$$
$$g_n\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, g_n\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big)$$
$$= \Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta g(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi g(s, \eta)\,ds\Big)_\Delta, \big(g(\xi, \eta)\big)_\Delta,$$
$$g\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, g\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big)$$

for $(x, y) \in \Delta$;

5° *moreover, for any* $(x, y, z^i, p^i, q^i, s^i, r^i_1, \ldots, r^i_\nu) \in \Delta \times [C(\Delta, E)]^4 \times E^\nu$, $i = 1, 2$, *we have the inequality*

$$(5)\quad \Big\|f\big(x, y, \big(z^1(\xi, \eta)\big)_\Delta, \big(p^1(\xi, \eta)\big)_\Delta, \big(q^1(\xi, \eta)\big)_\Delta, \big(s^1(\xi, \eta)\big)_\Delta, r^1_1, \ldots, r^1_\nu\big) -$$
$$- f\big(x, y, \big(z^2(\xi, \eta)\big)_\Delta, \big(p^2(\xi, \eta)\big)_\Delta, \big(q^2(\xi, \eta)\big)_\Delta, \big(s^2(\xi, \eta)\big)_\Delta, r^2_1, \ldots, r^2_\nu\big)\Big\|$$
$$\leqslant \Omega\big(x, y, \big(\|z^1(\xi, \eta) - z^2(\xi, \eta)\|\big)_\Delta, \big(\|p^1(\xi, \eta) - p^2(\xi, \eta)\|\big)_\Delta,$$
$$\big(\|q^1(\xi, \eta) - q^2(\xi, \eta)\|\big)_\Delta,$$
$$\big(\|s^1(\xi, \eta) - s^2(\xi, \eta)\|\big)_\Delta, \|r^1_1 - r^2_1\|, \ldots, \|r^1_\nu - r^2_\nu\|\big).$$

Assumption H_2. *Suppose that*

1° *there exists a non-negative and continuous function* $\bar{g}\colon \Delta \to R_+$ *being a solution of the inequality*

$$(6)\quad \Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta g(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi g(s, \eta)\,ds\Big)_\Delta, \big(g(\xi, \eta)\big)_\Delta,$$
$$g\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, g\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big) + h(x, y) \leqslant g(x, y),$$

where

$$h(x, y) = \sup_{0\leqslant\gamma\leqslant x}\ \sup_{0\leqslant\delta\leqslant y} \|f(\gamma, \delta, 0, \ldots, 0)\|;$$

2° *in the class of functions satisfying the condition* $0 \leqslant g(x, y) \leqslant \bar{g}(x, y)$, $(x, y) \in \Delta$, *the function* g, $g(x, y) \equiv 0$, $(x, y) \in \Delta$, *is the only upper semi-continuous solution of the equation*

$$(7) \quad g(x, y) = \Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta g(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi g(s, \eta)\,ds\Big)_\Delta,$$
$$\big(g(\xi, \eta)\big)_\Delta, g\big(a_1(x, y), \beta_1(x, y)\big), \ldots, g\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big).$$

Remark 1. It is easy to prove that conditions 1°, 2°, of H_2 are fulfilled if inequality (6) has the form

$$K_1\Big(\int_0^x\int_0^y g(s, t)\,ds\,dt\Big) + K_2\Big(\int_0^y g(x, t)\,dt\Big) + K_3\Big(\int_0^x g(s, y)\,ds\Big) + K_4\big(g(x, y)\big) +$$
$$+\sum_{i=1}^{\nu} M_i g\big(a_i(x, y), \beta_i(x, y)\big) + h(x, y) \leqslant g(x, y),$$

where K_j, $j = 1, 2, 3, 4$ and M_i, $i = 1, \ldots, \nu$, are non-negative constants and

$$K_1 ab + K_2 b + K_3 a + K_4 + \sum_{i=1}^{\nu} M_i < 1.$$

ASSUMPTION H_3. *Suppose that* Ω *is defined in assumption* H_1 *and*

1° *the functional* Ω *has the following Volterra property: if the functions* g *and* h, *upper semicontinuous and belonging to the domain of the functional* Ω, *fulfil the condition:*

$g(s, t) = h(s, t)$ *for* $0 \leqslant s \leqslant x$, $0 \leqslant t \leqslant y$ *and* $a_i(x, y) \leqslant x$, $\beta_i(x, y) \leqslant y$, $i = 1, \ldots, \nu$, *then*

$$\Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta g(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi g(s, \eta)\,ds\Big)_\Delta, \big(g(\xi, \eta)\big)_\Delta,$$
$$g\big(a_1(x, y), \beta_1(x, y)\big), \ldots, g\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big)$$
$$= \Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta h(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta h(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi h(s, \eta)\,ds\Big)_\Delta, \big(h(\xi, \eta)\big)_\Delta,$$
$$h\big(a_1(x, y), \beta_1(x, y)\big), \ldots, h\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big).$$

For the sake of emphasizing this property of functional Ω *we shall continue to denote the left-hand side of the above equation by the symbol*

$$\Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta g(s, t)\,ds\,dt\Big)_{\Delta_{xy}}, \Big(\int_0^\eta g(\xi, t)\,dt\Big)_{\Delta_{xy}}, \Big(\int_0^\xi g(s, \eta)\,ds\Big)_{\Delta_{xy}}, \big(g(\xi, \eta)\big)_{\Delta_{xy}},$$
$$g\big(a_1(x, y), \beta_1(x, y)\big), \ldots, g\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big);$$

in a similar way we define the Volterra properties of the operator f;

2° *there exists a function* $\Omega^*\colon \Delta\times R_+^{\nu+4}\to R_+^1$, *continuous and non-decreasing with respect to* $z, p, q, s, r_1, \ldots, r_\nu$, *which fulfils the condition* $\Omega^*(x, y, 0, \ldots, 0) \equiv 0$, *and the inequality*

$$\Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta g(s,t)\,ds\,dt\Big)_{\Delta_{xy}}, \Big(\int_0^\eta g(\xi,t)\,dt\Big)_{\Delta_{xy}}, \Big(\int_0^\xi g(s,\eta)\,ds\Big)_{\Delta_{xy}}, \big(g(\xi,\eta)\big)_{\Delta_{xy}},$$
$$g\big(a_1(x,y), \beta_1(x,y)\big), \ldots, g\big(a_\nu(x,y), \beta_\nu(x,y)\big)\Big)$$
$$\leqslant \Omega^*\Big(x, y, \int_0^x\int_0^y g(s,t)\,ds\,dt, \int_0^y g(x,t)\,dt, \int_0^x g(s,y)\,ds, g(x,y),$$
$$g(k_1x, l_1y), \ldots, g(k_\nu x, l_\nu y)\Big)$$

holds for $a_i(x,y)\leqslant k_ix$, $\beta_i(x,y)\leqslant l_iy$, $0\leqslant k_i\leqslant 1$, $0\leqslant l_i\leqslant 1$, $i = 1, \ldots, \nu$, *and non-decreasing* g, $g\in C(\Delta, R_+^1)$;

3° *there exists a function* $w\colon \Delta\to R_+^1$, *continuous and non-decreasing, which is a solution of the inequality*

$$\Omega^*\Big(x, y, \int_0^x\int_0^y g(s,t)\,ds\,dt, \int_0^y g(x,t)\,dt, \int_0^x g(s,y)\,ds, g(x,y), g(k_1x, l_1y), \ldots$$
$$\ldots, g(k_\nu x, l_\nu y)\Big) + h(x,y) \leqslant g(x,y),$$

where $h(x,y)$ *is defined in* (6) *and* k_i, l_i *are given constants,* $0\leqslant k_i\leqslant 1$, $0\leqslant l_i\leqslant 1$, $i = 1, \ldots, \nu$;

4° *in the class of functions satisfying the condition* $0\leqslant g(x,y)\leqslant w(x,y)$, $(x,y)\in\Delta$, *the function* g, $g(x,y)\equiv 0$, $(x,y)\in\Delta$, *is the only upper semi-continuous solution of the equation*

$$g(x,y) = \Omega^*\Big(x, y, \int_0^x\int_0^y g(s,t)\,ds\,dt, \int_0^y g(x,t)\,dt, \int_0^x g(s,y)\,ds, g(x,y),$$
$$g(k_1x, l_1y), \ldots, g(k_\nu x, l_\nu y)\Big).$$

Let us define the sequence $\{g_n\}$ by the relations

$$g_0(x,y) = \bar g(x,y),$$

(8)

$$g_{n+1}(x,y) = \Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta g_n(s,t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g_n(\xi,t)\,dt\Big)_\Delta, \Big(\int_0^\xi g_n(s,\eta)\,ds\Big)_\Delta,$$
$$\big(g_n(\xi,\eta)\big)_\Delta, g_n\big(a_1(x,y), \beta_1(x,y)\big), \ldots, g_n\big(a_\nu(x,y), \beta_\nu(x,y)\big)\Big),$$
$$(x,y)\in\Delta,\ n = 0, 1, \ldots,$$

where the function $\bar g$ is from assumption H_2.

LEMMA 1. *If conditions* 4° *of* H$_1$ *and* H$_2$ *are satisfied, then*

$$(9)\qquad 0 \leqslant g_{n+1}(x,y) \leqslant g_n(x,y) \leqslant \bar{g}(x,y), \qquad (x,y)\in \Delta,\ n = 0,1,\ldots,$$
$$g_n \xrightarrow{u} 0,$$

where the sign $\xrightarrow{u}$ *denotes uniform convergence.*

Proof. From relations (6) and (8) we get

$$g_1(x,y) = \Omega\Big(x,y,\Big(\int_0^\xi\int_0^\eta g_0(s,t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g_0(\xi,t)\,dt\Big)_\Delta, \Big(\int_0^\xi g_0(s,\eta)\,ds\Big)_\Delta,$$
$$\big(g_0(\xi,\eta)\big)_\Delta, g_0\big(\alpha_1(x,y),\beta_1(x,y)\big), \ldots, g_0\big(\alpha_\nu(x,y),\beta_\nu(x,y)\big)\Big)$$
$$\leqslant \Omega\Big(x,y,\Big(\int_0^\xi\int_0^\eta \bar{g}(s,t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta \bar{g}(\xi,t)\,dt\Big)_\Delta, \Big(\int_0^\xi \bar{g}(s,\eta)\,ds\Big)_\Delta,$$
$$\big(\bar{g}(\xi,\eta)\big)_\Delta, \bar{g}\big(\alpha_1(x,y),\beta_1(x,\eta)\big), \ldots, \bar{g}\big(\alpha_\nu(x,y),\beta_\nu(x,y)\big)\Big) + h(x,y)$$
$$\leqslant \bar{g}(x,y) = g_0(x,y), \qquad (x,y)\in\Delta.$$

Further, if we suppose that

$$g_n(x,y) \leqslant g_{n-1}(x,y) \leqslant \bar{g}(x,y), \qquad (x,y)\in\Delta,$$

then

$$g_{n+1}(x,y) = \Omega\Big(x,y,\Big(\int_0^\xi\int_0^\eta g_n(s,t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g_n(\xi,t)\,dt\Big)_\Delta,$$
$$\Big(\int_0^\xi g_n(s,\eta)\,ds\Big)_\Delta, \big(g_n(\xi,\eta)\big), g_n\big(\alpha_1(x,y),\beta_1(x,y)\big), \ldots$$
$$\ldots, g_n\big(\alpha_\nu(x,y),\beta_\nu(x,y)\big)\Big)$$
$$\leqslant \Omega\Big(x,y,\Big(\int_0^\xi\int_0^\eta g_{n-1}(s,t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g_{n-1}(\xi,t)\,dt\Big)_\Delta,$$
$$\Big(\int_0^\eta g_{n-1}(s,\eta)\,ds\Big)_\Delta, \big(g_{n-1}(\xi,\eta)\big)_\Delta, g_{n-1}\big(\alpha_1(x,y),\beta_1(x,y)\big), \ldots$$
$$\ldots, g_{n-1}\big(\alpha_\nu(x,y),\beta_\nu(x,y)\big)\Big) = g_n(x,y) \leqslant \bar{g}(x,y), \qquad (x,y)\in\Delta.$$

Since the sequence of continuous functions g_n is non-increasing and bounded from below, it is convergent to a certain upper semi-continuous function φ such that $0 \leqslant \varphi(x,y) \leqslant \bar{g}(x,y)$ for $(x,y)\in\Delta$. By Lebesgue's theorem and the continuity of the functional Ω it follows that the function φ satisfies equation (7).

Now from assumption H$_2$ we have $\varphi(x,y) \equiv 0$, $(x,y)\in\Delta$.

The uniform convergence of the sequence $\{g_n\}$ in Δ follows from Dini's theorem. Thus the proof of Lemma 1 is complete.

Let us define the sequence $\{\tilde{g}_n\}$ by the relations

$$\begin{aligned}(10)\quad & \tilde{g}_0(x,y) = w(x,y),\\ & \tilde{g}_{n+1}(x,y) = \Omega^*\Big(x, y, \int_0^x\int_0^y \tilde{g}_n(s,t)\,ds\,dt, \int_0^y \tilde{g}_n(x,t)\,dt, \int_0^x \tilde{g}_n(s,y)\,ds,\\ & \qquad \tilde{g}_n(x,y), \tilde{g}_n(k_1x, l_1y), \ldots, \tilde{g}_n(k_\nu x, l_\nu y)\Big), \quad (x,y)\in\Delta,\ n = 0,1,\ldots\end{aligned}$$

We then have

LEMMA 2. *If assumption* H_3 *is satisfied, and*

1° $0\leqslant a_i(x,y)\leqslant k_i x$, $0\leqslant \beta_i(x,y)\leqslant l_i y$, $0\leqslant k_i\leqslant 1$, $0\leqslant l_i\leqslant 1$, $i = 1,\ldots,\nu$, $(x,y)\in\Delta$,

2° *the function* Ω^* *is non-decreasing with respect to all variables, then*

(i) *the functions* $\tilde{g}_n$, $n = 0,1,\ldots$, *are non-decreasing with respect to* x *and* y,

(ii) $0\leqslant \tilde{g}_{n+1}(x,y)\leqslant \tilde{g}_n(x,y)\leqslant w(x,y)$, $(x,y)\in\Delta$, $n = 0,1,\ldots$, $\tilde{g}_n \xrightarrow{u} 0$, *in* Δ,

(iii) *the function* w *satisfies inequality* (6) *and if* $\bar{g}(x,y)\leqslant w(x,y)$, *then* $0\leqslant g_n(x,y)\leqslant \tilde{g}_n(x,y)$, $(x,y)\in\Delta$, $n = 0,1,\ldots$

Remark 2. The introduction of the comparative function Ω^* and sequence (10) connected with it obviously does not give us better estimations of the approximate solutions; still, it allows us to simplify the considerations referring to the equation of Volterra's type in an essential way and, in particular, it permits us to give effective conditions under which conditions 3° and 4° of assumptions H_3 are fulfilled.

2. The existence of a solution of equation (4). In order to prove the existence of a solution of equation (4) we shall show that the sequence $\{z_n\}$ defined by the relations

$$\begin{aligned}& z_0(x,y) \equiv 0,\\ (11)\quad & z_{n+1}(x,y) = f\Big(x, y, \Big(\int_0^\xi\int_0^\eta z_n(s,t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta z_n(\xi,t)\,dt\Big)_\Delta, \Big(\int_0^\xi z_n(s,\eta)\,ds\Big)_\Delta,\\ & \qquad \big(z_n(\xi,\eta)\big)_\Delta, z_n\big(a_1(x,y), \beta_1(x,y)\big), \ldots, z_n\big(a_\nu(x,y), \beta_\nu(x,y)\big)\Big),\\ & \qquad (x,y)\in\Delta,\ n = 0,1,\ldots,\end{aligned}$$

is uniformly convergent to a solution of equation (4).

We have

THEOREM 1. *If assumptions* H_1 *and* H_2 *are satisfied, then there exists in the set* Δ *a continuous solution* $\bar{z}$ *of equation* (4). *The sequence* $\{z_n\}$ *defined by* (11) *converges uniformly on* Δ *to* $\bar{z}$, *as* $n\to\infty$; *moreover, the estimations*

$$(12)\qquad \|\bar{z}(x,y) - z_n(x,y)\| \leqslant g_n(x,y),\quad (x,y)\in\Delta,\ n = 0,1,\ldots,$$

and

$$(13) \qquad \|\bar{z}(x, y)\| \leqslant \bar{g}(x, y), \qquad (x, y) \in \Delta,$$

hold true.

The solution $\bar{z}$ *of* (4) *is unique in the class of functions satisfying relation* (13).

Proof. We shall prove that sequence $\{z_n(x, y)\}$, $(x, y) \in \Delta$, fulfils the condition

$$(14) \qquad \|z_n(x, y)\| \leqslant \bar{g}(x, y), \qquad (x, y) \in \Delta, \ n = 0, 1, \ldots$$

Evidently

$$\|z_0(x, y)\| \equiv 0 \leqslant \bar{g}(x, y), \qquad (x, y) \in \Delta.$$

Let us suppose that inequality (14) is true for $n \geqslant 0$. By the definition of $z_n(x, y)$, $(x, y) \in \Delta$, and condition 5° of H_1, we have

$$\|z_{n+1}(x, y)\| = \Big\| f\Big(x, y, \Big(\int_0^\xi \int_0^\eta z_n(s, t)\, ds\, dt\Big)_\Delta, \Big(\int_0^\eta z_n(\xi, t)\, dt\Big)_\Delta, \Big(\int_0^\xi z_n(s, \eta)\, ds\Big)_\Delta,$$
$$\big(z_n(\xi, \eta)\big)_\Delta, z_n\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots$$
$$\ldots, z_n\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big) - f(x, y, 0, \ldots, 0) + f(x, y, 0, \ldots, 0)\Big\|$$
$$\leqslant \Omega\Big(x, y, \Big(\int_0^\xi \int_0^\eta \|z_n(s, t)\|\, ds\, dt\Big)_\Delta, \Big(\int_0^\eta \|z_n(\xi, t)\|\, dt\Big)_\Delta,$$
$$\Big(\int_0^\xi \|z_n(s, \eta)\|\, ds\Big)_\Delta, \big(\|z_n(\xi, \eta)\|\big)_\Delta, \|z_n\big(\alpha_1(x, y), \beta_1(x, y)\big)\|, \ldots$$
$$\ldots, \big\|z_n\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\big\|\Big) + h(x, y)$$
$$\leqslant \Omega\Big(x, y, \Big(\int_0^\xi \int_0^\eta \bar{g}(s, t)\, ds\, dt\Big)_\Delta, \Big(\int_0^\eta \bar{g}(\xi, t)\, dt\Big)_\Delta, \Big(\int_0^\xi \bar{g}(s, \eta)\, ds\Big)_\Delta,$$
$$\big(\bar{g}(\xi, \eta)\big)_\Delta, \bar{g}\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, \bar{g}\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big) +$$
$$+ h(x, y) \leqslant \bar{g}(x, y) \in \Delta.$$

Now we obtain (14) by induction.

Further, we prove that

$$(15) \qquad \|z_{n+r}(x, y) - z_n(x, y)\| \leqslant g_n(x, y), \qquad (x, y) \in \Delta, \ n = 0, 1, \ldots, \ r = 0, 1, \ldots$$

By (14) we have

$$\|z_r(x, y) - z_0(x, y)\| = \|z_r(x, y)\| \leqslant \bar{g}(x, y) = g_0(x, y), \qquad (x, y) \in \Delta, \quad r = 0, 1, \ldots$$

Further, if we suppose that (15) is true for $n, r \geqslant 0$, then

$$\|z_{n+r+1}(x, y) - z_{n+1}(x, y)\| = \Big\| f\Big(x, y, \Big(\int_0^\xi\int_0^\eta z_{n+r}(s, t)\,ds\,dt\Big)_\Delta,$$

$$\Big(\int_0^\eta z_{n+r}(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi z_{n+r}(s, \eta)\,ds\Big)_\Delta, \big(z_{n+r}(\xi, \eta)\big)_\Delta,$$

$$z_{n+r}\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, z_{n+r}\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big) -$$

$$- f\Big(x, y, \Big(\int_0^\xi\int_0^\eta z_n(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta z_n(\xi, t)\,dt\Big)_\Delta,$$

$$\Big(\int_0^\xi z_n(s, \eta)\,ds\Big)_\Delta, \big(z_n(\xi, \eta)\big)_\Delta, z_n\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots$$

$$\ldots, z_n\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big)\Big\|$$

$$\leqslant \Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta \|z_{n+r}(s, t) - z_n(s, t)\|\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta \|z_{n+r}(\xi, t) - z_n(\xi, t)\|\,dt\Big)_\Delta,$$

$$\Big(\int_0^\xi \|z_{n+r}(s, \eta) - z_n(s, \eta)\|\,ds\Big)_\Delta, \big(\|z_{n+r}(\xi, \eta) - z_n(\xi, \eta)\|\big)_\Delta,$$

$$\big\|z_{n+r}\big(\alpha_1(x, y), \beta_1(x, y)\big) - z_n\big(\alpha_1(x, y), \beta_1(x, y)\big)\big\|, \ldots$$

$$\ldots, \big\|z_{n+r}\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big) - z_n\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\big\|\Big)$$

$$\leqslant \Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta g_n(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g_n(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi g_n(s, \eta)\,ds\Big)_\Delta,$$

$$\big(g_n(\xi, \eta)\big)_\Delta, g_n\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, g_n\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big)$$

$$= g_{n+1}(x, y), \qquad (x, y) \in \Delta.$$

Now we obtain (15) by induction.

Because of Lemma 1 $g_n \xrightarrow{u} 0$ in Δ, we have from (15) $z_n \xrightarrow{u} \bar{z}$ in Δ. The continuity of $\bar{z}$ follows from the uniform convergence of the sequence $\{z_n\}$ and the continuity of all functions z_n.

If $r \to \infty$, then (15) gives estimation (12). Estimation (13) is implied by (14).

It is obvious that $\bar{z}$ is a solution of (4).

To prove that the solution $\bar{z}$ is a unique solution of (4) in the class pointed out above let us suppose that there exists another solution $\tilde{z}$ defined in Δ and such that $\bar{z}(x, y) \neq \tilde{z}(x, y)$ for $(x, y) \in \Delta$, and $\|\tilde{z}(x, y)\| \leqslant \bar{g}(x, y)$ for $(x, y) \in \Delta$.

We get

$$\|\tilde{z}(x, y) - z_n(x, y)\| \leqslant g_n(x, y), \qquad (x, y) \in \Delta, \ n = 0, 1, \ldots,$$

by induction, and hence it follows that $\bar{z}(x, y) \equiv \tilde{z}(x, y)$, $(x, y) \in \Delta$. This contradiction proves the uniqueness of $\bar{z}$ in the class of functions satisfying relation (13). Thus the proof of Theorem 1 is completed.

Now we can formulate an analogous theorem for an equation of Volterra's type, namely for equation (4), in which operator f has the Volterra property (see condition 1° of H_3).

THEOREM 2. *If assumption* H_1, H_3 *and conditions* 1°–2° *of Lemma* 2 *are fulfilled, then the assertion of Theorem* 1 *is true, and the estimations*

$$\|\bar{z}(x, y) - z_n(x, y)\| \leqslant \tilde{g}_n(x, y), \quad (x, y) \in \Delta, \ n = 0, 1, \ldots,$$

and

$$\|\bar{z}(x, y)\| \leqslant w(x, y), \quad (x, y) \in \Delta,$$

hold true.

Proof. We prove that assumption H_2 is fulfilled. By Lemma 2 we see that the function w satisfies inequality (6). Now we put $\tilde{g} = w$. Let g be an upper semicontinuous solution of (7) in the class $0 \leqslant g(x, y) \leqslant w(x, y)$, $(x, y) \in \Delta$. We get

$$0 \leqslant g(x, y) \leqslant \tilde{g}_n(x, y), \quad (x, y) \in \Delta, \ n = 0, 1, \ldots,$$

by induction, and because $\tilde{g}_n(x, y) \xrightarrow{u} 0$ for $(x, y) \in \Delta$, we have $g(x, y) \equiv 0$, $(x, y) \in \Delta$. Hence assumption H_2 is fulfilled.

Since all assumption of Theorem 1 are fulfilled, and $g_n(x, y) \leqslant \tilde{g}_n(x, y)$, $(x, y) \in \Delta$, Theorem 2 is proved.

Remark 3. Equation (1) may have various particular forms depending on the form of the operator F, eg.:

(a) If

$$\begin{aligned} F\big(x, y, u(\cdot, \cdot), u_x(\cdot, \cdot), u_y(\cdot, \cdot), u_{xy}(\cdot \cdot \cdot), u_{xy}(\alpha_1(x, y), \beta_1(x, y)), \ldots \\ \ldots, u_{xy}(\alpha_\nu(x, y), \beta_\nu(x, y))\big) \\ = F\big(x, y, u(\gamma_1(x, y), \delta_1(x, y)), u_x(\gamma_2(x, y), \delta_2(x, y)), \\ u_y(\gamma_3(x, y), \delta_3(x, y)), u_{xy}(\alpha(x, y), \beta(x, y))\big), \end{aligned}$$

then we obtain an equation of the form

$$\begin{aligned} u_{xy}(x, y) = F\big(x, y, u(\gamma_1(x, y), \delta_1(x, y)), u_x(\gamma_2(x, y), \delta_2(x, y)), \\ u_y(\gamma_3(x, y), \delta_3(x, y)), u_{xy}(\alpha(x, y), \beta(x, y))\big). \end{aligned}$$

This equation has been considered in [5] for $E = R^1$.

(b) If $F = F\big(x, y, u(x, y), u_x(x, y), u_y(x, y)\big)$, then we obtain the partial differential equation

$$u_{xy}(x, y) = F\big(x, y, u(x, y), u_x(x, y), u_y(x, y)\big),$$

and from Theorem 1 we get some results contained in [3], [7], where the functional Ω has the form

$$\Omega = \omega\Big(x, y, \int_0^x\int_0^y g(s,t)\,ds\,dt, \int_0^y g(x,t)\,dt, \int_0^x g(s,y)\,ds\Big).$$

(c) If $F = F\big(x, y, u(x,y), u_{xy}(x,y)\big)$, then we obtain the equation

$$u_{xy}(x,y) = F\big(x, y, u(x,y), u_{xy}(x,y)\big),$$

and for $E = R^1$ from Theorem 1 we get some results contained in [8], namely

$$\Omega = c_0\int_0^x\int_0^y g(s,t)\,ds\,dt + c_1 g(x,y), \qquad \text{where } c_0 \geqslant 0,\ 0 < c_1 < 1.$$

(d) If

$$F = F\Big(x, y, u(x,y), u_x(x,y), u_y(x,y), \int_0^x\int_0^y g\big(x, y, s, t, u(s,t), u_s(s,t), u_t(s,t)\big)\,ds\,dt\Big),$$

then we obtain an equation of the form

$$u_{xy}(x,y) = F\Big(x, y, u(x,y), u_x(x,y), u_y(x,y), \int_0^x\int_0^y g\big(x, y, s, t, u(s,t), u_s(s,t), u_t(s,t)\big)\,ds\,dt\Big),$$

which has been considered in [6].

3. Uniqueness theorem. Now we give conditions under which equation (4) has at most one solution; these conditions do not guarantee existence. We have

THEOREM 3. *If assumption* H_1 *is satisfied and the function* m, $m(x,y) \equiv 0$, $(x,y) \in \Delta$, *is the only non-negative, upper semicontinuous solution of the inequality*

$$(16)\qquad m(x,y) \leqslant \Omega\Big(x, y, \Big(\int_0^\xi\int_0^\eta m(s,t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta m(\xi,t)\,dt\Big)_\Delta, \Big(\int_0^\xi m(s,\eta)\,ds\Big)_\Delta, \big(m(\xi,\eta)\big)_\Delta, m\big(\alpha_1(x,y), \beta_1(x,y)\big), \ldots, m\big(\alpha_\nu(x,y), \beta_\nu(x,y)\big)\Big), \quad (x,y) \in \Delta,$$

then equation (4) *has at most one solution in the set* Δ.

Proof. Let us suppose that there exist two solutions, $\tilde{z}$ and $\tilde{\tilde{z}}$, of equation (4), defined in Δ and such that $\tilde{z}(x, y) \not\equiv \tilde{\tilde{z}}(x, y)$, $(x, y) \in \Delta$. Now from condition 5° of H_1 we have

$$\|\tilde{z}(x, y)-\tilde{\tilde{z}}(x, y)\| = \Big\|f\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} \tilde{z}(s, t)\,ds\,dt\Big)_{\Delta}, \Big(\int_0^{\eta} \tilde{z}(\xi, t)\,dt\Big)_{\Delta},$$
$$\Big(\int_0^{\xi} \tilde{z}(s, \eta)\,ds\Big)_{\Delta}, \big(\tilde{z}(\xi, \eta)\big)_{\Delta}, \tilde{z}\big(a_1(x, y), \beta_1(x, y)\big), \ldots$$
$$\ldots, \tilde{z}\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big) -$$
$$-f\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} \tilde{\tilde{z}}(s, t)\,ds\,dt\Big)_{\Delta}, \Big(\int_0^{\eta} \tilde{\tilde{z}}(\xi, t)\,dt\Big)_{\Delta},$$
$$\Big(\int_0^{\xi} \tilde{\tilde{z}}(s, \eta)\,ds\Big)_{\Delta}, \big(\tilde{\tilde{z}}(\xi, \eta)\big)_{\Delta}, \tilde{\tilde{z}}\big(a_1(x, y), \beta_1(x, y)\big), \ldots, \tilde{\tilde{z}}\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big)\Big\|$$
$$\leqslant \Omega\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} \|\tilde{z}(s, t)-\tilde{\tilde{z}}(s, t)\|\,ds\,dt\Big)_{\Delta},$$
$$\Big(\int_0^{\eta} \|\tilde{z}(\xi, t)-\tilde{\tilde{z}}(\xi, t)\|\,dt\Big)_{\Delta}, \Big(\int_0^{\xi} \|\tilde{z}(s, \eta)-\tilde{\tilde{z}}(s, \eta)\|\,ds\Big)_{\Delta},$$
$$\big(\|\tilde{z}(\xi, \eta)-\tilde{\tilde{z}}(\xi, \eta)\|\big)_{\Delta}, \big\|\tilde{z}\big(a_1(x, y), \beta_1(x, y)\big)-\tilde{\tilde{z}}\big(a_1(x, y), \beta_1(x, y)\big)\big\|, \ldots$$
$$\ldots, \big\|\tilde{z}\big(a_\nu(x, y), \beta_\nu(x, y)\big)-\tilde{\tilde{z}}\big(a_\nu(x, y), \beta_\nu(x, y)\big)\big\|\Big).$$

Putting

$$m(x, y) = \|\tilde{z}(x, y)-\tilde{\tilde{z}}(x, y)\|, \quad (x, y) \in \Delta,$$

we have from (16) that $m(x, y) \equiv 0$ for $(x, y) \in \Delta$, i.e. $\tilde{z}(x, y) \equiv \tilde{\tilde{z}}(x, y)$, $(x, y) \in \Delta$. This contradiction proves Theorem 3.

Remark 4. If assumption H_2 is satisfied, then the function m, $m(x, y) \equiv 0$ for $(x, y) \in \Delta$, is the only upper semicontinuous solution of (16) in the class $0 \leqslant m(x, y) \leqslant \bar{g}(x, y)$, $(x, y) \in \Delta$.

Indeed, we can prove by induction that

$$0 \leqslant m(x, y) \leqslant g_n(x, y), \quad (x, y) \in \Delta,\ n = 0, 1, \ldots,$$

and if $n \to \infty$, then, in view of Lemma 1, we have $m(x, y) \equiv 0$ for $(x, y) \in \Delta$.

Remark 5. If assumption H_1, conditions 1°–2° of H_3 and conditions 1°–2° of Lemma 2 are satisfied, and the function g, $g(x, y) \equiv 0$, $(x, y) \in \Delta$, is the only non-negative, non-decreasing, upper semicontinuous solution of the inequality

$$(17) \quad g(x, y) \leqslant \Omega^*\Big(x, y, \int_0^x\int_0^y g(s, t)\,ds\,dt, \int_0^y g(x, t)\,dt, \int_0^x g(s, y)\,ds,$$
$$g(x, y), g(k_1 x, l_1 y), \ldots, g(k_\nu x, l_\nu y)\Big), \quad (x, y) \in \Delta,$$

then equation (4) has at most one solution.

4. Continuous dependence of solutions on the right-hand side of equation (4). Let us consider the second equation

$$(18)\qquad p(x,y) = P\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} p(s,t)\,ds\,dt\Big)_{\Delta}, \Big(\int_0^{\eta} p(\xi,t)\,dt\Big)_{\Delta}, \Big(\int_0^{\xi} p(s,\eta)\,ds\Big)_{\Delta},$$
$$\big(p(\xi,\eta)\big)_{\Delta}, p\big(\bar{a}_1(x,y), \bar{\beta}_1(x,y)\big), \ldots, p\big(\bar{a}_\nu(x,y), \bar{\beta}_\nu(x,y)\big)\Big),$$

where the functions $P, \bar{a}_i, \bar{\beta}_i$, $i = 1, \ldots, \nu$, have the same properties as f, a_i, β_i, $i = 1, \ldots, \nu$, given in assumption H_1.

Now we have

THEOREM 4. *If assumption* H_1 *is satisfied, and*

1° $\bar{z}$ *and* $\bar{p}$ *are solutions of equations* (4) *and* (18) *respectively,*

2° *the sequence* $\{u_n(x,y)\}$ *defined by the relations*

$$u_{n+1}(x,y) = \Omega\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} u_n(s,t)\,ds\,dt\Big)_{\Delta}, \Big(\int_0^{\eta} u_n(\xi,t)\,dt\Big)_{\Delta},$$
$$\Big(\int_0^{\xi} u_n(s,\eta)\,ds\Big)_{\Delta}, \big(u_n(\xi,\eta)\big)_{\Delta}, u_n\big(a_1(x,y), \beta_1(x,y)\big), \ldots$$
$$\ldots, u_n\big(a_\nu(x,y), \beta_\nu(x,y)\big)\Big) + \bar{h}(x,y), \qquad n = 0, 1, \ldots,$$

function u_0 *being continuous and such that*

$$u_0(x,y) \geqslant \|\bar{z}(x,y)\| + \|\bar{p}(x,y)\|, \qquad (x,y)\in\Delta,$$

and

$$\bar{h}(x,y) = \Big\|f\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} \bar{p}(s,t)\,ds\,dt\Big)_{\Delta}, \Big(\int_0^{\eta} \bar{p}(\xi,t)\,dt\Big)_{\Delta}, \Big(\int_0^{\xi} \bar{p}(s,\eta)\,ds\Big)_{\Delta},$$
$$\big(\bar{p}(\xi,\eta)\big)_{\Delta}, \bar{p}\big(a_1(x,y), \beta_1(x,y)\big), \ldots, \bar{p}\big(a_\nu(x,y), \beta_\nu(x,y)\big)\Big) - \bar{p}(x,y)\Big\|,$$
$$(x,y)\in\Delta,$$

has the limit $\bar{u}(x,y)$ *for* $(x,y)\in\Delta$,
then

$$(19)\qquad \|\bar{z}(x,y) - \bar{p}(x,y)\| \leqslant \bar{u}(x,y), \qquad (x,y)\in\Delta.$$

Proof. Let

$$u(x,y) = \|\bar{z}(x,y) - \bar{p}(x,y)\|, \qquad (x,y)\in\Delta.$$

Thus for $(x, y) \in \Delta$ we have

$$u(x, y) \leqslant \Big\| f\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} \bar{z}(s, t)\,ds\,dt\Big)_{\Delta}, \Big(\int_0^{\eta} \bar{z}(\xi, t)\,dt\Big)_{\Delta}, \Big(\int_0^{\xi} \bar{z}(s, \eta)\,ds\Big)_{\Delta},$$

$$\big(\bar{z}(\xi, \eta)\big)_{\Delta}, \bar{z}\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, \bar{z}\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big) -$$

$$-f\Big(x, y; \Big(\int_0^{\xi}\int_0^{\eta} \bar{p}(s, t)\,ds\,dt\Big)_{\Delta}, \Big(\int_0^{\eta} \bar{p}(\xi, t)\,dt\Big)_{\Delta}, \Big(\int_0^{\xi} \bar{p}(s, \eta)\,ds\Big)_{\Delta},$$

$$\big(\bar{p}(\xi, \eta)\big)_{\Delta}, \bar{p}\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, \bar{p}\big(\alpha_1(x, y), \beta_\nu(x, y)\big)\Big)\Big\| +$$

$$+ \Big\| f\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} \bar{p}(s, t)\,ds\,dt\Big)_{\Delta}. \Big(\int_0^{\eta} \bar{p}(\xi, t)\,dt\Big)_{\Delta}, \Big(\int_0^{\xi} \bar{p}(s, \eta)\,ds\Big)_{\Delta},$$

$$\big(\bar{p}(\xi, \eta)\big)_{\Delta}, \bar{p}\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, \bar{p}\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big) - \bar{p}(x, y)\Big\|$$

$$\leqslant \Omega\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} \|\bar{z}(s, t) - \bar{p}(s, t)\|\,ds\,dt\Big)_{\Delta}, \Big(\int_0^{\eta} \|\bar{z}(\xi, t) - \bar{p}(\xi, t)\|\,dt\Big)_{\Delta},$$

$$\Big(\int_0^{\xi} \|\bar{z}(s, \eta) - \bar{p}(s, \eta)\|\,ds\Big)_{\Delta}, \big(\|\bar{z}(\xi, \eta) - \bar{p}(\xi, \eta)\|\big)_{\Delta},$$

$$\big\|\bar{z}\big(\alpha_1(x, y), \beta_1(x, y)\big) - \bar{p}\big(\alpha_1(x, y), \beta_1(x, y)\big)\big\|, \ldots$$

$$\ldots, \big\|\bar{z}\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big) - \bar{p}\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\big\|\Big) + \bar{h}(x, y).$$

Since

$$u(x, y) \leqslant \|\bar{z}(x, y)\| + \|\bar{p}(x, y)\| \leqslant u_0(x, y), \qquad (x, y) \in \Delta,$$

from the above consideration we get

$$u(x, y) \leqslant u_n(x, y), \qquad (x, y) \in \Delta,\ n = 0, 1, \ldots,$$

by induction.

Inequality (19) is implied by the above inequality as $n \to \infty$.

Remark 6. From the proof of Theorem 4 it follows that this theorem is true if there exists a non-negative and continuous function k_0 defined in the set Δ and satisfying the inequality

$$\Omega\Big(x, y, \Big(\int_0^{\xi}\int_0^{\eta} k_0(s, t)\,ds\,dt\Big)_{\Delta}, \Big(\int_0^{\eta} k_0(\xi, t)\,dt\Big)_{\Delta}, \Big(\int_0^{\xi} k_0(s, \eta)\,ds\Big)_{\Delta},$$

$$\big(k_0(\xi, \eta)\big)_{\Delta}, k_0\big(\alpha_1(x, y), \beta_1(x, y)\big), \ldots, k_0\big(\alpha_\nu(x, y), \beta_\nu(x, y)\big)\Big) +$$

$$+ \max[\bar{h}(x, y), u_0(x, y)] \leqslant k_0(x, y), \qquad (x, y) \in \Delta.$$

Indeed, now in the class of upper semicontinuous functions satisfying the condition $0 \leqslant g(x, y) \leqslant k_0(x, y)$, $(x, y) \in \Delta$, there exists a function $\bar{k}$ which is a solution of the equation

$$\Omega\Big(x, y, \Big(\int_0^\xi \int_0^\eta g(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta g(\xi, t)\,dt\Big)_\Delta, \Big(\int_0^\xi g(s, \eta)\,ds\Big)_\Delta, \big(g(\xi, \eta)\big)_\Delta,$$
$$g\big(a_1(x, y), \beta_1(x, y)\big), \ldots, g\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big) +$$
$$+\bar{h}(x, y) = g(x, y), \ (x, y) \in \Delta.$$

Put

$$k_{n+1}(x, y) = \Omega\Big(x, y, \Big(\int_0^\xi \int_0^\eta k_n(s, t)\,ds\,dt\Big)_\Delta, \Big(\int_0^\eta k_n(\xi, t)\,dt\Big)_\Delta,$$
$$\Big(\int_0^\xi k_n(s, \eta)\,ds\Big)_\Delta, \big(k_n(\xi, \eta)\big)_\Delta, k_n\big(a_1(x, y), \beta_1(x, y)\big), \ldots$$
$$\ldots, k_n\big(a_\nu(x, y), \beta_\nu(x, y)\big)\Big) + \bar{h}(x, y), \quad (x, y) \in \Delta, \ n = 0, 1, \ldots$$

We see that

$$u_n(x, y) \leqslant k_n(x, y), \quad k_{n+1}(x, y) \leqslant k_n(x, y), \quad (x, y) \in \Delta, \ n = 0, 1, \ldots,$$

and hence $u(x, y) \leqslant k_n(x, y)$, $(x, y) \in \Delta$, $n = 0, 1, \ldots$ ($u(x, y)$ is defined in the proof of Theorem 4). From the last inequality we get

$$u_n(x, y) \to \bar{u}(x, y), \quad (x, y) \in \Delta,$$
$$\text{and} \quad u(x, y) \leqslant \bar{u}(x, y) \leqslant \bar{k}(x, y), \quad (x, y) \in \Delta.$$

Theorem 4 implies for an equation of Volterra's type

THEOREM 5. *If the assumptions of Theorem 4 (except 2°) and conditions 1°–2° of Lemma 2 are satisfied, and the sequence $\{\tilde{u}_n\}$, defined by the relations*

$$\tilde{u}_0(x, y) = \sup_{0 \leqslant \gamma \leqslant x}\ \sup_{0 \leqslant \delta \leqslant y} \{\|\bar{z}(\gamma, \delta)\| + \|\bar{p}(\gamma, \delta)\|\}, \quad (x, y) \in \Delta,$$

$$\tilde{u}_{n+1}(x, y) = \Omega^*\Big(x, y, \int_0^x \int_0^y \tilde{u}_n(s, t)\,ds\,dt, \int_0^y \tilde{u}_n(x, t)\,dt, \int_0^x \tilde{u}_n(s, y)\,ds,$$
$$\tilde{u}_n(x, y), \tilde{u}_n(k_1 x, l_1 y), \ldots, \tilde{u}_n(k_\nu x, l_\nu y)\Big) + \sup_{0 \leqslant \gamma \leqslant x}\ \sup_{0 \leqslant \delta \leqslant y} \bar{h}(\gamma, \delta),$$

for $(x, y) \in \Delta$, $n = 0, 1, \ldots$, has the limit $\tilde{u}(x, y)$, $(x, y) \in \Delta$, then

$$\|\bar{z}(x, y) - \bar{p}(x, y)\| \leqslant \tilde{u}(x, y), \quad (x, y) \in \Delta. \tag{20}$$

Proof. It is obvious that the functions $\tilde{u}_n$ are non-decreasing for $(x, y) \in \Delta$, $n = 0, 1, \ldots$ Further, we get

$$u_n(x, y) \leqslant \tilde{u}_n(x, y), \quad (x, y) \in \Delta, \ n = 0, 1, \ldots,$$

by induction, where the sequence $\{u_n\}$ is defined in condition 2° of Theorem 4. Hence $u(x, y) \leqslant \tilde{u}_n(x, y)$, $(x, y) \in \Delta$, $n = 0, 1, \ldots$ ($u(x, y)$ is defined in the proof of Theorem 4), and if $n \to \infty$, then we have (20).

5. The case of the functional Ω being linear in r_i, $i = 1, \ldots, \nu$. We are now going to consider particular forms of the functional Ω permitting us to give effective conditions for fulfilling assumptions $\mathbf{H}_2$ or $\mathbf{H}_3$.

At first we assume $\Omega(x, y, z, p, q, s, r_1, \ldots, r_\nu) = \sum_{i=1}^{\nu} \lambda_i(x, y) r_i$, $\lambda_i(x, y) \geqslant 0$, $i = 1, \ldots, \nu$, $(x, y) \in \Delta$.

Now equation (1) is a purely functional one; the discusion of this case is necessary for further considerations.

Let

$$\begin{aligned} (21)\qquad & \alpha_0^{i_0}(x, y) = x, \quad \alpha_{n+1}^{i_0, \ldots, i_{n+1}}(x, y) = \alpha_n^{i_1, \ldots, i_{n+1}}\big(\alpha_{i_0}(x, y), \beta_{i_0}(x, y)\big), \\ & \beta_0^{i_0}(x, y) = y, \quad \beta_{n+1}^{i_0, \ldots, i_{n+1}}(x, y) = \beta_n^{i_1, \ldots, i_{n+1}}\big(\alpha_{i_0}(x, y), \beta_{i_0}(x, y)\big), \\ & \lambda_0^{i_0}(x, y) = \frac{1}{\nu}, \quad \lambda_{n+1}^{i_0, \ldots, i_{n+1}}(x, y) \\ & \qquad = \lambda_{i_0}(x, y) \lambda_n^{i_1, \ldots, i_{n+1}}\big(\alpha_{i_0}(x, y), \beta_{i_0}(x, y)\big), \end{aligned}$$

where $\alpha_i(x, y)$, $\beta_i(x, y)$ and $\lambda_i(x, y)$, $(x, y) \in \Delta$, $i = 1, \ldots, \nu$, are as in assumption $\mathbf{H}_1$.

It is obvious that $\alpha_n^{i_0, \ldots, i_n}(x, y) \in [0, a]$, $\beta_n^{i_0, \ldots, i_n}(x, y) \in [0, b]$ for $(x, y) \in \Delta$, $n = 0, 1, \ldots$

Now we formulate lemmas by which assumption $\mathbf{H}_2$ is fulfilled in this special case.

LEMMA 3. *For any function $h\colon \Delta \to R_+^1$ the condition*

$$(22)\qquad \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \lambda_n^{i_0, \ldots, i_n}(x, y) h\big(\alpha_n^{i_0, \ldots, i_n}(x, y), \beta_n^{i_0, \ldots, i_n}(x, y)\big) < \infty, \quad (x, y) \in \Delta,$$

is necessary and sufficient for the equation

$$(23)\quad g(x, y) = \sum_{i=1}^{\nu} \lambda_i(x, y) g\big(\alpha_i(x, y), \beta_i(x, y)\big) + h(x, y), \quad (x, y) \in \Delta,$$

to have a non-negative solution $\bar{g}$ defined in Δ.

If condition (22) *is fulfilled, then the function $\bar{g}$,*

$$(24)\quad \bar{g}(x, y) = \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \lambda_n^{i_0, \ldots, i_n}(x, y) h\big(\alpha_n^{i_0, \ldots, i_n}(x, y), \beta_n^{i_0, \ldots, i_n}(x, y)\big), \quad (x, y) \in \Delta,$$

is a solution of equation (23), *and*

$$(25)\quad \lim_{n\to\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\lambda_n^{i_0,\ldots,i_n}(x,y)\bar{g}\big(a_n^{i_0,\ldots,i_n}(x,y),\beta_n^{i_0,\ldots,i_n}(x,y)\big)=0,$$
$$(x,y)\in\Delta.$$

There is no other solution of equation (23) *in the class of functions* g *satisfying the condition* $0\leqslant g(x,y)\leqslant\bar{g}(x,y)$ $(x,y)\in\Delta$.

Proof. *Necessity.* If g is any non-negative solution of (23), then we get by induction the equations

$$(26)\quad g(x,y)=\sum_{n=0}^{m}\lambda_n^{i_0,\ldots,i_n}(x,y)h\big(a_n^{i_0,\ldots,i_n}(x,y),\beta_n^{i_0,\ldots,i_n}(x,y)\big)+$$
$$+\sum_{i_0=1}^{\nu}\cdots\sum_{i_{m+1}=1}^{\nu}\lambda_{m+1}^{i_0,\ldots,i_{m+1}}(x,y)g\big(a_{m+1}^{i_0,\ldots,i_{m+1}}(x,y),\beta_{m+1}^{i_0,\ldots,i_{m+1}}(x,y)\big),$$
$$m=0,1,\ldots,$$

whence

$$\sum_{n=0}^{m}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\lambda_n^{i_0,\ldots,i_n}(x,y)h\big(a_n^{i_0,\ldots,i_n}(x,y),\beta_n^{i_0,\ldots,i_n}(x,y)\big)\leqslant g(x,y),$$
$$(x,y)\in\Delta,$$

because g is non-negative.

By letting $m\to\infty$ we get (22).

Sufficiency. If (22) holds, then it is obvious that $\bar{g}$ defined by (24) satisfies equation (23); hence and according to (26) relation (25) is fulfilled. The uniqueness mentioned in the assertion of the Lemma follows from (25).

Remark 7. If $\nu=1$, $a(x,y)\overset{\mathrm{df}}{=}a_1(x,y)$, $\beta(x,y)\overset{\mathrm{df}}{=}\beta_1(x,y)$, $\lambda(x,y)\overset{\mathrm{df}}{=}\lambda_1(x,y)$, $(x,y)\in\Delta$, then the sequences $\{a_n(x,y)\}$, $\{\beta_n(x,y)\}$, $\{\lambda_n(x,y)\}$ defined by (21) are of the form

$$a_0(x,y)=x,\qquad a_{n+1}(x,y)=a\big(a_n(x,y),\beta_n(x,y)\big),$$
$$\beta_0(x,y)=y,\qquad \beta_{n+1}(x,y)=\beta\big(a_n(x,y),\beta_n(x,y)\big),$$
$$\lambda_0(x,y)=1,\qquad \lambda_{n+1}(x,y)=\prod_{i=0}^{n}\lambda\big(a_i(x,y),\beta_i(x,y)\big),$$
$$(x,y)\in\Delta,\ n=1,2,\ldots$$

Now (24) and (25) are of the form (see [5])

$$\bar{g}(x,y)=\sum_{n=0}^{\infty}\lambda_n(x,y)h\big(a_n(x,y),\beta_n(x,y)\big),\qquad (x,y)\in\Delta,$$

and

$$\lim_{n\to\infty}\lambda_n(x,y)\bar{g}\big(a_n(x,y),\beta_n(x,y)\big)=0,\qquad (x,y)\in\Delta.$$

LEMMA 4. *If*

1° $0 \leqslant \varphi_1(x, y) \leqslant \varphi_2(x, y)$, $(x, y) \in \Delta$,

2° $\sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \dots \sum_{i_n=1}^{\nu} \lambda_n^{i_0,\dots,i_n}(x, y)\varphi_2\big(a_n^{i_0,\dots,i_n}(x, y), \beta_n^{i_0,\dots,i_n}(x, y)\big)$, $(x, y) \in \Delta$,

then the functions $\bar{v}_i$,

$$\bar{v}_i(x, y) = \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \dots \sum_{i_n=1}^{\nu} \lambda_n^{i_0,\dots,i_n}(x, y)\varphi_i\big(a_n^{i_0,\dots,i_n}(x, y), \beta_n^{i_0,\dots,i_n}(x, y)\big),$$
$$(x, y) \in \Delta,\ i = 1, 2,$$

are non-negative solutions of the equations

$$v(x, y) = \sum_{i=1}^{\nu} \lambda_i(x, y) v\big(a_i(x, y), \beta_i(x, y)\big) + \varphi_j(x, y), \tag{27}$$
$$(x, y) \in \Delta,\ j = 1, 2,$$

respectively, and

$$\lim_{n\to\infty} \sum_{i_0=1}^{\nu} \dots \sum_{i_n=1}^{\nu} \lambda_n^{i_0,\dots,i_n}(x, y)\bar{v}_i\big(a_n^{i_0,\dots,i_n}(x, y), \beta_n^{i_0,\dots,i_n}(x, y)\big) = 0, \tag{28}$$
$$(x, y) \in \Delta,\ i = 1, 2.$$

Moreover, the functions $\bar{v}_i$, $i = 1, 2$, *are the unique solutions of* (27) *for* $i = 1, 2$, *respectively, in the class of functions satisfying the condition* $0 \leqslant v(x, y) \leqslant v_2(x, y)$, $(x, y) \in \Delta$.

The proof of the above lemma is similar to the proofs of Lemma 4 of [5] and of Lemma 5 of [1].

These considerations and Theorem 1 imply

THEOREM 6. *If assumption* H_1 *is satisfied, and*

1° $\Omega(x, y, z, p, q, s, r_1, \dots, r_\nu) = \sum_{i=1}^{\nu} \lambda_i(x, y) r_i$, $\lambda_i(x, y) \geqslant 0$, $i = 1, \dots, \nu$,

2° $\bar{g}(x, y) = \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \dots \sum_{i_n=1}^{\nu} \lambda_n^{i_0,\dots,i_n}(x, y) h\big(a_n^{i_0,\dots,i_n}(x, y), \beta_n^{i_0,\dots,i_n}(x, y)\big) < \infty$, $(x, y) \in \Delta$, *where* $h(x, y) = \sup_{0\leqslant\gamma\leqslant x} \sup_{0\leqslant\delta\leqslant y} \|f(\gamma, \delta, 0, \dots, 0)\|$, $(x, y) \in \Delta$, *and the function* $\bar{g}$ *is continuous, then there exists a solution* $\bar{z}$ *of equation* (4) *in* Δ *with the following properties*:

$$\|\bar{z}(x, y)\| \leqslant \bar{g}(x, y), \quad (x, y) \in \Delta,$$
$$\|\bar{z}(x, y) - z_n(x, y)\| \leqslant g_n(x, y), \quad (x, y) \in \Delta,\ n = 0, 1, \dots,$$

where

$$g_0(x, y) = \bar{g}(x, y),$$
$$g_{n+1}(x, y) = \sum_{m=n}^{\infty} \sum_{i_0=1}^{\nu} \dots \sum_{i_m=1}^{\nu} \lambda_m^{i_0,\dots,i_m}(x, y) h\big(a_m^{i_0,\dots,i_m}(x, y), \beta_m^{i_0,\dots,i_m}(x, y)\big),$$
$$(x, y) \in \Delta,\ n = 0, 1, \dots$$

The solution $\bar{z}$ is unique in the class of functions satisfying the inequality $\|z(x, y)\| \leqslant \bar{g}(x, y)$, $(x, y) \in \Delta$.

Remark 8. If the function $\bar{g}$ is not continuous, then we can prove only that there exists a solution of equation (4) which in general is not continuous.

Theorem 4 implies the following

THEOREM 7. *If assumption* H_1 *is satisfied, and*

1° $\Omega(x, y, z, p, q, s, r_1, \ldots, r_\nu) = \sum_{i=1}^{\nu} \lambda_i(x, y) r_i$, $(x, y) \in \Delta$,

2° *the functions* $\bar{z}$ *and* $\bar{p}$ *are solutions of equations* (4) *and* (18), *respectively,*

3° $\sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \ldots \sum_{i_n=1}^{\nu} \lambda_n^{i_0,\ldots,i_n}(x, y) c\big(\alpha_n^{i_0,\ldots,i_n}(x, y),\ \beta_n^{i_0,\ldots,i_n}(x, y)\big) < \infty$, $(x,y) \in \Delta$,

where

$$c(x, y) \geqslant \max\{\|\bar{z}(x, y)\| + \|\bar{p}(x, y)\|, \bar{h}(x, y)\}, \quad (x, y) \in \Delta,$$

and $\bar{h}(x, y)$ *is defined by condition* 2° *of Theorem* 4, *then*

$$\|\bar{z}(x, y) - \bar{p}(x, y)\| \leqslant \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \ldots \sum_{i_n=1}^{\nu} \lambda_n^{i_0,\ldots,i_n}(x, y) \bar{h}\big(\alpha_n^{i_0,\ldots,i_n}(x, y),\ \beta_n^{i_0,\ldots,i_n}(x, y)\big), \quad (x, y) \in \Delta.$$

6. Discussion of equation (4) of Volterra's type with a linear function Ω^*. Now we are going to consider the case where

$$\Omega^*(x, y, z, p, q, s, r_1, \ldots, r_\nu) = Kz + Mp + Nq + \sum_{i=1}^{\nu} \lambda_i r_i, (x, y) \in \Delta, \tag{29}$$

where K, M, N and λ_i, $i = 1, \ldots, \nu$, are non-negative constants.

Remark 9. The case where Ω^* depends also linearly on s may be reduced to the case considered here; indeed it is sufficient to put $r_0 = s$, $\alpha_0(x, y) = x$, $\beta_0(x, y) = y$ and to change adequately the range of the index i.

In this section we assume that the functions α_i, β_i, $i = 1, \ldots, \nu$, satisfy the conditions

$$0 \leqslant \alpha_i(x, y) \leqslant k_i x, \quad 0 \leqslant \beta_i(x, y) \leqslant l_i y, \quad 0 \leqslant k_i \leqslant 1, \quad 0 \leqslant l_i \leqslant 1, \quad (x, y) \in \Delta. \tag{30}$$

Now the sequences $\{\alpha_n^{i_0,\ldots,i_n}(x, y)\}$, $\{\beta_n^{i_0,\ldots,i_n}(x, y)\}$ and $\{\lambda_n^{i_0,\ldots,i_n}(x, y)\}$, $(x, y) \in \Delta$, defined by (21), satisfy the relations

$$0 \leqslant \alpha_n^{i_0,\ldots,i_n}(x, y) \leqslant x \prod_{r=0}^{n-1} k_{i_r}, \quad 0 \leqslant \beta_n^{i_0,\ldots,i_n}(x, y) \leqslant y \prod_{r=0}^{n-1} l_{i_r}, \tag{31}$$

$$\lambda_n^{i_0,\ldots,i_n}(x, y) = \frac{1}{\nu} \prod_{r=0}^{n-1} \lambda_{i_r}, \quad (x, y) \in \Delta,$$

where

$$\prod_{r=0}^{n-1} c_{i_r} \overset{\text{df}}{=} \begin{cases} 1 & \text{for } n = 0, \\ \prod_{r=0}^{n-1} c_{i_r} & \text{for } n \geqslant 1. \end{cases}$$

We have

LEMMA 5 [1]. *If* $t \in [0, +\infty)$ *and* $a \in [0, 1]$, *then*

(32) $$e^{t(a-1)} \leqslant a(1-e^{-t}) + e^{-t} \qquad (e^t = \exp t).$$

We put

$$S_k = \sum_{i=1}^{\nu} \lambda_i k_i, \quad S_l = \sum_{i=1}^{\nu} \lambda_i l_i, \quad S_{kl} = \sum_{i=1}^{\nu} \lambda_i k_i l_i.$$

LEMMA 6. *If*

1° *the function* H, $H(x, y) \overset{\text{df}}{=} \frac{1}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \dots \sum_{i_n=1}^{\nu} \left(\prod_{r=0}^{n-1} \lambda_{i_r} \right) h\left(x \prod_{r=0}^{n-1} k_{i_r}, y \prod_{r=0}^{n-1} l_{i_r} \right)$ $< \infty$, *is continuous for* $(x, y) \in \Delta$,

2° $0 \leqslant S_k < 1$, $0 \leqslant S_l < 1$,

3° $0 \leqslant k_i \leqslant 1$, $0 \leqslant l_i \leqslant 1$, $i = 1, \dots, \nu$,

4° *the function* h *is continuous, non-negative and non-decreasing in the set* Δ,

then

(a) *in the class of upper semicontinuous functions in* Δ *there exists a unique solution* g^* *of the equation*

(33) $$\begin{aligned} g(x, y) = {} & \frac{K}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \dots \sum_{i_n=1}^{\nu} \left(\prod_{r=0}^{n-1} \lambda_{i_r} \right) \int_0^{x \prod_{r=0}^{n-1} k_{i_r}} \int_0^{y \prod_{r=0}^{n-1} l_{i_r}} g(s, t)\, ds\, dt + \\ & + \frac{M}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=0}^{\nu} \dots \sum_{i_n=1}^{\nu} \left(\prod_{r=0}^{n-1} \lambda_{i_r} \right) \int_0^{y \prod_{r=0}^{n-1} l_{i_r}} g\left(x \prod_{r=0}^{n-1} k_{i_r}, t \right) dt + \\ & + \frac{N}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \dots \sum_{i_n=1}^{\nu} \left(\prod_{r=0}^{n-1} \lambda_{i_r} \right) \int_0^{x \prod_{r=0}^{n-1} k_{i_r}} g\left(s, y \prod_{r=0}^{n-1} l_{i_r} \right) ds + \\ & + \frac{1}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \dots \sum_{i_n=1}^{\nu} \left(\prod_{r=0}^{n-1} \lambda_{i_r} \right) h\left(x \prod_{r=0}^{n-1} k_{i_r}, y \prod_{r=0}^{n-1} l_{i_r} \right), \qquad (x, y) \in \Delta; \end{aligned}$$

this solution is continuous, non-negative and non-decreasing in the set Δ,

(b) *in the class of upper semicontinuous functions in Δ satisfying the condition* $0 \leqslant g(x,y) \leqslant g^*(x,y)$, $(x,y) \in \Delta$, *the function* g^* *is the unique, continuous, non-negative and non-decreasing solution of the equation*

$$(34)\quad g(x,y) = \sum_{i=1}^{\nu} \lambda_i g(k_i x, l_i y) + K \int_0^x \int_0^y g(s,t)\,ds\,dt + M \int_0^y g(x,t)\,dt + \\ + N \int_0^x g(s,y)\,ds + h(x,y), \qquad (x,y) \in \Delta,$$

(c) *in the class of upper semicontinuous functions in Δ satisfying the condition* $0 \leqslant g(x,y) \leqslant g^*(x,y)$, $(x,y) \in \Delta$, *the function* g, $g(x,y) \equiv 0$, $(x,y) \in \Delta$, *is the unique solution of the inequality*

$$(35)\quad g(x,y) \leqslant \sum_{i=1}^{\nu} \lambda_i g(k_i x, l_i y) + K \int_0^x \int_0^y g(s,t)\,ds\,dt + M \int_0^y g(x,t)\,dt + \\ + N \int_0^x g(s,y)\,ds, \qquad (x,y) \in \Delta.$$

Proof. Let A be the operator defined by the right-hand side of equation (33), and

$$\|g\|_* \overset{\mathrm{df}}{=} \max_{(x,y)\in\Delta} e^{-L(x+y)} |g(x,y)| \qquad \text{for } g \in C(\Delta, R^1_+),$$

where

$$L > \tfrac{1}{2}\left\{\frac{M}{1-S_l} + \frac{N}{1-S_k} + \left[\left(\frac{M}{1-S_l} + \frac{N}{1-S_k}\right)^2 + \frac{4K}{1-S_{kl}}\right]^{\frac{1}{2}}\right\}.$$

We obtain by induction

$$\frac{1}{\nu}\sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1} \lambda_{i_r} k_{i_r}\right) = S_k^n, \qquad \frac{1}{\nu}\sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1} \lambda_{i_r} l_{i_r}\right) = S_l^n,$$

$$\frac{1}{\nu}\sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1} \lambda_{i_r} k_{i_r} l_{i_r}\right) = S_{kl}^n.$$

Now from Lemma 5 we have for $g, z \in C(\Delta, R_+)$

$$\|Ag - Az\|_* \leqslant \frac{K}{\nu} \max_{(x,y)\in\Delta} e^{-L(x+y)} \Bigg| \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \left(\prod_{r=0}^{n-1} \lambda_{i_r}\right) \times \\ \times \int_0^{x \prod_{r=0}^{n-1} k_{i_r}} \int_0^{y \prod_{r=0}^{n-1} l_{i_r}} [g(s,t) - z(s,t)] e^{-L(s+t)} e^{L(s+t)}\,ds\,dt \Bigg| + \\ + \frac{M}{\nu} \max_{(x,y)\in\Delta} e^{-L(x+y)} \Bigg| \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \left(\prod_{r=0}^{n-1} \lambda_{i_r}\right) \times$$

$$\times \int\limits_0^{y\prod\limits_{r=0}^{n-1} l_{i_r}} \Big[g\Big(x\prod_{r=0}^{n-1} k_{i_r}, t\Big) - z\Big(x\prod_{r=0}^{n-1} k_{i_r}, t\Big)\Big] e^{-L(x\prod\limits_{r=0}^{n-1} k_{i_r}+t)} e^{L(x\prod\limits_{r=0}^{n-1} k_{i_r}+t)} dt\Big| +$$

$$+\frac{N}{\nu}\max_{(x,y)\in\Delta} e^{-L(x+y)} \Big|\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\Big(\prod_{r=0}^{n-1}\lambda_{i_r}\Big)\times$$

$$\times \int\limits_0^{x\prod\limits_{r=0}^{n-1} k_{i_r}} \Big[g\Big(s, y\prod_{r=0}^{n-1} l_{i_r}\Big) - z\Big(s, y\prod_{r=0}^{n-1} l_{i_r}\Big)\Big] e^{-L(s+y\prod\limits_{r=0}^{n-1} l_{i_r})} e^{L(s+y\prod\limits_{r=0}^{n-1} l_{i_r})} ds\Big|$$

$$\leqslant \frac{K}{\nu L^2}\|g-z\|_* \sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\Big(\prod_{r=0}^{n-1}\lambda_{i_r}\Big)\times$$

$$\times \max_{(x,y)\in\Delta}[e^{Lx(\prod\limits_{r=0}^{n-1} k_{i_r}-1)} - e^{-Lx}][e^{Ly(\prod\limits_{r=0}^{n-1} l_{i_r}-1)} - e^{-Ly}] +$$

$$+\frac{M}{\nu L}\|g-z\|_* \sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\Big(\prod_{r=0}^{n-1}\lambda_{i_r}\Big)\times$$

$$\times \max_{(x,y)\in\Delta} e^{Lx(\prod\limits_{r=0}^{n-1} k_{i_r}-1)}[e^{Ly(\prod\limits_{r=0}^{n-1} l_{i_r}-1)} - e^{-Ly}] +$$

$$+\frac{N}{\nu L}\|g-z\|_* \sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\Big(\prod_{r=0}^{n-1}\lambda_{i_r}\Big)\times$$

$$\times \max_{(x,y)\in\Delta} e^{Ly(\prod\limits_{r=0}^{n-1} l_{i_r}-1)}[e^{Lx(\prod\limits_{r=0}^{n-1} k_{i_r}-1)} - e^{-Lx}]$$

$$\leqslant \frac{K}{\nu L^2}\|g-z\|_* \sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\Big(\prod_{r=0}^{n-1}\lambda_{i_r}\Big)\times$$

$$\times \max_{(x,y)\in\Delta}\Big(\prod_{r=0}^{n-1} k_{i_r}\Big)\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)[1-e^{-Lx}][1-e^{-Ly}] +$$

$$+\frac{M}{\nu L}\|g-z\|_* \sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\Big(\prod_{r=0}^{n-1}\lambda_{i_r}\Big)\times$$

$$\times \max_{(x,y)\in\Delta}\Big(\prod_{r=0}^{n-1} l_{i_r}\Big)[1-e^{-Ly}] +$$

$$+\frac{N}{\nu L}\|g-z\|_*\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\Big(\prod_{r=0}^{n-1}\lambda_{i_r}\Big)\times$$

$$\times\max_{(x,y)\in\Delta}\Big(\prod_{r=0}^{n-1}k_{i_r}\Big)[1-e^{-Lx}]$$

$$=\frac{K}{L^2}\|g-z\|_*\sum_{n=0}^{\infty}S_{kl}^n[1-e^{-La}][1-e^{-Lb}]+$$

$$+\frac{M}{L}\|g-z\|_*\sum_{n=0}^{\infty}S_l^n[1-e^{-Lb}]+$$

$$+\frac{N}{L}\|g-z\|_*\sum_{n=0}^{\infty}S_k^n[1-e^{-La}]$$

$$\leqslant\left[\frac{1}{L^2}\cdot\frac{K}{1-S_{kl}}+\frac{1}{L}\left(\frac{M}{1-S_l}+\frac{N}{1-S_k}\right)\right]\|g-z\|_*.$$

Since

$$\left[\frac{1}{L^2}\cdot\frac{K}{1-S_{kl}}+\frac{1}{L}\left(\frac{M}{1-S_l}+\frac{N}{1-S_k}\right)\right]<1$$

for L satisfying the condition given before, then by the well-known Banach fixed-point theorem we infer that equation (33) has the unique solution g^* defined in Δ. This solution is the limit of the uniformly convergent sequence $\{z_n\}$ of the non-negative and continuous functions z_n defined by the relation

$$z_0(x,y)=0,\qquad (x,y)\in\Delta,$$
$$z_{n+1}(x,y)=Az_n(x,y),\qquad (x,y)\in\Delta,\ n=0,1,\ldots,$$

and therefore it is continuous, non-negative and non-decreasing because z_n are so. Further, it is easy to prove that any upper semicontinuous solution of equation (33) is the limit of the sequence $\{z_n\}$; therefore it is identical with the function g^*.

We shall prove that the function g^* satisfies equation (34). Indeed, since g^* fulfils equation (33), we have

$$R(x,y)\overset{\mathrm{df}}{=}g^*(x,y)-\sum_{i=1}^{\nu}\lambda_i g^*(k_ix,l_iy)-K\int_0^x\int_0^y g^*(s,t)\,ds\,dt-$$

$$-M\int_0^y g^*(x,t)\,dt-N\int_0^x g^*(s,y)\,ds-h(x,y)$$

$$= g^*(x, y) - \sum_{i=1}^{\nu} \lambda_i \Big[\frac{K}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \Big(\prod_{r=0}^{n-1} \lambda_{i_r} \Big) \times$$

$$\times \int_0^{xk_i \prod_{r=0}^{n-1} k_{i_r}} \int_0^{yl_i \prod_{r=0}^{n-1} l_{i_r}} g^*(s, t)\, ds\, dt +$$

$$+ \frac{M}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \Big(\prod_{r=0}^{n-1} \lambda_{i_r} \Big) \int_0^{yl_i \prod_{r=0}^{n-1} l_{i_r}} g^*\Big(xk_i \prod_{r=0}^{n-1} k_{i_r}, t \Big) dt +$$

$$+ \frac{N}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \Big(\prod_{r=0}^{n-1} \lambda_{i_r} \Big) \int_0^{xk_i \prod_{r=0}^{n-1} k_{i_r}} g^*\Big(s, yl_i \prod_{r=0}^{n-1} l_{i_r} \Big) ds +$$

$$+ \frac{1}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \Big(\prod_{r=0}^{n-1} \lambda_{i_r} \Big) h\Big(xk_i \prod_{r=0}^{n-1} k_{i_r}, yl_i \prod_{r=0}^{n-1} l_{i_r} \Big) \Big] -$$

$$- K \int_0^x \int_0^y g^*(s, t)\, ds\, dt - M \int_0^y g^*(x, t)\, dt - N \int_0^x g^*(s, y)\, ds - h(x, y)$$

$$= g^*(x, y) - K \Big(\frac{1}{\nu} \sum_{n=1}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \Big(\prod_{r=0}^{n-1} \lambda_{i_r} \Big) \times$$

$$\times \int_0^{x \prod_{r=0}^{n-1} k_{i_r}} \int_0^{y \prod_{r=0}^{n-1} l_{i_r}} g^*(s, t)\, ds\, dt + \int_0^x \int_0^y g^*(s, t)\, ds\, dt \Big) -$$

$$- M \Big(\frac{1}{\nu} \sum_{n=1}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \Big(\prod_{r=0}^{n-1} \lambda_{i_r} \Big) \times$$

$$\times \int_0^{y \prod_{r=0}^{n-1} l_{i_r}} g^*\Big(x \prod_{r=0}^{n-1} k_{i_r}, t \Big) dt + \int_0^y g^*(x, t)\, dt \Big) -$$

$$- N \Big(\frac{1}{\nu} \sum_{n=1}^{\nu} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \Big(\prod_{r=0}^{n-1} \lambda_{i_r} \Big) \times$$

$$\times \int_0^{x \prod_{r=0}^{n-1} k_{i_r}} g^*\Big(s, y \prod_{r=0}^{n-1} l_{i_r} \Big) ds + \int_0^x g^*(s, y)\, ds \Big) -$$

$$-\left(\frac{1}{\nu}\sum_{n=1}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1}\lambda_{i_r}\right)h\left(x\prod_{r=0}^{n-1}k_{i_r}, y\prod_{r=0}^{n-1}l_{i_r}\right)+h(x, y)\right)$$

$$=g^*(x, y)-\frac{K}{\nu}\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1}\lambda_{i_r}\right)\int_0^{x\prod_{r=0}^{n-1}k_{i_r}}\int_0^{y\prod_{r=0}^{n-1}l_{i_r}} g^*(s, t)\,ds\,dt-$$

$$-\frac{M}{\nu}\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1}\lambda_{i_r}\right)\int_0^{y\prod_{r=0}^{n-1}l_{i_r}} g\left(x\prod_{r=0}^{n-1}k_{i_r}, t\right)dt-$$

$$-\frac{N}{\nu}\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1}\lambda_{i_r}\right)\int_0^{x\prod_{r=0}^{n-1}k_{i_r}} g^*\left(s, y\prod_{r=0}^{n-1}l_{i_r}\right)ds-$$

$$-\frac{1}{\nu}\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1}\lambda_{i_r}\right)h\left(x\prod_{r=0}^{n-1}k_{i_r}, y\prod_{r=0}^{n-1}l_{i_r}\right)\equiv 0;$$

thus g^* is a solution of equation (34).

Now we prove that upper semicontinuous solution g of equation (34) satisfying the condition $0\leqslant g(x, y)\leqslant g^*(x, y)$, $(x, y)\in\Delta$, is a solution of equation (33).

Let g_0 be an upper semicontinuous solution of equation (34) satisfying the condition $0\leqslant g_0(x, y)\leqslant g^*(x, y)$, $(x, y)\in\Delta$. Put

$$\varphi_1(x, y) = K\int_0^x\int_0^y g_0(s, t)\,ds\,dt+M\int_0^y g_0(x, t)\,dt+N\int_0^x g_0(s, y)\,ds+h(x, y).$$

Now for $(x, y)\in\Delta$ we have

$$(36)\quad s(x, y)\stackrel{\text{df}}{=}\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1}\lambda_{i_r}\right)\varphi_1\left(x\prod_{r=0}^{n-1}k_{i_r}, y\prod_{r=0}^{n-1}l_{i_r}\right)$$

$$=\frac{K}{\nu}\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1}\lambda_{i_r}\right)\int_0^{x\prod_{r=0}^{n-1}k_{i_r}}\int_0^{y\prod_{r=0}^{n-1}l_{i_r}} g_0(s, t)\,ds\,dt+$$

$$+\frac{M}{\nu}\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\left(\prod_{r=0}^{n-1}\lambda_{i_r}\right)\int_0^{y\prod_{r=0}^{n-1}l_{i_r}} g_0\left(x\prod_{r=0}^{n-1}k_{i_r}, t\right)dt+$$

$$+\frac{N}{\nu}\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\Big(\prod_{r=0}^{n-1}\lambda_{i_r}\Big)\int\limits_0^{x\prod_{r=0}^{n-1}k_{i_r}} g_0\Big(s, y\prod_{r=0}^{n-1}l_{i_r}\Big)ds+$$

$$+\frac{1}{\nu}\sum_{n=0}^{\infty}\sum_{i_0=1}^{\nu}\cdots\sum_{i_n=1}^{\nu}\Big(\prod_{r=0}^{n-1}\lambda_{i_r}\Big)\times$$

$$\times h\Big(x\prod_{r=0}^{n-1}k_{i_r}, y\prod_{r=0}^{n-1}l_{i_r}\Big) = (Ag_0)(x, y).$$

Hence

$$s(x, y) \leqslant Kab\cdot\max_{(x,y)\in\Delta}|g_0(x, y)|\sum_{n=0}^{\infty}S_{kl}^n+Mb\cdot\max_{(x,y)\in\Delta}|g_0(x, y)|\sum_{n=0}^{\infty}S_l^n+$$

$$+Na\cdot\max_{(x,y)\in\Delta}|g_0(x, y)|\sum_{n=0}^{\infty}S_k^n+H(x, y) < \infty,$$

and from Lemma 3 it follows that the equation

$$g(x, y) = \sum_{i=1}^{\nu}\lambda_i g(k_i x, l_i y)+\varphi_1(x, y), \qquad (x, y)\in\Delta, \tag{37}$$

has a unique solution in the class $0\leqslant g(x, y)\leqslant (Ag_0)(x, y)$, and this solution is the function s, $s(x, y) = (Ag_0)(x, y)$.

Further, we put

$$\varphi_2(x, y) = K\int_0^x\int_0^y g^*(s, t)\,ds\,dt+M\int_0^y g^*(x, t)\,dt+N\int_0^x g^*(s, y)\,ds+h(x, y).$$

It is obvious that equation (37) with $\varphi_2(x, y)$ instead of $\varphi_1(x, y)$ also has a unique solution in the class $0\leqslant g(x, y)\leqslant (Ag^*)(x, y) = g^*(x, y)$.

Now from Lemma 4 it follows that the function s, $s(x, y) = (Ag_0)(x, y)$, is a unique solution of (37) in the class $0\leqslant g(x, y)\leqslant g^*(x, y)$, $(x, y)\in\Delta$.

Since g_0 is also a solution of (37) in the class $0\leqslant g(x, y)\leqslant g^*(x, y)$, $(x, y)\in\Delta$, then $s(x, y) = g_0(x, y)$, $(x, y)\in\Delta$. Hence g_0 is a solution of (33).

Since each upper semicontinuous solution of (34) from the class $0\leqslant g(x, y)\leqslant g^*(x, y)$, $(x, y)\in\Delta$ is a solution of (33), the function g^* is a unique solution of (33), and g^* satisfies equation (34), then the function g^* is a unique solution of (34) in the class pointed out above.

This completes the proof of part (b).

Now we prove that the function g, $g(x, y) \equiv 0$, $(x, y) \in \Delta$, is a unique upper semicontinuous solution of the equation

$$(38) \qquad g(x, y) = \sum_{i=1}^{\nu} \lambda_i g(k_i x, l_i y) + K \int_0^x \int_0^y g(s, t) ds dt + M \int_0^y g(x, t) + \\ + N \int_0^x g(s, y) ds, \qquad (x, y) \in \Delta,$$

satisfying the condition $0 \leqslant g(x, y) \leqslant g^*(x, y)$, $(x, y) \in \Delta$.

Let g_0 be an upper semicontinuous solution of (38) fulfilling this condition. According to our considerations in proving (b), we see that g_0 is a solution of equation (33) with $h = 0$, but the only solution of that equation is the function g, $g(x, y) \equiv 0$, $(x, y) \in \Delta$; therefore $g_0(x, y) \equiv 0$, $(x, y) \in \Delta$.

Now (c) is implied by Remark 4.

Thus the proof of Lemma 6 is completed.

Remark 10. If the function Ω^* does not depend on p, q, s, then assumption 2° of Lemma 6 may be replace by the following one: $0 \leqslant S_{kl} < 1$.

These considerations, Lemma 2 and Theorem 2 imply

THEOREM 8. *If assumption* H_1 *is satisfied and*

1° *conditions* (29) *and* (30) *are fulfilled,*

2° $$H(x, y) \stackrel{\mathrm{df}}{=} \frac{1}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \Big(\prod_{r=0}^{n-1} \lambda_{i_r} \Big) h \Big(x \prod_{r=0}^{n-1} k_{i_r}, y \prod_{r=0}^{n-1} l_{i_r} \Big) < \infty, \\ (x, y) \in \Delta,$$

where

$$h(x, y) = \sup_{0 \leqslant \gamma \leqslant x} \sup_{0 \leqslant \delta \leqslant y} \|f(\gamma, \delta, 0, \ldots, 0)\|,$$

and the function H *is continuous for* $(x, y) \in \Delta$,

3° $0 \leqslant S_k < 1$, $0 \leqslant S_l < 1$,

4° $0 \leqslant k_i \leqslant 1$, $0 \leqslant l_i \leqslant 1$, $i = 1, \ldots, \nu$,

then there exists a unique and continuous solution $\bar{z}$ *of equation* (4) *with the following properties*:

$$\|\bar{z}(x, y)\| \leqslant g^*(x, y), \qquad (x, y) \in \Delta,$$
$$\|\bar{z}(x, y) - \bar{z}_n(x, y)\| \leqslant g_n(x, y), \qquad (x, y) \in \Delta,\ n = 0, 1, \ldots,$$

where

$$g_0(x, y) = g^*(x, y), \qquad (x, y) \in \Delta, \qquad g^*(x, y) \text{ is defined in Lemma 6},$$

$$g_{n+1}(x, y) = \sum_{i=1}^{\nu} \lambda_i g_n(k_i x, l_i y) + K \int_0^x \int_0^y g_n(s, t) ds dt + M \int_0^y g_n(x, t) dt + \\ + N \int_0^y g_n(s, y) ds, \qquad (x, y) \in \Delta,\ n = 0, 1, \ldots$$

The solution $\bar{z}$ is unique in the class of functions satisfying the inequality $\|z(x, y)\| \leqslant g^*(x, y)$, $(x, y) \in \Delta$.

Remark 11. Condition 2° of Theorem 8 is fulfilled if

$$\|f(x, y, 0, \ldots, 0)\| \leqslant B(x+y), \qquad (x, y) \in \Delta, \; B = \mathrm{const} \geqslant 0;$$

now

$$H(x, y) \leqslant B\left(\frac{x}{1-S_k} + \frac{y}{1-S_l}\right).$$

Theorem 5 implies the following

THEOREM 9. *If assumptions of Theorem* 8 (*except* 2°) *are satisfied and if*

1° *the functions $\bar{z}$ and $\bar{p}$ are solutions of equations* (4) *and* (18), *respectively,*

2° $$H(x, y) \overset{\mathrm{df}}{=} \frac{1}{\nu} \sum_{n=0}^{\infty} \sum_{i_0=1}^{\nu} \cdots \sum_{i_n=1}^{\nu} \left(\prod_{r=0}^{n-1} \lambda_{i_r}\right) \psi\left(x \prod_{r=0}^{n-1} k_{i_r}, y \prod_{r=0}^{n-1} l_{i_r}\right) < \infty, \qquad (x, y) \in \Delta,$$

where

$$\psi(x, y) \geqslant \max\{\sup_{0 \leqslant \gamma \leqslant x} \sup_{0 \leqslant \delta \leqslant y} [\|\bar{z}(\gamma, \delta)\| + \|\bar{p}(\gamma, \delta)\|], \bar{h}(x, y)\}, \qquad (x, y) \in \Delta,$$

3° *$\bar{h}(x, y)$ is defined by condition 2° of Theorem 4, and the function H is continuous in Δ,*

then

(a) *there exists a continuous, non-negative and non-decreasing solution $\tilde{g}$ of the equation*

$$g(x, y) = \sum_{i=1}^{\nu} \lambda_i g(k_i x, l_i y) + K \int_0^x \int_0^y g(s, t)\, ds\, dt + M \int_0^y g(x, t)\, dt + N \int_0^x g(s, y)\, ds + \psi(x, y), \qquad (x, y) \in \Delta,$$

(b) *the sequence $\{\tilde{g}_n\}$,*

$$\tilde{g}_0(x, y) = \tilde{g}(x, y), \qquad (x, y) \in \Delta,$$

$$\tilde{g}_{n+1}(x, y) = \sum_{i=1}^{\nu} \lambda_i \tilde{g}_n(k_i x, l_i y) + K \int_0^x \int_0^y \tilde{g}_n(s, t)\, ds\, dt + M \int_0^y \tilde{g}_n(x, t)\, dt + N \int_0^x \tilde{g}_n(s, y)\, ds + \bar{h}(x, y), \qquad (x, y) \in \Delta, \; n = 0, 1, \ldots,$$

has a limit g^*, and the function g^* is continuous, non-negative and non-decreasing,

$$g^*(x, y) \leqslant \tilde{g}(x, y), \qquad (x, y) \in \Delta,$$

(c) *the estimation*

$$\|\bar{z}(x, y) - \bar{p}(x, y)\| \leqslant g^*(x, y), \qquad (x, y) \in \Delta,$$

holds true.

Remark 12. Our consideration can easily be extended to an appropriate equation with an unknown function u depending on n independent variables. It is also obvious that the results of this paper hold if $\Delta = [0, a] \times [0, b]$ is replaced by $\Delta^\infty = [0, +\infty) \times [0, +\infty)$ and if uniform convergence on Δ is replaced by such convergence on compact sets included in Δ^∞.

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