ANNALES
POLONICI MATHEMATICI
XIX (1967)

A difference method for certain hyperbolic systems of non-linear partial differential equations of the first order

by Z. Kowalski (Kraków)

§ 1. Let us suppose that the partial differential system of the first order is of the form

$$(1.1)\qquad \frac{\partial u_i}{\partial \xi} = f_i\left(\xi, x, u, \frac{\partial u_i}{\partial x_1}, \ldots, \frac{\partial u_i}{\partial x_p}\right), \quad i = 1, \ldots, n,$$

where $\xi \in R^1$, $x = (x_1, \ldots, x_p) \in R^p$, $u = (u_1, \ldots, u_n) \in R^n$.

(1.1) will be called the *diagonal system* and will be solved with the aid of the difference equation

$$(1.2)\qquad v_i^{\omega(M)} = v_i^M + kf_i\left(\xi^\mu, x^m, v^M, \frac{v_i^M - v_i^{1(M)}}{h}, \ldots, \frac{v_i^M - v_i^{p(M)}}{h}\right),$$

where v_i^M $(i = 1, 2, \ldots, n)$ denotes the approximate value of the solution at the nodal point M with coordinates (ξ^μ, x^m), cf. fig. 1, and § 4, equation (4.2).

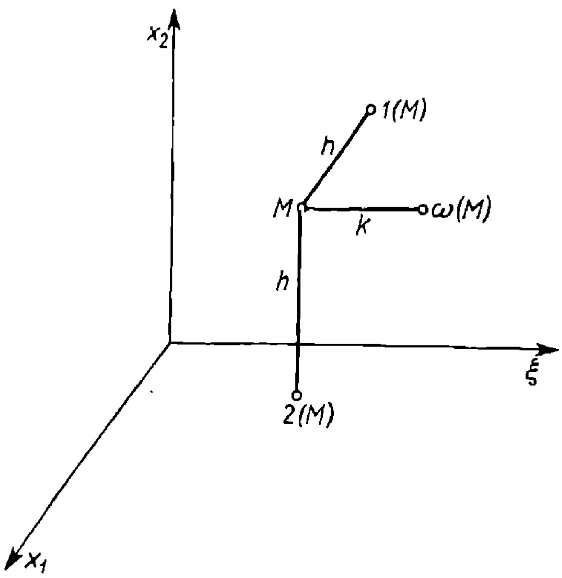

Fig. 1. The nodal points M, $\omega(M)$, $1(M)$, ..., $p(M)$ in the case $p = 2$

The difference scheme (1.2) can be applied to digital computers since $v_i^{\omega(M)}$ can be obtained with the aid of the preceding values v_i^M, $v_i^{1(M)}$, ..., $v_i^{p(M)}$ only.

The purpose of this paper is to prove, under suitable assumptions, the convergence of the difference method (1.2), and to derive some error estimates, cf. Theorem 1.

In the proofs we use, as in [1], the method of difference inequalities.

§ 2. Let us denote by E the set of points of the real $(p+1)$-dimensional space R^{p+1}:

$$E\colon\ 0 \leqslant \xi \leqslant a,\ 0 \leqslant x_j \leqslant a, \quad a > 0\ (j = 1, \ldots, p)\,. \tag{2.1}$$

We shall consider all nodal points in the set E, with coordinates defined by

$$\begin{gathered} \xi^\mu = \mu k, \quad x_j^\nu = \nu h \quad (\mu = 0, 1, \ldots;\ \nu = 0, 1, \ldots;\ j = 1, \ldots, p)\,, \\ 0 < h = \text{const},\ 0 < k = \text{const}, \quad \text{for } (\xi^\mu, x_1^{m_1}, \ldots, x_p^{m_p}) \in E\,, \end{gathered} \tag{2.2}$$

m_j $(j = 1, \ldots, p)$ being suitable natural numbers.

There is a one-to-one correspondence between the nodal points (2.2) in the set E, and their indices:

$$(\mu, m_1, m_2, \ldots, m_p) \quad \text{for} \quad (\xi^\mu, x_1^{m_1}, \ldots, x_p^{m_p}) \in E\,. \tag{2.3}$$

This leads to the following notations:

$$M = (\mu, m_1, m_2, \ldots, m_p)\,, \tag{2.4}$$

for the sequence of indices resulting from (2.3), or

$$M = (\mu, m)\,, \tag{2.5}$$

where

$$m = (m_1, m_2, \ldots, m_p)\,, \tag{2.6}$$

the coordinates of the nodal points (2.2) being denoted by

$$(\xi^\mu, x^m)\,, \tag{2.7}$$

for

$$x^m = (x_1^{m_1}, \ldots, x_p^{m_p})\,. \tag{2.8}$$

In this paper we shall deal with nodal points (2.2) or (2.7) in the set E, characterized by corresponding sequences M, cf. (2.5), of indices.

We shall consider also the nodal points in the set E, characterized by the following sequences of indices:

$$\omega(M) = (\mu+1, m), \quad j(M) = \big(\mu, j(m)\big)\,, \tag{2.9}$$

where

$$(2.10)\quad j(m) = (m_1, \ldots, m_{j-1}, m_j-1, m_{j+1}, \ldots, m_p) \quad (j = 1, 2, \ldots, p).$$

It can be seen from (2.9) and (2.10) that

$$(2.11)\qquad j(\mu, m) = \big(\mu, j(m)\big) \quad \text{for} \quad j = 1, 2, \ldots, p.$$

Let us suppose that, to each nodal point (2.2) in the set E characterized by the sequence M, there corresponds a sequence of n numbers:

$$(2.12)\qquad v_i^M \quad (i = 1, 2, \ldots, n).$$

The numbers (2.12) being given, we can compute the differences

$$(2.13)\qquad v_i^{M\sim} = \frac{1}{k}(v_i^{\omega(M)} - v_i^M) \quad (i = 1, 2, \ldots, n),$$

and

$$(2.14)\qquad \dot{v}_i^{Mj} = \frac{1}{h}(v_i^M - v_i^{j(M)}) \quad (i = 1, \ldots, n;\ j = 1, \ldots, p),$$

and introduce the p-dimensional vectors

$$(2.15)\qquad v_i^{M\Delta} = (v_i^{M1}, v_i^{M2}, \ldots, v_i^{Mp}) \quad (i = 1, 2, \ldots, n).$$

Those differences will be used instead of the derivatives in the system (1.1) in a following order: (2.13) will replace the derivatives with respect to the time variable ξ on the left-hand member of (1.1), and (2.14)—the derivatives with respect to space variables x_j ($j = 1, 2, \ldots, p$).

§ 3. The main theorem of the paper will be proved under the following

ASSUMPTIONS H. (1) Let us suppose that the scalar functions $f_i(\xi, x, u, q)$ $(i = 1, \ldots, n)$, $x = (x_1, \ldots, x_p)$, $u = (u_1, \ldots, u_n)$, $q = (q_1, \ldots, q_p)$, are of the class C^1 in the domain D defined by

$$(3.1)\quad D\colon\ 0 \leqslant \xi \leqslant a,\ 0 \leqslant x_j \leqslant a,\ -\infty < u_i < +\infty,\ -\infty < q_j < +\infty$$
$$(j = 1, \ldots, p;\ i = 1, \ldots, n)\ (a > 0).$$

(2) The derivatives of the functions f_i satisfy the relations:

$$(3.2)\quad \left|\frac{\partial f_i}{\partial u_\lambda}\right| \leqslant L, \quad \frac{\partial f_i}{\partial q_j} \leqslant 0 \quad (j = 1, \ldots, p;\ i = 1, \ldots, n;\ \lambda = 1, \ldots, n),$$

the intervals h and k, cf. (2.2), being chosen so as to give

$$(3.3)\qquad \sum_{j=1}^{p} \frac{\partial f_i}{\partial q_j} + \frac{h}{k} \geqslant 0 \quad \text{for} \quad (\xi, x, u, q) \in D.$$

(3) The scalar functions $u_i(\xi, x)$ $(i = 1, \dots, n)$ of the class C^1 are the solutions of the diagonal system of partial differential equations:

$$\frac{\partial u_i}{\partial \xi} = f_i\left(\xi, x, u, \frac{\partial u_i}{\partial x}\right) \quad (i = 1, 2, \dots, n), \tag{3.4}$$

in the domain E, cf. (2.1), the vector $\partial u_i/\partial x$ being defined as

$$\frac{\partial u_i}{\partial x} = \left(\frac{\partial u_i}{\partial x_1}, \dots, \frac{\partial u_i}{\partial x_p}\right). \tag{3.5}$$

We assume in addition that $u_i(\xi, x)$ satisfy the following boundary conditions:

$$\begin{aligned} &u_i(0, x) = \varphi_{i0}(x), \\ &u_i(\xi, x) = \varphi_{ij}(\xi, x) \quad \text{for} \quad (\xi, x) \in E, x_j = 0, \\ &\qquad (j = 1, \dots, p;\ i = 1, \dots, n). \end{aligned} \tag{3.6}$$

§ 4. The following considerations deal with the approximate solution v_i^M $(i = 1, \dots, n)$, cf. § 2, of the system (3.4), which will be defined only at the nodal points (2.2).

The boundary conditions for the numbers v_i^M $(i = 1, \dots, n)$ are:

$$\begin{aligned} &v_i^M = \varphi_{i0}(x^m) \quad \text{for} \quad M = (0, m), \\ &v_i^M = \varphi_{ij}(\xi^\mu, x_1^{m_1}, \dots, x_j^0, \dots, x_p^{m_p}), \quad \text{for} \quad \mu = 0, 1, \dots; \\ &\qquad j = 1, \dots, p;\ i = 1, \dots, n; \text{ and } M = (\mu, m_1, \dots, 0, \dots, m_p). \end{aligned} \tag{4.1}$$

The values v_i^M $(i = 1, \dots, n)$ at the remaining nodal points we define successively, starting from (4.1), with the aid of the difference equation

$$v_i^{M\sim} = f_i(\xi^\mu, x^m, v^M, v_i^{M\Delta}) \quad (i = 1, \dots, n), \tag{4.2}$$

where

$$v^M = (v_1^M, v_2^M, \dots, v_n^M). \tag{4.3}$$

The approximate solution v_i^M being defined, we shall consider also the numbers u_i^M $(i = 1, \dots, n)$, which will represent the values of the solution $u_i(\xi, x)$ $(i = 1, \dots, n)$ of the system (3.4) at the nodal points (2.2).

Accordingly, we define the corresponding differences for u_i^M in the same way as for v_i^M, cf. (2.13), (2.14), the boundary values for u_i^M being the consequence of the boundary values for the solution $u_i(\xi, x)$:

$$\begin{aligned} &u_i^M = \varphi_{i0}(x^m) \quad \text{for} \quad M = (0, m), \\ &u_i^M = \varphi_{ij}(\xi^\mu, x_1^{m_1}, \dots, x_j^0, \dots, x_p^{m_p}) \quad \text{for} \quad \mu = 0, 1, \dots; \\ &\qquad j = 1, \dots, p;\ i = 1, \dots, n; \text{ and } M = (\mu, m_1, \dots, 0, \dots, m_p). \end{aligned} \tag{4.4}$$

It can be seen that the numbers u_i^M satisfy the equation

$$u_i^{M\sim} = f_i(\xi^\mu, x^m, u^M, u_i^{M\Delta}) + \eta_i^M \qquad (i = 1, \ldots, n), \tag{4.5}$$

and the condition

$$\max_M |\eta_i^M| \to 0 \quad \text{as} \quad h \to 0 \qquad (i = 1, \ldots, n), \tag{4.6}$$

at the nodal points $M = (\mu, m)$ in the domain E for $m_j \geqslant 1$ $(j = 1, \ldots, p)$.

In fact, (4.5) follows from (3.4), since the solution $u_i(\xi, x)$ of system (3.4) is of the class C^1.

We shall make use of the definitions

$$\varepsilon_i(h) = \max_M |\eta_i^M| \quad \text{for} \quad M \in E \qquad (i = 1, \ldots, n), \tag{4.7}$$

$$\varepsilon(h) = \sum_{i=1}^{n} \varepsilon_i(h), \tag{4.8}$$

and of the relations

$$\varepsilon_i(h) \to 0 \quad \text{as} \quad h \to 0 \qquad (i = 1, \ldots, n), \tag{4.9}$$

$$\varepsilon(h) \to 0 \quad \text{as} \quad h \to 0, \tag{4.10}$$

which are satisfied because of (4.6), (4.7) and (4.8).

Remark 1. (a) The solution of problem (3.4)-(3.6) is also the solution of a problem of Cauchy for equation (3.4) with initial condition $u_i(0, x) = \varphi_{i0}(x)$ $(i = 1, \ldots, n)$.

(b) We can assume $\partial f_i/\partial q_j \leqslant A$ $(A = \text{const})$ in (3.2), since $\partial f_i/\partial q_j \leqslant 0$ can be obtained from $\partial f_i/\partial q_j \leqslant A$ with the aid of a transformation.

§ 5. Now we shall give without proof a lemma on linear difference inequalities.

LEMMA 1. *Let us suppose that the numbers s^μ $(\mu = 0, 1, \ldots)$ satisfy the non-homogeneous linear difference inequality*

$$s^{\mu\sim} \leqslant K s^\mu + \varepsilon \qquad (\mu = 0, 1, \ldots), \tag{5.1}$$

and the initial condition $s^0 = 0$, the difference $s^{\mu\sim}$ being defined by

$$s^{\mu\sim} = \frac{1}{H}(s^{\mu+1} - s^\mu) \qquad (\mu = 0, 1, \ldots), \tag{5.2}$$

where $0 < H = const$, $0 < K = const$, $0 < \varepsilon = const$.

Under these assumptions

$$s^\mu \leqslant \frac{\varepsilon}{K}(e^{KH\mu} - 1) \qquad (\mu = 0, 1, \ldots). \tag{5.3}$$

This lemma can be proved by induction.

§ 6. LEMMA 2. *Let us assume that the values of the solution u_i^M and approximation v_i^M at the nodal points of the domain E, cf.* (2.1), *satisfy relations* (4.4), (4.5) *and* (4.1), (4.2), *respectively.*

Suppose also that the assumptions H *are fulfilled, and let us write*

(6.1) $$r_i^M = u_i^M - v_i^M ,$$

(6.2) $$s_i^\mu = \max_m r_i^{\mu,m}, \quad z_i^\mu = \min_m r_i^{\mu,m} \quad (i = 1, \ldots, n;\ \mu = 0, 1, \ldots),$$

at the nodal points of the domain E.

Under these assumptions, the numbers s_i^μ and z_i^μ satisfy the conditions

(6.3) $$s_i^\mu \geqslant 0, \quad z_i^\mu \leqslant 0,$$

the initial conditions $s_i^0 = 0$, $z_i^0 = 0$, and the non-homogeneous linear difference inequalities

(6.4) $$\begin{aligned} & s_i^{\mu\sim} \leqslant L \sum_{j=1}^{n} s_j^\mu + \varepsilon_i(h), \\ & z_i^{\mu\sim} \geqslant L \sum_{j=1}^{n} z_j^\mu - \varepsilon_i(h) \quad (i = 1, \ldots, n;\ \mu = 0, 1, \ldots), \end{aligned}$$

$\varepsilon_i(h)$ being defined by (4.7).

Proof. Since the boundary values for u^M and v^M are equal, it follows, in virtue of (6.1), that

(6.5) $r_i^M = 0$, at the boundary nodal points $M = (0, m)$ and $M = (\mu, m_1, \ldots, 0, \ldots, m_p)$,

cf. (4.1) and (4.4). Therefore, the greatest value s_i^μ of the numbers r_i^M must be non-negative, $s_i^\mu \geqslant 0$, the initial value s_i^0 being zero. In a similar way we obtain $z_i^\mu \leqslant 0$ and $z_i^0 = 0$, which completes the proof of (6.3).

We shall now prove (6.4). The maximal values $s_i^{\mu+1}$ and s_i^μ are realized at certain nodal points $(\mu+1, a(i))$ and $(\mu, b(i))$, respectively:

(6.6) $$s_i^{\mu+1} = \max_m r_i^{\mu+1,m} = r_i^{\mu+1,a(i)},$$

(6.7) $$s_i^\mu = \max_m r_i^{\mu,m} = r_i^{\mu,b(i)} \quad (i = 1, \ldots, n),$$

$a(i)$ and $b(i)$ being defined as $a(i) = (a_1^i, \ldots, a_p^i)$, $b(i) = (b_1^i, \ldots, b_p^i)$.

Accordingly, the difference

(6.8) $$s_i^{\mu\sim} = \frac{1}{k}(s_i^{\mu+1} - s_i^\mu),$$

can be written as

$$s_i^{\mu\sim} = \frac{1}{k}\left(r_i^{\mu+1,a(i)} - r_i^{\mu,a(i)}\right) + \frac{1}{k}\left(r_i^{\mu,a(i)} - r_i^{\mu,b(i)}\right). \tag{6.9}$$

It will be sufficient now to consider the right-hand member of (6.9). If, for some j: $1 \leqslant j \leqslant p$, we have $a_j^i = 0$, then inequalities (6.4) are evident; therefore let us assume $a_j^i \geqslant 1$ $(j = 1, \dots, p)$.

We shall show that (6.9) can now be written in an equivalent form:

$$\begin{aligned} s_i^{\mu\sim} = \eta_i^{\mu,a(i)} &+ \sum_{j=1}^{n} \frac{\partial f_i}{\partial u_j}(\sim) r_j^{\mu,a(i)} + \\ &+ \frac{1}{h}\sum_{j=1}^{p} \frac{\partial f_i}{\partial q_j}(\sim)\left[r_i^{\mu,a(i)} - r_i^{\mu,j(a(i))}\right] + \frac{1}{k}\left(r_i^{\mu,a(i)} - r_i^{\mu,b(i)}\right), \end{aligned} \tag{6.10}$$

the derivatives $\partial f_i/\partial q_j$ being taken at a suitable point $(\sim)$.

In fact, from definition (6.1) it follows that

$$\frac{1}{k}\left(r_i^{\mu+1,a(i)} - r_i^{\mu,a(i)}\right) = \frac{1}{k}\left(u_i^{\mu+1,a(i)} - u_i^{\mu,a(i)}\right) - \frac{1}{k}\left(v_i^{\mu+1,a(i)} - v_i^{\mu,a(i)}\right); \tag{6.11}$$

therefore

$$\begin{aligned} &\frac{1}{k}\left(r_i^{\mu+1,a(i)} - r_i^{\mu,a(i)}\right) \\ &= \eta_i^{\mu,a(i)} + f_i\left(\xi^\mu, x^{a(i)}, u^{\mu,a(i)}, u_i^{(\mu,a(i))\Delta}\right) - f_i\left(\xi^\mu, x^{a(i)}, v^{\mu,a(i)}, v_i^{(\mu,a(i))\Delta}\right), \end{aligned} \tag{6.12}$$

because of (4.2) and (4.5). Now we can apply the mean value theorem to the right-hand member of (6.12), and we get by (6.1), (2.15) and (2.9):

$$\begin{aligned} &\frac{1}{k}\left(r_i^{\mu+1,a(i)} - r_i^{\mu,a(i)}\right) \\ &= \eta_i^{\mu,a(i)} + \sum_{j=1}^{n} \frac{\partial f_i}{\partial u_j}(\sim) r_j^{\mu,a(i)} + \frac{1}{h}\sum_{j=1}^{p} \frac{\partial f_i}{\partial q_j}(\sim)\left[r_i^{\mu,a(i)} - r_i^{\mu,j(a(i))}\right], \end{aligned} \tag{6.13}$$

the derivatives being taken at a suitable point $(\sim)$. Combining (6.13) with (6.9) we obtain the desired formula (6.10).

All that remains to be verified now is the majorization of the right-hand member in (6.10) so as to obtain (6.4). This will be made by the following argument.

First, we observe that $r_i^{\mu,b(i)}$ denotes the greatest value (6.7); therefore

$$r_i^{\mu,j(a(i))} \leqslant r_i^{\mu,b(i)}, \tag{6.14}$$

and consequently

$$r_i^{\mu,a(i)} - r_i^{\mu,j(a(i))} \geqslant r_i^{\mu,a(i)} - r_i^{\mu,b(i)} . \tag{6.15}$$

We multiply both sides of (6.15) by $\partial f_i/\partial q_j$ and obtain by summation

$$\sum_{j=1}^{p} \frac{\partial f_i}{\partial q_j}(\sim)[r_i^{\mu,a(i)} - r_i^{\mu,j(a(i))}] \leqslant \sum_{j=1}^{p} \frac{\partial f_i}{\partial q_j}(\sim)[r_i^{\mu,a(i)} - r_i^{\mu,b(i)}] , \tag{6.16}$$

since the derivatives $\partial f_i/\partial q_j$ are non-positive because of assumption (3.2). Relation (6.16) and equality (6.10) imply that

$$s_i^{\mu\sim} \leqslant \eta_i^{\mu,a(i)} + \sum_{j=1}^{n} \frac{\partial f_i}{\partial u_j}(\sim) r_j^{\mu,a(i)} + (r_i^{\mu,a(i)} - r_i^{\mu,b(i)}) \frac{1}{h}\left[\sum_{j=1}^{p} \frac{\partial f_i}{\partial q_j}(\sim) + \frac{h}{k}\right]. \tag{6.17}$$

Now we can delete the last term on the right-hand member of (6.17), since it is non-positive. In fact, $r_i^{\mu,b(i)}$ denotes by definition (6.7) the greatest value; therefore we have

$$r_i^{\mu,a(i)} - r_i^{\mu,b(i)} \leqslant 0 , \tag{6.18}$$

and the intervals h and k are chosen according to (3.3).

The first and second terms on the right-hand member of (6.17) can be majorized with the aid of definition (4.7), (6.7), and so we get

$$s_i^{\mu\sim} \leqslant L \sum_{j=1}^{n} s_j^{\mu} + \varepsilon_i(h) \qquad (i = 1, \ldots, n) , \tag{6.19}$$

which concludes the proof of the first part of (6.4).

The second part of (6.4) can be proved in a similar way. It is sufficient only to use the definitions

$$z_i^{\mu+1} = \min_m r_i^{\mu+1,m} = r_i^{\mu+1,c(i)} , \tag{6.20}$$

$$z_i^{\mu} = \min_m r_i^{\mu,m} = r_i^{\mu,d(i)} \qquad (i = 1, \ldots, n) , \tag{6.21}$$

in place of (6.6) and (6.7), the sense of the subsequent inequalities being reversed.

This completes the proof of Lemma 2.

§ 7. Theorem 1. *Let us suppose that*

(i) *the right-hand members* $f_i(\xi, x, u, q)$ *of the diagonal system* (3.4) *satisfy assumptions* H,

(ii) *the values of the solution* u_i^M *and approximation* v_i^M *are defined at the nodal points of the set* E *by* (4.5), (4.4) *and* (4.2), (4.1), *respectively,*

(iii) *the function* $\varepsilon(h)$ *is defined by* (4.8) *and the error* r_i^M *by* (6.1).

Under these assumptions

(1) *the error estimate:*

$$|r_i^M| \leqslant \frac{\varepsilon(h)}{nL}(e^{nLk\mu}-1) \quad (i=1,\dots,n)\,, \tag{7.1}$$

holds at the nodal points M in the set E,

(2) *the difference method* (4.2) *is convergent, i.e.,*

$$\lim_{h\to 0} r_i^M = 0 \quad (i=1,\dots,n)\,. \tag{7.2}$$

Proof. Condition (7.2) follows from (7.1), since $\varepsilon(h)\to 0$, as $h\to 0$, cf. (4.10); therefore, we shall prove (7.1).

To this end, let us consider the sums

$$S^\mu = \sum_{i=1}^{n} s_i^\mu, \quad Z^\mu = \sum_{i=1}^{n} z_i^\mu\,. \tag{7.3}$$

Obviously, we have

$$S^\mu \geqslant 0\,, \quad Z^\mu \leqslant 0\,, \tag{7.4}$$

and

$$Z^\mu \leqslant z_i^\mu, \quad s_i^\mu \leqslant S^\mu \quad (i=1,\dots,n)\,, \tag{7.5}$$

since $s_i^\mu \geqslant 0$ and $z_i^\mu \leqslant 0$, because of (6.3).

Now we sum relations (6.4) and obtain two linear non-homogeneous difference inequalities for S^μ and Z^μ:

$$S^{\mu\sim} \leqslant nLS^\mu + \varepsilon(h)\,, \tag{7.6}$$

$$Z^{\mu\sim} \geqslant nLZ^\mu - \varepsilon(h)\,, \tag{7.7}$$

the initial conditions

$$S^0 = 0\,, \quad Z^0 = 0 \tag{7.8}$$

being granted in view of $s_i^0 = 0$, $z_i^0 = 0$, cf. Lemma 2.

We shall now prove that (7.6), (7.7) and (7.8) imply two estimates for S^μ and Z^μ:

$$S^\mu \leqslant \frac{\varepsilon(h)}{nL}(e^{nLk\mu}-1)\,, \tag{7.9}$$

$$Z^\mu \geqslant -\frac{\varepsilon(h)}{nL}(e^{nLk\mu}-1) \quad (\mu = 0, 1, \dots)\,. \tag{7.10}$$

In fact, $S^\mu \geqslant 0$ satisfies the assumptions of Lemma 1 because of (7.6) and of the condition $S^0 = 0$. This means that (7.9) holds true in view of Lemma 1.

(7.10) can be obtained in a similar way. For that purpose, let us observe that $(-Z^{\mu}) \geqslant 0$ fulfils the inequality

$$(7.11) \qquad (-Z^{\mu})^{\sim} \leqslant nL(-Z^{\mu}) + \varepsilon(h),$$

cf. (7.7), and the initial condition $(-Z^0) = 0$. Whence, from (7.11) and Lemma 1, it follows that

$$(7.12) \qquad -Z^{\mu} \leqslant \frac{\varepsilon(h)}{nL}(e^{nLk\mu} - 1) \qquad (\mu = 0, 1, \ldots),$$

which completes the proof of (7.10).

We can now prove (7.1) by the following argument: from definition (6.2), we obtain

$$(7.13) \qquad z_i^{\mu,m} \leqslant r_i^{\mu,m} \leqslant s_i^{\mu,m} \qquad (i = 1, \ldots, n;\ \mu = 0, 1, \ldots);$$

therefore,

$$(7.14) \qquad Z^{\mu} \leqslant r_i^{\mu,m} \leqslant S^{\mu} \qquad (i = 1, \ldots, n;\ \mu = 0, 1, \ldots),$$

because of (7.13) and (7.5).

From (7.14), (7.9) and (7.10) follows the desired estimate (7.1).

This completes the proof of Theorem 1.

References

[1] Z. Kowalski, *A difference method for the non-linear partial differential equation of the first order*, Ann. Polon. Math. 18 (1966), pp. 235-242.

Reçu par la Rédaction le 2. 11. 1966