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On some class of non-linear functional equations

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Abstract. In the paper the functional equation

$$z(x) = F\Big(x, z\big(a_1(x, z(\gamma_1(x)))\big), \ldots, z\big(a_p(x, z(\gamma_p(x)))\big)\Big)$$

is considered. An unknown function z is supposed to be defined in a bounded metric space with the values in another complete metric space. Solutions of the equation considered are sought in the class of functions having the modulus of continuity appropriately defined. Under suitable assumptions on known functions F, a_i, γ_i, the existence, uniqueness and convergence of successive approximations is established.

In the present paper we considered a non-linear equation of the form

$$(1) \qquad z(x) = F\Big(x, z\big(a_1(x, z(\gamma_1(x)))\big), \ldots, z\big(a_p(x, z(\gamma_p(x)))\big)\Big) \stackrel{\mathrm{df}}{=} (\mathfrak{F}z)(x),$$

where the functions $F\colon M_1 \times M_2^p \to M_2$, $a_i\colon M_1 \times M_2 \to M_1$, $\gamma_i\colon M_1 \to M_1$, $i = 1, \ldots, p$, are given and (M_1, ϱ_1) is a bounded metric space (i.e., for an arbitrarily fixed $x_0 \in M_1$ there exists a number $a > 0$ such that $\varrho_1(x, x_0) \leqslant a$ for any $x \in M_1$), and (M_2, ϱ_2) is a complete metric space.

The particular cases of equation (1), where the functions a_i, $i = 1, \ldots, p$, does not depend on the last variable, were considered by many authors, see e.g. [1], [2], [4]–[6].

In [4] equation (1) was discussed in the case where M_1 is a real linear space and M_2 is a Banach space.

In this paper we shall consider the problem of the existence, uniqueness and the convergence of successive approximations and the continuous dependence of solutions on the right-hand side of equation (1). We shall search for the solutions of equation (1) in the class of functions having the "modulus of continuity" defined adequately (see the class $D(M_1, M_2, \lambda)$). We shall use the comparative method (see [5], [7]).

1. General assumptions and theorems. We introduce

ASSUMPTION A. Suppose that

1° there exist functions $\omega\colon R_+\to R_+ \overset{\mathrm{df}}{=} [0, +\infty)$, $\omega(0) = 0$, and $\Omega\colon I\times R_+^p\to R_+$, $I \overset{\mathrm{df}}{=} [0, a]$, which are non-decreasing (with respect to each variable); ω is continuous and Ω is continuous with respect to the last p variables, $\Omega(t, 0, \ldots, 0) \equiv 0$, and

$$\varrho_2\big(F(x, z_1, \ldots, z_p),\ F(\bar{x}, z_1, \ldots, z_p)\big) \leqslant \omega\big(\varrho_1(x, \bar{x})\big),$$

$$\varrho_2\big(F(x, z_1, \ldots, z_p),\ F(x, \bar{z}_1, \ldots, \bar{z}_p)\big) \leqslant \Omega\big(\varrho_1(x, x_0),\ \varrho_2(z_1, \bar{z}_1), \ldots, \varrho_2(z_p, \bar{z}_p)\big)$$

for any $x, \bar{x}\in M_1$, $z_i, \bar{z}_i\in M_2$, $i = 1, \ldots, p$,

2° there exist non-decreasing functions $m_i, r_i\colon R_+\to R_+$, and $s_i\colon I\times R_+\to R_+$; $m_i(0) = r_i(0) = 0$, $s_i(t, 0) \equiv 0$, $i = 1, \ldots, p$, the functions m_i, r_i are continuous and the functions s_i are continuous with respect to the second variable and

$$\varrho_1\big(a_i(x, z),\ a_i(\bar{x}, z)\big) \leqslant m_i\big(\varrho_1(x, \bar{x})\big),$$

$$\varrho_1\big(a_i(x, z),\ a_i(x, \bar{z})\big) \leqslant s_i\big(\varrho_1(x, x_0),\ \varrho_2(z, \bar{z})\big),$$

$$\varrho_1\big(\gamma_i(x),\ \gamma_i(\bar{x})\big) \leqslant r_i\big(\varrho_1(x, \bar{x})\big), \quad i = 1, \ldots, p,$$

for any $x, \bar{x}\in M_1$, $z, \bar{z}\in M_2$,

3° there exist non-decreasing functions $\delta_i, \sigma_i\colon I\to R_+$, such that

$$\varrho_1\big(a_i(x, z), x_0\big) \leqslant \delta_i\big(\varrho_1(x, x_0)\big),$$

$$\varrho_1\big(\gamma_i(x), x_0\big) \leqslant \sigma_i\big(\varrho_1(x, x_0)\big), \quad i = 1, \ldots, p,$$

for any $x\in M_1$, $z\in M_2$.

ASSUMPTION B(z_0). Suppose that

1° there exists a continuous and non-decreasing function $\lambda\colon R_+\to R_+$ which is a solution of the inequality

$$\Omega\Big(a, \lambda\Big(m_1(t) + s_1\big(a, \lambda\big(r_1(t)\big)\big)\Big), \ldots, \lambda\Big(m_p(t) + s_p\big(a, \lambda\big(r_p(t)\big)\big)\Big)\Big) + \omega(t) \leqslant \lambda(t), \quad t\in I,\ \lambda(0) = 0, \tag{2}$$

2° for a fixed function $z_0\colon M_1\to M_2$ there exists a non-decreasing solution $\bar{u}\colon I\to R_+$ of the inequality

$$\Omega\Big(t, \lambda\Big(s_1\big(t, u\big(\sigma_1(t)\big)\big)\Big), \ldots, \lambda\Big(s_p\big(t, u\big(\sigma_p(t)\big)\big)\Big)\Big) + h(t) \leqslant u(t), \quad t\in I,$$

where

$$h(t) = \sup_{x\in K(x_0, t)} \varrho_2\big(z_0(x), (\mathfrak{F}z_0)(x)\big) < +\infty, \quad t\in I,$$

$$K(x_0, t) \overset{\mathrm{df}}{=} [x\colon x\in M_1,\ \varrho_1(x, x_0) \leqslant t], \quad t\in I,$$

3° in the class of functions satisfying the condition $0 \leqslant u \leqslant \bar{u}$, the function $u = 0$ is the only solution of the equation

$$u(t) = \Omega\Big(t, \lambda\big(s_1(t, u(\sigma_1(t)))\big), \ldots, \lambda\big(s_p(t, u(\sigma_p(t)))\big)\Big), \qquad t \in I.$$

We introduce the class of functions

$$D(M_1, M_2, \lambda) \overset{\mathrm{df}}{=} \big[z\colon z\colon M_1 \to M_2,\ \varrho_2\big(z(x), z(\bar{x})\big) \leqslant \lambda\big(\varrho_1(x, \bar{x})\big),\ x, \bar{x} \in M_1\big],$$

where the function λ is defined by condition 1° of Assumption $\mathrm{B}(z_0)$.

LEMMA 1. *If conditions* 1°, 2° *of Assumption A and condition* 1° *of Assumption* $\mathrm{B}(z_0)$ *are satisfied, then the operator* $\mathfrak{F}$ *defined by the right-hand side of equation* (1) *maps* $D(M_1, M_2, \lambda)$ *into itself.*

Proof. If $z \in D(M_1, M_2, \lambda)$ and $v(x) = (\mathfrak{F}z)(x)$, then we have

$$\begin{aligned}
&\varrho_2\big(v(x), v(\bar{x})\big) \\
&\leqslant \omega\big(\varrho_1(x, \bar{x})\big) + \Omega\Big(\varrho_1(x, x_0),\ \varrho_2\Big(z\big(a_1(x, z(\gamma_1(x)))\big), z\big(a_1(\bar{x}, z(\gamma_1(\bar{x})))\big)\Big), \ldots \\
&\qquad \ldots, \varrho_2\Big(z\big(a_p(x, z(\gamma_p(x)))\big), z\big(a_p(\bar{x}, z(\gamma_p(\bar{x})))\big)\Big)\Big) \\
&\leqslant \omega\big(\varrho_1(x, \bar{x})\big) + \Omega\Big(\varrho_1(x, x_0),\ \lambda\Big(m_1\big(\varrho_1(x, \bar{x})\big) + \\
&\qquad + s_1\Big(\varrho_1(x, x_0),\ \varrho_2\big(z(\gamma_1(x)), z(\gamma_1(\bar{x}))\big)\Big)\Big), \ldots \\
&\qquad \ldots, \lambda\Big(m_p\big(\varrho_1(x, \bar{x})\big) + s_p\Big(\varrho_1(x, x_0),\ \varrho_2\big(z(\gamma_p(x)), z(\gamma_p(\bar{x}))\big)\Big)\Big)\Big) \\
&\leqslant \omega\big(\varrho_1(x, \bar{x})\big) + \Omega\Big(a, \lambda\Big(m_1\big(\varrho_1(x, \bar{x})\big) + s_1\Big(a, \lambda\big(r_1(\varrho_1(x, \bar{x}))\big)\Big)\Big), \ldots \\
&\qquad \ldots, \lambda\Big(m_p\big(\varrho_1(x, \bar{x})\big) + s_p\Big(a, \lambda\big(r_p(\varrho_1(x, \bar{x}))\big)\Big)\Big)\Big) \leqslant \lambda\big(\varrho_1(x, \bar{x})\big),
\end{aligned}$$

since λ is a solution of inequality (2). Hence it follows that $v \in D(M_1, M_2, \lambda)$ The lemma is proved.

Remark 1. If the functions a_i, $i = 1, \ldots, p$, are independent of the second variable or the functions Ω, s_i, $i = 1, \ldots, p$, are independent of the first variable, then the assumption that M_1 is a bounded space is superfluous (see [5]).

Let us now define the sequence $\{u_n\}$ by the relations

$$\begin{aligned}
&u_0(t) = \bar{u}(t), \\
(3)\qquad &u_{n+1}(t) = \Omega\Big(t, \lambda\big(s_1(t, u_n(\sigma_1(t)))\big), \ldots, \lambda\big(s_p(t, u_n(\sigma_p(t)))\big)\Big), \\
&\qquad\qquad n = 0, 1, \ldots, \qquad t \in I.
\end{aligned}$$

By induction we can easily prove the following

LEMMA 2. *If Assumption* A *and* $B(z_0)$ *are satisfied, then* $0 \leqslant u_{n+1} \leqslant u_n \leqslant \bar{u}$, $n = 0, 1, \ldots,$

$$u_n \rightrightarrows 0 \quad \text{for } n \to \infty, \quad t \in I,$$

where the sign $\rightrightarrows$ *denotes uniform convergence in* I.

Now let us construct the sequence $\{z_n\}$ by the relations

$$z_{n+1} = \mathfrak{F} z_n, \quad n = 0, 1, \ldots, \tag{4}$$

where the operator $\mathfrak{F}$ is defined by the right-hand side of equation (1) and z_0 is an arbitrarily fixed function from $D(M_1, M_2, \lambda)$ (see Assumption $B(z_0)$).

LEMMA 3. *If Assumptions* A *and* $B(z_0)$ *are satisfied,* $z_0 \in D(M_1, M_2, \lambda)$ *and the sequence* $\{z_n\}$ *is defined by* (4), *then the estimations*

$$\sup_{x \in K(x_0, t)} \varrho_2\big(z_n(x), z_0(x)\big) \leqslant \bar{u}(t), \quad n = 0, 1, \ldots, \; t \in I, \tag{5}$$

and

$$\sup_{x \in K(x_0, t)} \varrho_2\big(z_{n+k}(x), z_n(x)\big) \leqslant u_n(t), \quad n, k = 0, 1, \ldots, \; t \in I \tag{6}$$

hold true.

Proof. It is obvious that (5) holds for $n = 0$. If we suppose that (5) holds for some $n > 0$, then by the assumptions we have for $x \in K(x_0, t)$

$$\begin{aligned}
&\varrho_2\big(z_{n+1}(x), z_0(x)\big) \\
&\quad \leqslant \varrho_2\big((\mathfrak{F}z_n)(x), (\mathfrak{F}z_0)(x)\big) + \varrho_2\big(z_0(x), (\mathfrak{F}z_0)(x)\big) \\
&\quad \leqslant \Omega\Big(\varrho_1(x, x_0), \varrho_2\Big(z_n\big(a_1(x, z_n(\gamma_1(x)))\big), z_0\big(a_1(x, z_0(\gamma_1(x)))\big)\Big), \ldots \\
&\qquad\qquad \ldots, \varrho_2\Big(z_n\big(a_p(x, z_n(\gamma_p(x)))\big), z_0\big(a_p(x, z_0(\gamma_p(x)))\big)\Big)\Big) + h(t) \\
&\quad \leqslant \Omega\Big(\varrho_1(x, x_0), \lambda\Big(s_1\Big(\varrho(x, x_0), \varrho_2\big(z_n(\gamma_1(x)), z_0(\gamma_1(x))\big)\Big)\Big), \ldots \\
&\qquad\qquad \ldots, \lambda\Big(s_p\Big(\varrho_1(x, x_0), \varrho_2\big(z_n(\gamma_p(x)), z_0(\gamma_p(x))\big)\Big)\Big)\Big) + h(t) \\
&\quad \leqslant \Omega\Big(t, \lambda\Big(s_1\Big(t, \sup_{x \in K(x_0, \sigma_1(t))} \varrho_2\big(z_n(x), z_0(x)\big)\Big)\Big), \ldots \\
&\qquad\qquad \ldots, \lambda\Big(s_p\Big(t, \sup_{x \in K(x_0, \sigma_p(t))} \varrho_2\big(z_n(x), z_0(x)\big)\Big)\Big)\Big) + h(t) \\
&\quad \leqslant \Omega\Big(t, \lambda\Big(s_1\big(t, \bar{u}(\sigma_1(t))\big)\Big), \ldots, \lambda\Big(s_p\big(t, \bar{u}(\sigma_p(t))\big)\Big)\Big) + h(t) \leqslant \bar{u}(t).
\end{aligned}$$

Hence

$$\sup_{x \in K(x_0, t)} \varrho_2\big(z_{n+1}(x), z_0(x)\big) \leqslant \bar{u}(t).$$

Now (5) follows by induction.

Now we prove (6). From (5) it follows that (6) holds for $n = 0$, $k = 0, 1, \ldots$ Further, if we suppose that (6) is true for $n, k \geqslant 0$, then for $x \in K(x_0, t)$ we have

$$\begin{aligned} &\varrho_2\big(z_{n+k+1}(x), z_{n+1}(x)\big) \\ &\quad \leqslant \Omega\Big(t, \lambda\Big(s_1\Big(t, \sup_{x \in K(x_0, \sigma_1(t))} \varrho_2\big(z_{n+k}(x), z_n(x)\big)\Big)\Big), \ldots \\ &\qquad\qquad \ldots, \lambda\Big(s_p\Big(t, \sup_{x \in K(x_0, \sigma_p(t))} \varrho_2\big(z_{n+k}(x), z_n(x)\big)\Big)\Big)\Big) \\ &\quad \leqslant \Omega\Big(t, \lambda\Big(s_1\Big(t, u_n\big(\sigma_1(t)\big)\Big)\Big), \ldots, \lambda\Big(s_p\Big(t, u_n\big(\sigma_p(t)\big)\Big)\Big)\Big) = u_{n+1}(t). \end{aligned}$$

Now we obtain (6) by induction. Thus the proof of Lemma 3 is completed.

THEOREM 1. *If Assumption* A *and* B (z_0) *are satisfied,* $z_0 \in D(M_1, M_2, \lambda)$, *then there exists a solution* $\bar{z} \in D(M_1, M_2, \lambda)$ *of equation* (1) *which is the limit of the sequence* $\{z_n\}$ *defined by* (4) *and the estimations*

$$\sup_{x \in K(x_0, t)} \varrho_2\big(\bar{z}(x), z_n(x)\big) \leqslant u_n(t), \qquad n = 0, 1, \ldots, \ t \in I, \tag{7}$$

hold true. The solution $\bar{z}$ *belongs to the class* $D^*(M_1, M_2, \lambda, \bar{u})$, *where*

$$\begin{aligned} D^*(M_1, M_2, \lambda, \bar{u}) = \Big[z\colon z \in D(M_1, M_2, \lambda), \sup_{x \in K(x_0, t)} \varrho_2\big(z(x), z_0(x)\big) \\ \leqslant \bar{u}(t),\ t \in I\Big], \end{aligned}$$

and it is the unique solution of equation (1) *in this class.*

Proof. The convergence of the sequence $\{z_n\}$ and estimations (7) follow from Lemmas 1, 2 and 3. By the estimation

$$0 \leqslant \sup_{x \in K(x_0, t)} \varrho_2\big((\mathfrak{F}\bar{z})(x), \bar{z}(x)\big) \leqslant 2u_n(t), \qquad n = 0, 1, \ldots, \ t \in I,$$

it follows that the function $\bar{z}$ satisfies equation (1). Obviously $\bar{z} \in D^*(M_1, M_2, \lambda, \bar{u})$.

Now we prove that the solution $\bar{z}$ is a unique solution of (1) in the above-mentioned class. Let us suppose that there exists another solution $\tilde{z} \in D^*(M_1, M_2, \lambda, \bar{u})$. By induction we get

$$\sup_{x \in K(x_0, t)} \varrho_2\big(\tilde{z}(x), z_n(x)\big) \leqslant u_n(t), \qquad n = 0, 1, \ldots, \ t \in I;$$

hence it follows that $\tilde{z} = \bar{z}$. Theorem 1 is proved.

2. Continuous dependence of solutions on the right-hand side of equation (1). Let us consider the second equation:

$$(8)\qquad v(x) = \tilde{F}\Big(x, v\big(\tilde{a}_1(x, v(\tilde{\gamma}_1(x)))\big), \ldots, v\big(\tilde{a}_p(x, v(\tilde{\gamma}_p(x)))\big)\Big) = (\tilde{\mathfrak{F}}v)(x),$$

where $\tilde{F}, \tilde{a}_i, \tilde{\gamma}_i$ are of the same kind as F, a_i, γ_i, $i = 1, \ldots, p$.

Let $\bar{v} \in D(M_1, M_2, \lambda)$ be a solution of equation (8). Assume that

$$\varphi(t) = \sup_{x \in K(x_0,t)} \varrho_2\big((\mathfrak{F}\bar{v})(x), \bar{v}(x)\big) < +\infty, \qquad t \in I,$$

where $\mathfrak{F}$ is defined by the right-hand side of equation (1), and let there exist $\psi\colon I \to R_+$ such that

$$\sup_{x \in K(x_0,t)} \varrho_2\big(\bar{z}(x), \bar{v}(x)\big) \leqslant \psi(t), \qquad t \in I.$$

Put

$$\tilde{h}(t) = \max\{\psi(t), \varphi(t), h(t)\}.$$

THEOREM 2. *If Assumptions* A *and* $B(z_0)$ *are satisfied with* h (*in* $B(z_0)$) *replaced by* $\tilde{h}$, *then there exists a non-negative solution* $\tilde{u}$ *of the equation*

$$(9)\qquad u(t) = \Omega\Big(t, \lambda\big(s_1(t, u(\sigma_1(t)))\big), \ldots, \lambda\big(s_p(t, u(\sigma_p(t)))\big)\Big) + \varphi(t), \quad t \in I,$$

such that

$$\sup_{x \in K(x_0,t)} \varrho_2\big(\bar{z}(x), \bar{v}(x)\big) \leqslant \tilde{u}(t), \qquad t \in I, \ x \in M_1.$$

Proof. Let $\tilde{u}_0$ be the solution of the inequality

$$\Omega\Big(t, \lambda\big(s_1(t, u(\sigma_1(t)))\big), \ldots, \lambda\big(s_p(t, u(\sigma_p(t)))\big)\Big) + \tilde{h}(t) \leqslant u(t).$$

Put

$$\tilde{u}_{n+1}(t) = \Omega\Big(t, \lambda\big(s_1(t, \tilde{u}_n(\sigma_1(t)))\big), \ldots, \lambda\big(s_p(t, \tilde{u}_n(\sigma_p(t)))\big)\Big) + \varphi(t), \quad n = 0, 1, \ldots$$

By induction we get $0 \leqslant \tilde{u}_{n+1} \leqslant \tilde{u}_n$, $n = 0, 1, \ldots$ Hence it follows that the sequence $\{\tilde{u}_n\}$ is convergent to $\tilde{u}$, which satisfies equation (9).

Further, we easily find by induction that

$$\sup_{x \in K(x_0,t)} \varrho_2\big(\bar{z}(x), \bar{v}(x)\big) \leqslant \tilde{u}_n(t), \qquad n = 0, 1, \ldots, \ t \in I.$$

Now if $n \to \infty$, we infer the assertion of the theorem.

3. The case of the comparative functions being linear. We are now going to consider the case where the functions ω, Ω, m_i, r_i, s_i, $i = 1, \ldots, p$ are linear; it permits us to give effective conditions under which Assumption $B(z_0)$ is fulfilled. First of all we introduce some notations which will be useful further on.

Let a non-decreasing functions $l_i\colon I\to R_+$, $\beta_i\colon I\to I$, $i=1,\ldots,p$, be given.

We define

$$(Lu)(t) \overset{\mathrm{df}}{=} \sum_{i=1}^{p} l_i(t)\,u\big(\beta_i(t)\big), \qquad t\in I.$$

Put $L^n \overset{\mathrm{df}}{=} LL^{n-1}$, $n=0,1,\ldots$; $L^0=J$ denotes the identity operator. From the definition of the operator L it follows that

$$(L^n u)(t) = \sum_{i_1=1}^{p} \cdots \sum_{i_n=1}^{p} l_n^{i_1,\ldots,i_n}(t)\,u\big(\beta_n^{i_1,\ldots,i_n}(t)\big),$$

where

$$\beta_1^i(t) = \beta_i(t), \qquad \beta_{n+1}^{i_1,\ldots,i_{n+1}}(t) \overset{\mathrm{df}}{=} \beta_n^{i_1,\ldots,i_n}\big(\beta_{i_{n+1}}(t)\big),$$

$$l_1^i(t) = l_i(t), \qquad l_{n+1}^{i_1,\ldots,i_{n+1}}(t) \overset{\mathrm{df}}{=} l_{i_{n+1}}(t)\,l_n^{i_1,\ldots,i_n}\big(\beta_{i_{n+1}}(t)\big),$$

$$i, i_n = 1,\ldots,p, \quad n=0,1,\ldots$$

Put

$$Mu \overset{\mathrm{df}}{=} \sum_{n=0}^{\infty} L^n u,$$

with the point-wise convergence of the series in I.

We quote the following

LEMMA 4 [3]. *For any function $h\colon I\to R_+$ the condition*

$$(10)\qquad (Mh)(t) < +\infty, \qquad t\in I,$$

is necessary and sufficient for the equation

$$(11)\qquad u = Lu + h$$

to have a non-negative solution $\bar{u}$.

If condition (10) *is fulfilled, then the function $\bar{u}$,*

$$(12)\qquad \bar{u} = Mh,$$

is a solution of equation (11), *and*

$$L^n\bar{u}\to 0 \quad \text{if } n\to\infty.$$

There is no other solution of equation (11) in the class of functions

$$G(I, R_+, \bar{u}) \overset{\mathrm{df}}{=} [u\colon u\colon I\to R_+,\ [\![u]\!] < +\infty],$$

where

$$[\![u]\!] \overset{\mathrm{df}}{=} \inf[c\colon |u| \leqslant c\bar{u}].$$

LEMMA 5 [3]. *If $\bar{u}$ is of form* (12) *and the function* $u \in G(I, R_+, \bar{u})$ *satisfies the inequality* $u \leqslant Lu$, *then* $u = 0$.

Now we assume that the functions occurring in Assumptions A and B (z_0) are of the form

$$\omega(t) = \overline{\omega} t, \quad \Omega(t, z_1, \ldots, z_p) = \sum_{i=1}^{p} \bar{l}_i(t) z_i, \quad m_i(t) = \overline{m}_i t, \tag{13}$$
$$s_i(t, z) = \bar{s}_i(t) z, \quad r_i(t) = \bar{r}_i t, \quad i = 1, \ldots, p,$$

where $\bar{l}_i$, $\bar{s}_i$: $I \to R_+$ are non-decreasing functions, and $\overline{\omega}$, $\overline{m}_i$, $\bar{r}_i$ are non-negative constants.

Let

$$\Lambda_m = \sum_{i=1}^{p} \bar{l}_i^* \overline{m}_i, \quad \Lambda_s = \sum_{i=1}^{p} \bar{l}_i^* \bar{s}_i^* \bar{r}_i,$$

where $\bar{l}_i^* = \bar{l}_i(a)$, $\bar{s}_i^* = \bar{s}_i(a)$.

In this case the class of functions $D(M_1, M_2, \lambda)$ is replaced by the class

$$\overline{D}(M_1, M_2, \bar{\lambda}) = \big[z\colon z\colon M_1 \to M_2,\ \varrho_2\big(z(x), z(\bar{x})\big) \leqslant \bar{\lambda}\varrho_1(x, \bar{x}),\ x, \bar{x} \in M_1\big],$$

where the constant $\bar{\lambda}$ satisfies the condition $\lambda_1 \leqslant \bar{\lambda} \leqslant \lambda_2$, and λ_1, λ_2 are non-negative roots of the equation

$$\Lambda_s \lambda^2 + (\Lambda_m - 1)\lambda + \overline{\omega} = 0$$

if $\Lambda_s \neq 0$, but $\bar{\lambda}$ satisfies the condition $\bar{\lambda} \geqslant \lambda_* = \dfrac{\overline{\omega}}{1 - \Lambda_m}$ if $\Lambda_s = 0$.

Lemma 1 implies

LEMMA 6. *If conditions* 1°, 2° *of Assumption* A *and* (13) *are satisfied and if* $\Lambda_m < 1$, $(\Lambda_m - 1)^2 - 4\Lambda_s \overline{\omega} > 0$, *then the operator* $\mathfrak{F}$ *defined by the right-hand side of equation* (1) *maps* $\overline{D}(M_1, M_2, \bar{\lambda})$ *into itself.*

Proof. To prove this lemma it is sufficient to prove that condition 1° of Assumption B(z_0) is fulfilled. In this case inequality (2) is satisfied by any function $\bar{\lambda} t$, where the constant $\bar{\lambda}$ is a solution of the inequality

$$\Big\{\sum_{i=1}^{p} \bar{l}_i(a) \bar{s}_i(a) \bar{r}_i\Big\}\lambda^2 + \Big\{\sum_{i=1}^{p} \bar{l}_i(a) \overline{m}_i\Big\}\lambda + \overline{\omega} \leqslant \lambda,$$

i.e., where $\bar{\lambda} \in [\lambda_1, \lambda_2]$. Thus Lemma 6 is proved.

Put

$$l_i(t) = \bar{l}_i(t)[1 + \bar{\lambda}\bar{s}_i(t)],$$
$$\beta_i(t) = \max[\delta_i(t), \sigma_i(t)], \quad i = 1, \ldots, p, \tag{14}$$
$$h(t) = \sup_{x \in K(x_0, t)} \varrho_2\big(z_0(x), (\mathfrak{F} z_0)(x)\big),$$

where z_0 is an arbitrarily fixed element of $\bar{D}(M_1, M_2, \bar{\lambda})$.

From Lemmas 4, 5 and 6 we infer

THEOREM 3. *If Assumption* A *and* (13) *are satisfied, and condition* (10) *holds with* l_i, β_i, *and* h *defined by* (14) *and if* $\Lambda_m < 1$, $(\Lambda_m - 1)^2 - 4\Lambda_s \overline{\omega} > 0$, *then there exists a solution* $\bar{z} \in \bar{D}(M_1, M_2, \bar{\lambda})$ *of equation* (1) *which is the limit of the sequence* $\{z_n\}$ *defined by* (4) *with* $z_0 \in \bar{D}(M_1, M_2, \bar{\lambda})$, *and the estimations*

$$\sup_{x \in K(x_0, t)} \varrho_2\big(\bar{z}(x), z_n(x)\big) \leqslant \bar{u}_n(t), \qquad n = 0, 1, \ldots, \ t \in I, \tag{15}$$

hold true, where

$$\bar{u}_0 = \bar{u}, \qquad \bar{u}_{n+1} = L\bar{u}_n, \qquad n = 0, 1, \ldots,$$

and $\bar{u}$ *is defined in Lemma* 4.

The solution $\bar{z}$ *belongs to the class of functions*

$$Z(M_1, M_2, \bar{\lambda}, \bar{u}) \overset{\mathrm{df}}{=} \big[z\colon\ z \in \bar{D}(M_1, M_2, \bar{\lambda}),\ d(z, z_0) < +\infty\big],$$

where

$$d(z, z_0) = \inf\big[c;\ \sup_{x \in K(x_0, t)} \varrho_2\big(z(x), z_0(x)\big) \leqslant c\bar{u}(t),\ t \in I\big],$$

and it is the unique solution of equation (1) *in this class.*

Proof. To prove the existence of a solution of equation (1) we first prove the following estimations:

$$\sup_{x \in K(x_0, t)} \varrho_2\big(z_n(x), z_0(x)\big) \leqslant \bar{u}(t), \qquad n = 0, 1, \ldots, \ t \in I,\ x \in M_1, \tag{16}$$

$$\sup_{x \in K(x_0, t)} \varrho_2\big(z_{n+k}(x), z_n(x)\big) \leqslant \bar{u}_n(t),\ n = 0, 1, \ldots, \ t \in I,\ x \in M_1. \tag{17}$$

It is obvious that (16) holds for $n = 0$. If we suppose that (16) holds for some $n > 0$, then according to Lemma 6 we have, for $x \in M_1$, $\varrho_1(x, x_0) \leqslant t$

$$\begin{aligned}
&\varrho_2\big(z_{n+1}(x), z_0(x)\big) \\
&\quad \leqslant \sum_{i=1}^{p} \bar{l}_i\big(\varrho_1(x, x_0)\big)\, \varrho_2\Big(z_n\Big(a_i\Big(x, z_n\big(\gamma_i(x)\big)\Big)\Big), z_0\Big(a_i\Big(x, z_0\big(\gamma_i(x)\big)\Big)\Big)\Big) + h(t) \\
&\quad \leqslant \sum_{i=1}^{p} \bar{l}_i(t)\big[\lambda\bar{s}_i(t)\, \varrho_2\big(z_n\big(\gamma_i(x)\big), z_0\big(\gamma_i(x)\big)\big) + \bar{u}\big(\delta_i(t)\big)\big] + h(t) \\
&\quad \leqslant (\bar{L}\bar{u})(t) + h(t) = \bar{u}(t).
\end{aligned}$$

Hence

$$\sup_{x \in K(x_0, t)} \varrho_2\big(z_{n+1}(x), z_0(x)\big) \leqslant \bar{u}(t), \qquad t \in I.$$

Now (16) follows by induction.

From (16) it follows that (17) holds for $n = 0$, $k = 0, 1, \ldots$ Further, if we suppose that (17) is true for $n, k \geqslant 0$, then for $x \in M_1$, $\varrho_1(x, x_0) \leqslant t$, in view of Lemma 6 we get

$$\begin{aligned}
&\varrho_2\big(z_{n+k+1}(x), z_{n+1}(x)\big)\\
&\quad\leqslant \sum_{i=1}^{p} \bar{l}_i(t)\Big[\bar{\lambda}\varrho_1\Big(a_i\big(x, z_{n+k}(\gamma_i(x))\big), a_i\big(x, z_n(\gamma_i(x))\big)\Big) + \bar{u}_n\Big(a_i\big(x, z_k(\gamma_i(x))\big)\Big)\Big]\\
&\quad\leqslant \sum_{i=1}^{p} \bar{l}_i(t)\big[\bar{\lambda}\bar{s}_i(t)\bar{u}_n\big(\sigma_i(t)\big) + \bar{u}_n\big(\delta_i(t)\big)\big]\\
&\quad\leqslant (L\bar{u}_n)(t) = \bar{u}_{n+1}(t).
\end{aligned}$$

Now we obtain (17) by induction.

From Lemmas 4 and 5 it follows that $\bar{u}_n \to 0$ if $n \to \infty$. Hence and by (17) it follows that the sequence $\{z_n\}$ is convergent to the solution $\bar{z}$ of equation (1). If $k \to \infty$, then (17) gives estimation (15). Obviously $\bar{z} \in Z(M_1, M_2, \lambda, \bar{u})$.

To prove that the solution $\bar{z}$ is a unique solution of (1) in $Z(M_1, M_2, \bar{\lambda}, \bar{u})$ let us suppose that there exists another solution $\tilde{z} \in Z(M_1, M_2, \bar{\lambda}, \bar{u})$. It is easy to prove that $u^*(t) = \sup\limits_{x \in K(x_0, t)} \varrho_2\big(\bar{z}(x), \tilde{z}(x)\big) \in G(I, R_+, \bar{u})$ and $u^* \leqslant Lu^*$. Hence and from Lemma 5 it follows that $\bar{z} = \tilde{z}$. Thus the proof of the theorem is complete.

Theorem 2 implies

THEOREM 4. *If Assumption* A, (13) *and condition* (10) *are satisfied with h replaced by $\tilde{h}$ and if $\Lambda_m < 1$, $(\Lambda_m - 1)^2 - 4\Lambda_s\varpi > 0$, then*

$$\sup_{x \in K(x_0, t)} \varrho_2\big(\bar{z}(x), \bar{v}(x)\big) \leqslant \tilde{u}(t), \qquad t \in I,$$

where $\bar{z}, \bar{v} \in \bar{D}(M_1, M_2, \bar{\lambda})$ are the solution of equations (1) *and* (8), *respectively, and $\tilde{u}$ is a non-negative solution of equation*

$$u = Lu + \varphi.$$

Remark 2. Now we give some effective conditions under which condition (10) is fulfilled.

a) If we assume that $\bar{l}_i(t) \leqslant \tilde{l}_i t$, $\delta_i(t) \leqslant \bar{\delta}_i t$, $\sigma_i(t) \leqslant \bar{\sigma}_i t$, and $\bar{\delta}_i, \bar{\sigma}_i < 1$, $h(t) \leqslant Ht$, $\tilde{l}_i, H \in R_+$, then condition (10) is fulfilled.

b) If $\delta_i(t) \leqslant \bar{\delta}_i t$, $\sigma_i(t) \leqslant \bar{\sigma}_i t$, $i = 1, \ldots, p$, and $h(t) \leqslant Ht^q$, where $\bar{\delta}_i$, $\bar{\sigma}_i, H, q$ are non-negative constants, then condition (10) is fulfilled if

$$\sum_{i=1}^{p} \bar{l}_i^*[1 + \bar{\lambda}\bar{s}_i^*]\beta_i^q < 1, \tag{18}$$

where $\bar{\beta}_i = \max[\bar{\sigma}_i, \bar{\delta}_i]$, $i = 1, \ldots, p$.

Remark 3. From the form of the function $\bar{u}$ (see (12)) and the definition of the sequence $\{\bar{u}_n\}$ it follows that

$$\bar{u}_{n+1} = \sum_{i=n}^{\infty} L^i n, \quad n = 0, 1, \ldots$$

Remark 4. By the use of the Banach fixed point theorem in [4] the existence and uniqueness theorem is proved under the conditions including $\sum_{i=1}^{p} \bar{l}_i^* < 1$, which is obviously a stronger condition than (18).

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