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The degenerate B-splines as a basis in the space of algebraic polynomials

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Abstract. The polynomials appearing in the definition of Bernstein polynomials are the degenerate B-splines. Clearly, they form a basis in the corresponding space of polynomials. The eigenvectors and the eigenvalues of the corresponding Gram matrix are found explicitely. It turns out that the eigenvectors are simply the discrete Chebyshev orthogonal polynomials and in the same time these vectors are obtained as vector coefficients of the representation of the orthogonal Legendre polynomials in the basis in question.

1. Introduction. Consider on the real line two knots -1 and $+1$, each of multiplicity k. Then the B-splines corresponding to such knot-sequence (cf. [5]) are simply the following polynomials

$$N_{i,k}(t) = \binom{k-1}{i}\left(\frac{1+t}{2}\right)^{k-1-i}\left(\frac{1-t}{2}\right)^{i}, \quad i = 0, 1, \ldots, k-1. \tag{1.1}$$

It is well known that $(N_{i,k};\ i = 0, \ldots, k-1)$ is a basis in $\mathscr{P}_k$ the space of algebraic polynomials of order k (of degree not exceeding $k-1$). Representations of classical orthogonal polynomials in this bases are known (see [3]). It was pointed out in [1], formula (3.2), that for the L^2 condition number for (1.1) we have formula

$$\sup_{\underline{a}} \frac{\|\underline{a}\|_2}{\|f_{\underline{a}}\|_2} \cdot \sup_{\underline{a}} \frac{\|f_{\underline{a}}\|_2}{\|\underline{a}\|_2} = \binom{2k-1}{k}^{1/2}, \tag{1.2}$$

where the norms are the $L^2\langle -1, 1\rangle$ and the l_k^2 norms, respectively, and $f_{\underline{a}} = \sum_{i=0}^{k-1} a_i N_{i,k}$, $\underline{a} = (a_0, \ldots, a_{k-1})$.

The extremal polynomial up to multiplicative constant for the first factor in (1.2) is the k-th Legendre polynomial. Since the polynomials (1.1) form a non-negative partition of unity it follows that the second factor in (1.2) is equal to $\sqrt{2/k}$.

The inverse of the quantity in (1.2) appears to be equal to the square root of the minimal eigenvalue of the Gram matrix $\underline{G}_k$ of the basis (1.1). This has its natural extension in Theorem 3.18.

Since the Legendre polynomials are the extreme ones in Theorem 3.18 it was interesting to look at the coefficient vectors $\underline{u}_k^{(0)},\ldots,\underline{u}_k^{(k-1)}$ (cf. (2.7)), and it is rather surprising that they appear to be orthogonal (see Proposition 3.9). Consequently, they are proportional to the discrete Chebyshev orthogonal polynomials (cf. Corollary 3.11). Moreover, they are eigenvectors for $\underline{G}_k$ (cf. Theorem 3.14).

Section 4 contains some explicite formulas related to the discussion presented in previous sections.

We would like to mention that the interesting properties, e.g. the orthogonality of $\underline{u}_k^{(0)},\ldots,\underline{u}_k^{(k-1)}$, were discovered on a personal computer.

2. Preliminaries. We start with the Legendre orthogonal polynomials P_0, $P_1,\ldots$ normalized by $P_\nu(1)=1$. Then $(P_\nu, P_\nu)=(\nu+\frac{1}{2})^{-1}$, where $(f,g) = \int_{-1}^{+1} f\cdot g$. Define $\hat{P}_\nu = \sqrt{\nu+\frac{1}{2}}\,P_\nu$. For $k\geqslant 1$ we have (cf. [3])

$$P_{k-1}(t)=\sum_{j=0}^{k-1}(-1)^{k-1+j}\binom{k-1}{j}\binom{k-1+j}{j}\left(\frac{1+t}{2}\right)^j, \tag{2.1}$$

$$P_{k-1}(t)=\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}N_{j,k}(t). \tag{2.2}$$

Let $(N_{j,k}^*;\ j=0,\ldots,k-1)$ be the basis in $\mathscr{P}_k$ which is dual to $(N_{i,k};\ i=0,\ldots,k-1)$, i.e. $(N_{i,k}, N_{j,k}^*)=\delta_{i,j}$. Then (cf. [5], pp. 125–126 and 128), for $i,j=0,\ldots,k-1)$ we have

$$\int_{-1}^{+1}\left(\frac{1+s}{2}\right)^j N_{i,k}^*(s)\,ds=\binom{k-1-j}{i}\Big/\binom{k-1}{i}=f_k^{(j)}(i), \tag{2.3}$$

$$\int_{-1}^{+1}\left(\frac{1+s}{2}\right)^j N_{i,k}(s)\,ds=\frac{2}{k}\binom{k}{i+1}\Big/\binom{k+j}{i+1}=g_k^{(j)}(i), \tag{2.4}$$

where $f_k^{(j)}$ and $g_k^{(j)}$ are polynomials of degree j determined by the following conditions

$$f_k^{(j)}(l)=0 \quad \text{for } l=k-j,\ldots,k-1,\ f_k^{(j)}(0)=1, \tag{2.5}$$

$$g_k^{(j)}(l)=0 \quad \text{for } l=k,\ldots,k+j-1,\ g_k^{(j)}(0)=\frac{2}{k+j}. \tag{2.6}$$

For $\nu = 0, \dots, k-1$ we introduce

$$\hat{P}_\nu = \sum_{i=0}^{k-1} u_{i,k}^{(\nu)} N_{i,k}, \quad \underline{u}_k^{(\nu)} = (u_{0,k}^{(\nu)}, \dots, u_{k-1,k}^{(\nu)}), \tag{2.7}$$

$$\hat{P}_\nu = \sum_{i=0}^{k-1} v_{i,k}^{(\nu)} N_{i,k}^*, \quad \underline{v}_k^{(\nu)} = (v_{0,k}^{(\nu)}, \dots, v_{k-1,k}^{(\nu)}). \tag{2.8}$$

Note that

$$u_{0,k}^{(\nu)} = \hat{P}_\nu(1) = \sqrt{\nu + \tfrac{1}{2}}. \tag{2.9}$$

PROPOSITION 2.10. *$(\underline{u}_k^{(\nu)}, \underline{v}_k^{(\mu)};\ \nu, \mu = 0, \dots, k-1)$ is a biorthogonal system in R^k.*

Proof.

$$\begin{aligned}\langle \underline{u}_k^{(\nu)}, \underline{v}_k^{(\mu)} \rangle &= \sum_{i=0}^{k-1} u_{i,k}^{(\nu)} v_{i,k}^{(\mu)} \\ &= \sum_{i=0}^{k-1} (\hat{P}_\nu, N_{i,k}^*)(\hat{P}_\mu, N_{i,k}) = (\hat{P}_\nu, \hat{P}_\mu) = \delta_{\mu,\nu}.\end{aligned}$$

It follows by (2.2) that

$$u_{i,k}^{(k-1)} = \sqrt{k - \tfrac{1}{2}}\,(-1)^i \binom{k-1}{i}, \quad i = 0, \dots, k-1. \tag{2.11}$$

The identity $2(t+s) = (1+t)(1+s) - (1-t)(1-s)$ implies

$$\left(\frac{t+s}{2}\right)^{k-1} = \sum_{i=0}^{k-1} \frac{(-1)^i}{\binom{k-1}{i}} N_{i,k}(t) N_{i,k}(s), \tag{2.12}$$

whence

$$\int_{-1}^{+1} \left(\frac{t+s}{2}\right)^{k-1} N_{i,k}^*(s)\,ds = \frac{(-1)^i}{\binom{k-1}{i}} N_{i,k}(t), \quad i = 0, \dots, k-1. \tag{2.13}$$

Now (2.13) implies

$$\frac{(-1)^i}{\binom{k-1}{i}} (P_{k-1}, N_{i,k}) = \int_{-1}^{+1} \left(\frac{t}{2}\right)^{k-1} P_{k-1}(t)\,dt \cdot \int_{-1}^{+1} N_{i,k}^*(s)\,ds.$$

Since $(N_{i,k};\ i = 0, \dots, k-1)$ is a partition of unity it follows that

$$\int_{-1}^{+1} N_{i,k}^* = 1$$

(cf. also (2.3)) and by the definition of the Legendre polynomials, using (2.1), we obtain

$$\int_{-1}^{+1} \left(\frac{t}{2}\right)^{k-1} P_{k-1}(t)\,dt = \frac{2}{k\binom{2k-1}{k}}.$$

Thus,

$$\hat{P}_{k-1} = \frac{2\sqrt{k-\frac{1}{2}}}{k\binom{2k-1}{k}} \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} N_{i,k}^{*}, \tag{2.14}$$

i.e. by (2.11) and (2.8)

$$v_{i,k}^{(k-1)} = \frac{2}{k\binom{2k-1}{k}} u_{i,k}^{(k-1)}. \tag{2.15}$$

The Gram matrix for $(N_{i,k};\ i = 0,\dots,k-1)$ is denoted by $\underline{G}_k$, i.e. $\underline{G}_k = \left(G_{i,j;k} = (N_{i,k}, N_{j,k});\ i, j = 0,\dots,k-1\right)$. It follows immediately that

$$\underline{v}_k^{(\nu)} = \underline{G}_k \circ \underline{u}_k^{(\nu)} \quad \text{for } \nu = 0,\dots,k-1, \tag{2.16}$$

where $\circ$ denotes the composition of matrices, and $\underline{v}_k^{(\nu)}$, $\underline{u}_k^{(\nu)}$ are treated as column matrices. Comparing this and (2.15) we can conclude that $\underline{u}_k^{(k-1)}$ is the eigenvector for $\underline{G}_k$ corresponding to the eigenvalue

$$\lambda_k^{(k-1)} = 2\cdot\left[k\binom{2k-1}{k}\right]^{-1}.$$

In the next section it will be proved, that also the other elements of $(\underline{u}_k^{(0)},\dots,\underline{u}_k^{(k-1)})$ are the eigenvectors for $\underline{G}_k$.

3. Main result. For the later convenience, for given $f \in \mathscr{P}_k$ write

$$f = \sum_{i=0}^{k-1} u_{i,k}(f) N_{i,k}, \qquad u_{i,k}(f) = (f, N_{i,k}^{*}), \tag{3.1}$$

$$f = \sum_{i=0}^{k-1} v_{i,k}(f) N_{i,k}^{*}, \qquad v_{i,k}(f) = (f, N_{i,k}). \tag{3.2}$$

Clearly, $u_{i,k}^{(\nu)} = u_{i,k}(\hat{P}_\nu)$, $v_{i,k}^{(\nu)} = v_{i,k}(\hat{P}_\nu)$. The duality implies

$$(f, g) = \sum_{i=0}^{k-1} u_{i,k}(f)\, v_{i,k}(g) \qquad \text{for } f, g \in \mathscr{P}_k. \tag{3.3}$$

We now define two mappings from $\mathscr{P}_k$ onto R^k:

$$U_k(f) = \underline{u}_k(f) = (u_{0,k}(f), \ldots, u_{k-1,k}(f)), \tag{3.4}$$

$$V_k(f) = \underline{v}_k(f) = (v_{0,k}(f), \ldots, v_{k-1,k}(f)). \tag{3.5}$$

Let us distinguish the following subspaces of R^k for $j = 1, \ldots, k$

$$E_k^{(j)} = \{\underline{x} = (x_0, \ldots, x_{k-1}) \in R^k;\ x_i = f(i) \text{ for some } f \in \mathscr{P}_j\}. \tag{3.6}$$

PROPOSITION 3.7. *For* $E_k^{(j)}$, $j = 1, \ldots, k$ *defined in* (3.6) *we have*:

(i) $U_k(\mathscr{P}_j) = V_k(\mathscr{P}_j) = E_k^{(j)}$,

(ii) $E_k^{(j-1)} \subsetneq E_k^{(j)}$ *with* $E_k^{(0)}$ *being the trivial subspace,*

(iii) $\dim E_k^{(j)} = j$.

Proof. Only property (i) needs to be proved. Let $T_i(t) = ((1+t)/2)^i$. Equalities (2.3) and (2.4) give

$$U_k(T_i) = \underline{f}_k^{(i)} = (f_k^{(i)}(0), \ldots, f_k^{(i)}(k-1)), \tag{3.8'}$$

$$V_k(T_i) = \underline{g}_k^{(i)} = (g_k^{(i)}(0), \ldots, g_k^{(i)}(k-1)), \tag{3.8''}$$

where $\underline{f}_k^{(i)}$, $\underline{g}_k^{(i)} \in \mathscr{P}_{i+1}$ are defined by (2.5), (2.6). Since

$$\mathscr{P}_j = \operatorname{span}\{T_0, \ldots, T_{j-1}\} = \operatorname{span}\{\underline{f}_k^{(0)}, \ldots, \underline{f}_k^{(j-1)}\} = \operatorname{span}\{\underline{g}_k^{(0)}, \ldots, \underline{g}_k^{(j-1)}\}$$

the proof follows by (3.8′) and (3.8″).

PROPOSITION 3.9. *The vectors* $\underline{u}_k^{(0)}, \ldots, \underline{u}_k^{(k-1)}$ *are orthogonal in* R^k.

Proof. Since $U_k(\hat{P}_i) = \underline{u}_k^{(i)}$ it follows by Proposition 3.7

$$\underline{u}_k^{(i)} \in E_k^{(i+1)} \quad \text{for } i = 0, \ldots, k-1. \tag{3.10}$$

Now, (3.3) and the definition of the Legendre polynomials imply

$$0 = (\hat{P}_i, g) = \langle U_k(\hat{P}_i), V_k(g)\rangle = \langle \underline{u}_k^{(i)}, V_k(g)\rangle$$

for $g \in \mathscr{P}_i$ and therefore, by Proposition 3.7, $\underline{u}_k^{(i)} \perp E_k^{(i)}$, what together with (3.10) completes the proof.

COROLLARY 3.11. *The vectors* $\underline{u}_k^{(0)}, \ldots, \underline{u}_k^{(k-1)}$ *are proportional to the discrete Chebyshev orthogonal polynomials and therefore they are given by a formula analogous to the Rodrigues formula* (see [3]), *i.e.*

$$u_{i,k}^{(\nu)} = \sqrt{\nu + \tfrac{1}{2}}\, \frac{1}{\binom{k-1}{\nu}} \Delta^\nu \binom{i}{\nu}\binom{k-1+\nu-i}{\nu} = \sqrt{\nu+\tfrac{1}{2}}\, \frac{(-1)^\nu}{\binom{k-1}{\nu}} \Delta^\nu \binom{i}{\nu}\binom{i-k}{\nu}, \tag{3.12}$$

where Δ *is the forward difference in* i *with step* 1 (*the coefficient can be determined by* (2.9)). *Thus, we can write also*

$$u_{i,k}^{(\nu)} = \sqrt{\nu+\tfrac{1}{2}}\, \frac{1}{\binom{k-1}{\nu}} \sum_{j=0}^{\nu} (-1)^{\nu+j} \binom{\nu}{j}\binom{i+j}{\nu}\binom{i+j-k}{\nu}. \tag{3.13}$$

THEOREM 3.14. *$\underline{u}_k^{(0)}, \dots, \underline{u}_k^{(k-1)}$ are the eigenvectors for $\underline{G}_k$ and the ν-th eigenvalue is given by formula*

$$\lambda_{\nu,k} = \int_{-1}^{+1} \left(\frac{1+s}{2}\right)^{k-1} P_\nu(s)\,ds = \frac{2}{k}\binom{k-1}{\nu}\Big/\binom{k+\nu}{k} \quad \textit{for } \nu = 0, \dots, k-1. \tag{3.15}$$

Proof. Propositions 2.10, 3.9 and (2.16) imply that for some $\lambda_{\nu,k} > 0$ the equality

$$\underline{G}_k \circ \underline{u}_k^{(\nu)} = \lambda_{\nu,k} \underline{u}_k^{(\nu)}$$

holds for $\nu = 0, \dots, k-1$. To prove (3.15), introduce the kernel

$$G_k(t, s) = \sum_{i=0}^{k-1} N_{i,k}(t) N_{i,k}(s).$$

Now, the i-th coordinate of $\underline{v}_k^{(\nu)}$ equals to

$$(\underline{G}_k \circ \underline{u}_k^{(\nu)})_i = \sum_{j=0}^{k-1} (N_{i,k}, N_{j,k})(\hat{P}_\nu, N_{j,k}^*) = (N_{i,k}, \hat{P}_\nu).$$

On the other hand it is equal to $\lambda_{\nu,k} u_{i,k}^{(\nu)} = \lambda_{\nu,k}(N_{i,k}^*, \hat{P}_\nu)$. Thus $(N_{i,k}, P_\nu) = \lambda_{\nu,k}(N_{i,k}^*, P_\nu)$ for $i = 0, \dots, k-1$, whence

$$\int_{-1}^{+1} G_k(t, s) P_\nu(s)\,ds = \lambda_{\nu,k} P_\nu(t).$$

Letting $t = 1$, since $P_\nu(1) = 1$ and $G_k(1, s) = ((1+s)/2)^{k-1}$, we obtain the first equality in (3.15). The second equality in (3.15) can be proved directly integrating by parts.

COROLLARY 3.16. *For $\nu = 0, \dots, k-1$ the following equalities hold*

$$\underline{v}_k^{(\nu)} = \lambda_{\nu,k} \underline{u}_k^{(\nu)} \quad \textit{and} \quad \lambda_{\nu,k} = \|\underline{u}_k^{(\nu)}\|_2^{-2}.$$

Moreover, $\lambda_{0,k} > \lambda_{1,k} > \dots > \lambda_{k-1,k} > 0$.

COROLLARY 3.17. *The following equality holds for $t \in R$*

$$\left(\frac{1+t}{2}\right)^{k-1} = \sum_{\nu=0}^{k-1} \lambda_{\nu,k}(\nu + \tfrac{1}{2}) P_\nu(t).$$

The next theorem in case $j = k-1$ was announced in [1].

THEOREM 3.18. *The following equality*

$$\inf\left\{\frac{\|f\|_2}{\|\underline{u}_k(f)\|_2};\ f \in \mathscr{P}_{j+1}\right\} = \lambda_{j,k} \tag{3.19}$$

holds for $0 \leqslant j < k$, and the inf *is attained for f proportional to P_j only.*

Proof. Using the orthonormal Legendre polynomials $\hat{P}_\nu$, Proposition 3.9 and Corollary 3.16, we can write for $f \in \mathcal{P}_{j+1}$

$$\|\underline{u}_k(f)\|_2^2 = \Big\| \sum_{\nu=0}^{j} \underline{u}_k(\hat{P}_\nu)(\hat{P}_\nu, f)\Big\|_2^2 = \sum_{\nu=0}^{j} \lambda_{\nu,k}^{-1}(\hat{P}_\nu, f)^2$$
$$\leqslant \lambda_{j,k}^{-1} \sum_{\nu=0}^{j} (\hat{P}_\nu, f)^2 = \lambda_{j,k}^{-1} \|f\|_2^2.$$

Moreover, the inequalities of Corollary 3.16 imply that for $f \in \mathcal{P}_{j+1}$ the equality takes place iff $(\hat{P}_\nu, f) = 0$ for $0 \leqslant \nu < j$, i.e. iff f is proportional to $\hat{P}_j$.

Corollary 3.20. *Let* $0 \leqslant j < k$. *If* $f = \sum_{i=0}^{k-1} a_i N_{i,k} \in \mathcal{P}_{j+1}$, *then*

$$\sqrt{\lambda_{j,k}}\, \|\underline{a}\|_2 \leqslant \|f\|_2 \leqslant \sqrt{2/k}\, \|\underline{a}\|_2,$$

where the constants $\sqrt{\lambda_{j,k}}$ *and* $\sqrt{2/k}$ *are the best possible.*

Corollary 3.21. *Let* $0 \leqslant j < k$. *If* $f = \sum_{i=0}^{k-1} b_i N_{i,k;2} \in \mathcal{P}_{j+1}$, *where* $N_{i,k;2} = \sqrt{k/2}\, N_{i,k}$, *then*

$$\sqrt{\frac{(k-1)_j}{(k+j)_j}}\, \|\underline{b}\|_2 \leqslant \|f\|_2 \leqslant \|\underline{b}\|_2,$$

where $(n)_j = n \cdot \ldots \cdot (n-j+1)$. *Notice, for fixed* j,

$$\frac{(k-1)_j}{(k+j)_j} \to 1 \quad \text{as } k \to \infty.$$

4. More formulas for the eigenvectors and the matrices. The following recurrence relation was proved in [1]

$$N_{i,k+1}^* = \frac{k-i}{k} N_{i,k}^* + \frac{i}{k} N_{i-1,k}^* + \sqrt{k+\tfrac{1}{2}}\,(-1)^i \binom{k}{i} \hat{P}_k. \tag{4.1}$$

Since $u_{i,k}^{(\nu)} = (N_{i,k}^*, \hat{P}_\nu)$ for $\nu = 0, \ldots, k-1$, (4.1) implies

$$u_{i,k+1}^{(\nu)} = \frac{k-i}{k} u_{i,k}^{(\nu)} + \frac{i}{k} u_{i-1,k}^{(\nu)}, \qquad \nu = 0, \ldots, k-1, \tag{4.2}$$

where $u_{-1,k}^{(\nu)} = 0$. Now, (4.2) and (2.11) allow to compute $\underline{u}_k^{(\nu)}$ for all k and $\nu = 0, \ldots, k-1$.

Proposition 4.3. *For the entries of the inverse to the Gram matrix* $\underline{G}_k$ *we have the formula*

$$(\underline{G}_k^{-1})_{i,j} = (N_{i,k}^*, N_{j,k}^*) = \sum_{\nu=0}^{k-1} u_{i,k}^{(\nu)} u_{j,k}^{(\nu)}, \qquad i, j = 0, \ldots, k-1.$$

Proof. This is simply the Parseval identity for the Legendre orthonormal set $(\hat{P}_0, \ldots, \hat{P}_{k-1})$.

PROPOSITION 4.4. *For the entries of the Gram matrix $\underset{\sim}{G}_k$ we have the formula*

$$G_{i,j,k} = (N_{i,k}, N_{j,k}) = \frac{2}{2k-1)}\binom{k-1}{i}\binom{k-1}{j}\Big/\binom{2k-2}{i+j}, \quad i,j = 0, \ldots, k-1.$$

Proof. It follows by the equalities

$$N_{i,k}(t)\, N_{j,k}(t) = \binom{k-1}{i}\binom{k-1}{j} N_{i+j,2k-1}(t)\Big/\binom{2k-2}{i+j}$$

and

$$\int_{-1}^{+1} N_{i,k}(t)\, dt = 2/k.$$

PROPOSITION 4.5 *For $v, i = 0, \ldots, k-1$ we have*

$$u_{i,k}^{(v)} = \sqrt{v+\tfrac{1}{2}}\, \frac{1}{\binom{k-1}{i}} \sum_{j=0}^{v} (-1)^{v+j} \binom{v}{j}\binom{v+j}{j}\binom{k-j-1}{i} \tag{4.6}$$

and

$$v_{i,k}^{(v)} = \sqrt{v+\tfrac{1}{2}}\, \frac{2}{k}\binom{k}{i+1} \sum_{j=0}^{v} (-1)^{v+j} \binom{v}{j}\binom{v+j}{j}\Big/\binom{k+j}{i+1}. \tag{4.7}$$

Proof. Since we have (2.1), formulas (4.6) and (4.7) follow from (2.3) and (2.4), respectively.

Remark 4.8. Simple transformation of (4.6) leads to the following familiar formula for the discrete Chebyshev polynomials (cf. e.g. [2], [4]),

$$\frac{1}{\sqrt{v+\frac{1}{2}}}\, u_{i,k}^{(v)} = \sum_{j=0}^{v} (-1)^j \binom{v}{j}\binom{v+j}{j} \frac{\binom{i}{j}}{\binom{k-1}{j}}.$$

PROPOSITION 4.9. *For $k \geqslant 1$ we have*

$$\frac{2}{2k-1} \sum_{j=0}^{k-1} \frac{\binom{k-1}{j}^2}{\binom{2k-2}{2j}} = \frac{2}{k} \sum_{v=0}^{k-1} \frac{\binom{k-1}{v}}{\binom{k+v}{k}}.$$

Proof. Use the following two equivalent expressions for the trace of $\underset{\sim}{G}_k$,

$$\mathrm{Tr}\, \underset{\sim}{G}_k = \sum_{j=0}^{k-1} (N_{j,k}, N_{j,k}) = \sum_{v=0}^{k-1} \lambda_{v,k}.$$

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