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Sharp estimation of even coefficients of bounded symmetric univalent functions

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Abstract. The fundamental result of the paper is

THEOREM. *Let N be an arbitrary fixed natural even number. Let $S_R(M)$, $M > 1$, be the family of functions $F(z) = z+\sum_{n=2}^{\infty} A_{nF} z^n$ holomorphic, univalent, with real coefficients and such that $F \in S_R(M)$ implies $|F(z)| \leqslant M$ for $|z| < 1$. Then there exists a constant M_N, $M_N > 1$, such that for all $M > M_N$ and every function $F \in S_R(M)$ the estimation*

$$A_{NF} \leqslant P_N(M)$$

is true, where

$$P_N(M) = N+\sum_{k=2}^{N} \left[(-1)^{k+1} \frac{2^k}{M^{k-1}} \frac{1\cdot 3\cdot \ldots \cdot (2k-1)}{(k+1)!} \cdot \sum_{\substack{(m_1,m_2,\ldots,m_k) \\ m_1+m_2+\ldots+m_k=N \\ 1\leqslant m_j\leqslant N,\, j=1,2,\ldots,k}} m_1\cdot m_2\cdot \ldots \cdot m_k\right].$$

The only function for which with a given M, $M > M_N$, equality holds in the above estimation is the Pick Function $w = P(z, M)$, $P(0, M) = 0$, given by the formula

$$\frac{w}{(1-w/M)^2} = \frac{z}{(1-z)^2}.$$

1. Introduction. Let S be the class of functions

$$F(z) = z+\sum_{n=2}^{\infty} A_{nF} z^n \tag{1}$$

which are holomorphic and univalent in the disc $K = \{z\colon |z| < 1\}$. We shall denote by $S_R(M)$, $M > 1$, the subclass of functions of the family S satisfying the conditions:

1° $|F(z)| \leqslant M$ for $z \in K$,

2° $A_{nF} = \bar{A}_{nF}$ for $n = 2, 3, \ldots$

Of course, for every function $F \in S_R(M)$ and any $z \in K$ we have $F(z) = \overline{F(\bar{z})}$.

It is known (e.g. [5], [8]) that:

$$A_{2F} \leqslant 2(1-M^{-1}), \quad M > 1,\ F \in S_R(M), \tag{2}$$

$$A_{4F} \leqslant 2(2-10M^{-1}+15M^{-2}-7M^{-3}), \quad M > 11, \quad F \in S_R(M). \tag{3}$$

Equality in estimations (2) and (3) holds only for the Pick function $w = P(z, M)$ given by the equation

$$w(1-wM^{-1})^{-2} = z(1-z)^{-2} \tag{4}$$

$(P(0, M) = 0, P(z, M) \in S_R(M))$. Moreover, estimation (3) is not true when M is sufficiently close to unity. This follows from a well-known result obtained by L. Siewierski ([10]–[12]) and, in some other way, by M. Schiffer and O. Tammi [9], concerning the class of bounded functions with arbitrary coefficients, sufficiently close to identity. This result constitutes a solution to the Charzyński–Tammi hypothesis.

In 1931 J. Dieudonné [1] proved that in the family $S_R \equiv S_R(\infty)$ $(S_R(M) \subset S_R, M > 1)$ the estimation

$$|A_{nF}| \leqslant n, \quad n = 2, 3, \ldots,$$

holds, equality taking place only for the function

$$F(z) = \frac{z}{(1-\varepsilon z)^2}, \quad \varepsilon = \pm 1.$$

The above result of Dieudonné as well as estimations (2) and (3) became a premise of the following hypothesis: for every $N = 2, 4, 6, \ldots$ there exists a constant M_N, $M_N > 1$, such that for all $M > M_N$ and every function $F \in S_R(M)$ the estimation

$$A_{NF} \leqslant P_N(M) \tag{5}$$

holds true, where $P_N(M)$ is the N-th coefficient in Taylor expansion (1) of the function $P(z, M)$. This hypothesis, formulated by Z. J. Jakubowski, was first posed in paper [16].

Note that inequality (5) is not true for any odd N since in the class $S_R(M)$, $M > 1$, the sharp estimation ([6], [3], [4], [13])

$$A_{3F} \leqslant 1+2\lambda^2-4\lambda M^{-1}+M^{-2} \quad \text{for } e \leqslant M < +\infty$$

is known, where λ is the greater of the roots of the equation $\lambda \log \lambda = -M^{-1}$. As is seen, the function $w = P(z, M)$ is not an extremal one in this case.

In papers [16], [17] we succeeded in proving estimation (5) for M sufficiently large when $N = 6$ and $N = 8$. In the present paper we prove the validity of the hypothesis for an arbitrary even N. It is worth marking that the method applied here allowed us to avoid a complicated integration of the differential-functional equation (cf. e.g. [8]).

2. The equation of extremal functions. Let N be any fixed natural even number. Consider in the family $S_R(M)$, $M > 1$, the functional

$$H(F) = A_{NF}. \tag{6}$$

This functional is continuous and class compact, so there exists at least one function for which functional (6) attains its maximum. Denote by $\mathcal{F}_M = \mathcal{F}_M(N)$ the family of all functions extremal with respect to the maximum of functional (6) in the class $S_R(M)$.

It follows from the main theorem ([2], [8]) that if $F(z) = z+\sum_{n=2}^{\infty} A_{nF} z^n$ is an arbitrary function of the family $\mathcal{F}_M$, then the function

$$w = f(z) = \frac{1}{M} F(z) = \sum_{n=1}^{\infty} a_{nf} z^n, \quad a_{1f} = \frac{1}{M}, \quad a_{nf} = \frac{1}{M} A_{nF}, \quad n = 2, 3, \ldots,$$

satisfies the differential-functional equation of the form:

$$\left(\frac{zw'}{w}\right)^2 \mathfrak{M}(w) = \mathfrak{N}(z), \quad 0 < |z| < 1, \tag{7}$$

where

$$\begin{aligned}
\mathfrak{M}(w) &= \sum_{j=2}^{N} D_{j-1}\left(w^{j-1}+\frac{1}{w^{j-1}}\right)-2\mathcal{P},\\
\mathfrak{N}(z) &= \sum_{j=1}^{N} E_{j-1}\left(z^{j-1}+\frac{1}{z^{j-1}}\right)-2\mathcal{P},\\
D_j &= 2a_{nF}^{(j+1)}, \quad j = 1,2,\ldots,N-1,\\
E_j &= 2(N-j)a_{N-j,f}, \quad j = 1,2,\ldots,N-1,\\
E_0 &= (N-1)a_{NF},\\
\mathcal{P} &= \min_{0\leqslant x\leqslant 2\pi} \left[\sum_{j=2}^{N} D_{j-1} \cos(j-1)x\right],
\end{aligned} \tag{8}$$

$$f^m(z) = \sum_{n=m}^{\infty} a_{nf}^{(m)} z^n, \quad m = 2, 3, \ldots$$

Inserting (8) into the differential-functional equation (7), and then dividing both sides by 2, we obtain the equation:

$$\left(\frac{zw'}{w}\right)^2 M(w) = N(z), \quad 0 < |z| < 1, \tag{9}$$

where

$$\begin{aligned}
M(w) &= \sum_{j=2}^{N} a_{Nf}^{j}\left(w^{j-1}+\frac{1}{w^{j-1}}\right)-\mathcal{P},\\
N(z) &= (N-1)a_{Nf}+\sum_{j=2}^{N} (N-j+1)a_{N-j+1,f}\left(z^{j-1}+\frac{1}{z^{j-1}}\right)-\mathcal{P}.
\end{aligned} \tag{10}$$

3. Auxiliary theorems. Before examining the differential-functional equation (9) we prove a few lemmas.

Let us introduce a one-parameter family $\mathfrak{P} = (P(z, M)), M \in (1, +\infty)$ of functions $w = P(z, M), z \in K$, satisfying equation (4) and the condition $P(0, M) = 0, M \in (1, +\infty)$. Each function of the family $\mathfrak{P}$ can be represented in the form

$$(11) \qquad P(z, M) = M\frac{2z+M(1-z)^2-(1-z)\sqrt{M[4z+M(1-z)^2]}}{2z}, \qquad z \in K, \; M \in (1, +\infty),$$

where $\sqrt{M^2} = M$. The branch of the root exists in the disc K because for all $M > 1$, $4z+M(1-z)^2 \neq 0$. It is known that for every M, $M \in (1, +\infty)$, $P(z, M) \in S_R(M)$.

LEMMA 1. *Let $P(z, M)$ be an arbitrary function of the family $\mathfrak{P}$, F_0 — a Koebe function defined by the formula*

$$(12) \qquad F_0(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n, \qquad z \in K.$$

Then, for every $M > 1$ and every $z \in K$ such that $|F_0(z)| < M/4$, the equality

$$(13) \qquad P(z, M) = F_0(z)+\sum_{k=2}^{\infty}(-1)^{k+1}\frac{2^k}{M^{k-1}}\cdot\frac{1\cdot 3\cdot\ldots\cdot(2k-1)}{(k+1)!}F_0^k(z)$$

holds.

Proof. From formulae (11) and (12) we obtain that every function $P(z, M) \in \mathfrak{P}$ can be represented in the form

$$P(z, M) = M\left[1+\frac{M}{2}F_0^{-1}(z)-\frac{M}{2}F_0^{-1}(z)\sqrt{1+\frac{4}{M}F_0(z)}\right], \qquad z \in K,$$

where $\sqrt{1} = 1$. Using the expansion of the function $(1+w)^\alpha$ into a Taylor series, we hence have for $|F_0(z)| < M/4, z \in K$:

$$(14) \qquad P(z, M) = M\left[1+\frac{M}{2}F_0^{-1}(z)-\frac{M}{2}F_0^{-1}(z)\Big(\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}\frac{4^k}{M^k}F_0^k(z)\Big)\right].$$

Isolating the first three addends of the sum in (14), we ultimately obtain after reduction and change of the limits of summation

$$P(z, M) = F_0(z)-\sum_{k=2}^{\infty}\binom{\frac{1}{2}}{k+1}\frac{2^{2k+1}}{M^{k-1}}F_0^k(z), \qquad z \in K, |F_0(z)| < \tfrac{1}{4}M,$$

which gives formula (13).

Remark 1. It follows from the Weierstrass criterion and the estimation $|F_0(z)| \leqslant r/(1-r)^2$, $|z| \leqslant r$, that for every fixed $M > 1$ there exists an $r_M \in (0, 1)$ $\left(r_M = 1+\frac{2}{M}-2\sqrt{\frac{1}{M}+\frac{1}{M^2}}\right)$ such that series (13) is almost uniformly convergent in the disc $|z| < r_M$.

Now we prove

LEMMA 2. *For every number $\varepsilon > 0$ and any closed set $\Delta \subset K$ there exists a constant $\hat{M}, \hat{M} > 1$, such that for all $M > \hat{M}$ and every function $P(z, M) \in \mathfrak{P}$*

$$|P(z, M) - F_0(z)| < \varepsilon, \quad z \in \Delta,$$

where F_0 is the Koebe function (12).

Proof. Paying respect to (12) we have, by Lemma 1,

$$|P(z, M) - F_0(z)| = \frac{1}{M} \Big| \sum_{k=2}^{\infty} (-1)^{k+1} \frac{2^k}{M^{k-2}} \cdot \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{(k+1)!} F_0^k(z) \Big|,$$
$$z \in K, |F_0(z)| < \tfrac{1}{4} M.$$

Let us take any $\varepsilon > 0$ and a closed set $\Delta \subset K$. Let $L_\Delta = \sup_{z \in \Delta} |F_0(z)|$; obviously $L_\Delta < +\infty$. Denote by S the sum of the convergent series $\sum_{k=2}^{\infty} \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2^k (k+1)!}$. Adopt $\hat{M} = \max \{4L_\Delta, 2^4 \cdot S \cdot L_\Delta^2/\varepsilon\}$. Then, for $M > \hat{M} \geqslant 4L_\Delta$, we have

$$\left| \frac{2^k}{M^{k-2}} \cdot \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{(k+1)!} F_0^k(z) \right| < 2^4 L_\Delta^2 \frac{1 \cdot 3 \ \ldots \cdot (2k-1)}{2^k (k+1)!}.$$

So, in virtue of the Weierstrass criterion, the series $\sum_{k=2}^{\infty} (-1)^{k+1} \frac{2^k}{M^{k-2}} \times \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{(k+1)!} F_0^k(z)$ is uniformly convergent in Δ when $M > \hat{M}$ and the modulus of its sum is not greater than $2^4 \cdot L_\Delta^2 \cdot S$. From the above we have for $M > \hat{M}$ and $z \in \Delta$:

$$|P(z, M) - F_0(z)| \leqslant \frac{1}{M} \cdot 2^4 \cdot L_\Delta^2 \cdot S < \varepsilon,$$

which completes the proof of the lemma.

COROLLARY 1. *Let $P(z, M) \in \mathfrak{P}$,*

$$P(z, M) = z + \sum_{n=2}^{\infty} P_n(M) z^n, \quad z \in K.$$

Then, for every $M > 1$,

$$P_n(M) = n + \sum_{k=2}^{n} \Big[(-1)^{k+1} \frac{2^k}{M^{k-1}} \cdot \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{(k+1)!} \times \sum_{\substack{(m_1, m_2, \ldots, m_k) \\ m_1 + m_2 + \ldots + m_k = n \\ 1 \leqslant m_j \leqslant n, j = 1, 2, \ldots, k}} m_1 \cdot m_2 \cdot \ldots \cdot m_k \Big], \quad n = 2, 3, \ldots, \tag{15}$$

and, in consequence,

$$\lim_{M \to +\infty} P_n(M) = n, \quad n = 2, 3, \ldots \tag{16}$$

Proof. By Remark 1 and formula (13) we have for every $M > 1$

$$(17)\quad P_n(M) = \frac{P^{(n)}(0, M)}{n!} = \frac{F_0^{(n)}(0)}{n!} + \sum_{k=2}^{\infty} (-1)^{k+1} \frac{2^k}{M^{k-1}} \times \\ \times \frac{1\cdot 3\cdot\ldots\cdot(2k-1)}{(k+1)!\,n!} [F_0^k(z)]^{(n)}|_{z=0}, \quad n = 2, 3, \ldots$$

Thus, in order to prove formula (15), it suffices to calculate $[F_0^k(z)]^{(n)}|_{z=0}$ for any $n = 2, 3, \ldots$ and any $k = 1, 2, \ldots$, $k \leqslant n$. From the generalized Leibniz formula we get

$$[F_0^k(z)]^{(n)} = \sum_{\substack{(m_1, m_2, \ldots, m_k) \\ m_1+m_2+\ldots+m_k = n \\ 0\leqslant m_j \leqslant n,\, j=1,2,\ldots,k}} \frac{n!}{m_1!\,m_2!\cdot\ldots\cdot m_k!} \cdot F_0^{(m_1)}(z)\cdot F_0^{(m_2)}(z)\cdot\ldots\cdot F_0^{(m_k)}(z),$$

$$n = 0, 1, \ldots,\ k = 1, 2, \ldots,$$

$$[F_0(z)]^{(0)} \equiv F_0(z),$$

and thus

$$(18)\quad [F_0^k(z)]^{(n)}|_{z=0} = \sum_{\substack{(m_1, m_2, \ldots, m_k) \\ m_1+m_2+\ldots+m_k = n \\ 0\leqslant m_j \leqslant n,\, j=1,2,\ldots,k}} \frac{n!}{m_1!\,m_2!\cdot\ldots\cdot m_k!} \times \\ \times F_0^{(m_1)}(0)\cdot F_0^{(m_2)}(0)\cdot\ldots\cdot F_0^{(m_k)}(0), \quad n = 0, 1, \ldots,\ k = 1, 2, \ldots$$

Note that $F_0^{(0)}(0) = 0$; therefore in the sum on the right-hand side of (18) all addends with indices $(m_1, m_2, \ldots, m_k)$ containing at least one zero vanish. And so, we may assume that $1 \leqslant m_j \leqslant n$, $j = 1, 2, \ldots, k$. Since $[F_0(z)]^{(m)}|_{z=0} = m\cdot m!$, $m \geqslant 1$, we have

$$[F_0^k(z)]^{(n)}|_{z=0} = n! \sum_{\substack{(m_1, m_2, \ldots, m_k) \\ m_1+m_2+\ldots+m_k = n \\ 1\leqslant m_j \leqslant n,\, j=1,2,\ldots,k}} m_1\cdot m_2\cdot\ldots\cdot m_k, \quad n = 1, 2, \ldots,\ k = 1, 2, \ldots$$

From that and from (17) we get (15). Note that the convergence (16) could be obtained immediately from Lemma 2 and the Weierstrass theorem.

LEMMA 3 (cf. [16]). *Let N be any fixed natural even number, $(M_h)_{h=1,2,\ldots}$ — an arbitrary sequence of real numbers, $M_h > 1$, $h = 1, 2, \ldots$, such that $\lim_{h\to+\infty} M_h = +\infty$. From each of the families $\mathscr{F}_{M_h} = \mathscr{F}_{M_h}(N)$, $h = 1, 2, \ldots$, of functions extremal with respect to the maximum of functional* (6) *in the respective classes $S_R(M_h)$, $h = 1, 2, \ldots$, let us choose arbitrarily a function $F_h(z) = z + \sum_{n=2}^{\infty} A_{nh} z^n$ and consider the sequence $(F_h)_{h=1,2,\ldots}$. Then, for every* *number* $\varepsilon > 0$ *and any closed set* $\Delta \subset K$, *there exists an* h_0 *such that for all* $h > h_0$ *and* $z \in \Delta$:

$$|F_h(z) - F_0(z)| < \varepsilon,$$

where F_0 *is the Koebe function* (12).

The proof is a consequence of the inequality $P_N(M_h) \leqslant A_{Nh} \leqslant N$, Corollary 1, the fact that F_0 is the only function in the family S_R for which $A_{NF} = N$ [1] (N natural even), and of the Vitali–Osgood theorem.

Remark 2. It follows immediately from the Weierstrass theorem that for every sequence of extremal functions $(F_h)_{h=1,2,\dots}$ defined in Lemma 3, and for any $n, n = 2, 3, \dots, \lim_{h \to +\infty} A_{nh} = n$.

COROLLARY 2. *If, for every* $M, M \in (1, +\infty)$, *any function* F *of form* (1) $(A_{nF} = A_{nF}(M), n = 2, 3, \dots)$ *belongs to the family* $\mathscr{F}_M$, *then, for every* n, $n = 2, 3, \dots$, *and any* $\varepsilon > 0$, *there exists an* M_n *such that for all* $M > M_n$ *and every function* $F \in \mathscr{F}_M$

$$|A_{nF} - n| < \varepsilon.$$

The proof follows immediately from the arbitrariness of choice of the sequence $(M_h)_{h=1,2,\dots}$ and of the sequence of extremal functions, corresponding to it, in Lemma 3.

LEMMA 4 (cf. [16]). *Let* $(M_h)_{h=1,2,\dots}$ *be an arbitrary sequence of real numbers,* $M_h > 1, h = 1, 2, \dots$, *such that* $\lim_{h \to +\infty} M_h = +\infty$. *From each of the families* $\mathscr{F}_{M_h}$, $h = 1, 2, \dots$, *of functions extremal with respect to the maximum of functional* (6) *in the respective classes* $S_R(M_h)$, $h = 1, 2, \dots$, *let us choose arbitrarily a function* $F_h(z) = z + \sum_{n=2}^{\infty} A_{nh} z^n$ *and consider the sequence* $(F_h^m)_{h=1,2,\dots}$, *where* m *is a fixed natural number,* $m \geqslant 2$. *Then, for every number* $\varepsilon > 0$ *and any closed set* $\Delta \subset K$, *there exists an* h_0 *such that for all* $h > h_0$ *and* $z \in \Delta$:

$$|F_h^m(z) - F_0^m(z)| < \varepsilon,$$

where F_0 *is the Koebe function* (12).

The proof is a consequence of the close-to-common boundedness of the sequence $(F_h^m)_{h=1,2,\dots}$ in the disc K, Lemma 3, and of the Vitali–Osgood theorem.

Remark 3. Let $(F_h^m)_{h=1,2,\dots}$ be an arbitrary sequence defined in Lemma 4. Denote

$$F_h^m(z) = \sum_{n=m}^{\infty} A_{nh}^{(m)} z^n, \quad h = 1, 2, \dots, m \geqslant 2,$$

$$F_0^m(z) = \sum_{n=m}^{\infty} A_{n0}^{(m)} z^n, \quad m \geqslant 2.$$

It follows immediately from the Weierstrass theorem and Lemma 4 that

$$\lim_{h\to+\infty} A_{nh}^{(m)} = A_{n0}^{(m)}, \quad n = m, m+1, \ldots, \ m \geqslant 2.$$

COROLLARY 3. *Let m, $m \geqslant 2$, be an arbitrary fixed natural number. For every $M \in (1, +\infty)$ and any function $F \in \mathcal{F}_M$ denote*

$$(19) \quad F^m(z) = \sum_{n=m}^{\infty} A_{nF}^{(m)} z^n, \quad z \in K \ (A_{nF}^{(m)} = A_{nF}^{(m)}(M), \ n = m, m+1, \ldots).$$

Then, for every n, $n = m, m+1, \ldots$, and every number $\varepsilon > 0$, there exists an M_n such that for all $M > M_n$ and every function $F \in \mathcal{F}_M$

$$|A_{nF}^{(m)} - A_{n0}^{(m)}| < \varepsilon,$$

where the numbers $A_{n0}^{(m)}$, $n = m, m+1, \ldots$, stand for the coefficients of the function F_0^m defined in Remark 3.

The proof follows directly from the arbitrariness of choice of the sequence $(M_h)_{h=1,2,\ldots}$ in Lemma 4 and of the sequence of functions $(F_h^m)_{h=1,2,\ldots}$, corresponding to it.

LEMMA 5 [16]. *Let*

1° *n be an arbitrary fixed natural number,*

2° *$(C_{0t}, C_{1t}, \ldots, C_{nt})_{t\in(1,+\infty)}$ – a given family of $(n+1)$ – sequences of sets of real numbers,*

3° *$\mathscr{C}_t = \{(c_0(t), c_1(t), \ldots, c_n(t)) \colon c_0(t) \in C_{0t}, c_1(t) \in C_{1t}, \ldots, c_n(t) \in C_{nt}\}$, $t \in (1, +\infty)$,*

4° *there exists a sequence of numbers $(c_0, c_1, \ldots, c_n)$ which satisfies the condition: for every $\eta > 0$ there exists a t_0 such that for all $t > t_0$, if $(c_0(t), c_1(t), \ldots, c_n(t)) \in \mathscr{C}_t$, then $\max_{0\leqslant k\leqslant n} |c_k(t) - c_k| < \eta$,*

5° $W_0(z) = c_0 z^n + c_1 z^{n-1} + \ldots + c_n$,

$$\mathscr{W}_t = \{W(z, t) \colon W(z, t) = c_0(t) z^n + c_1(t) z^{n-1} + \ldots + c_n(t)\}, \quad t \in (1, +\infty).$$

Then, for an arbitrary number $\varepsilon > 0$ and every closed and bounded set Δ, there exists a t' such that for all $t > t'$ and every $W(z, t) \in \mathscr{W}_t$

$$|W(z, t) - W_0(z)| < \varepsilon$$

for every $z \in \Delta$.

The proof is immediate.

Remark 4. If assumptions 1°–5° of Lemma 5 are satisfied and

6° $W_0(z) \not\equiv 0$,

7° $W_0(z)$ has a k-tuple zero at a certain point z_0,

then it follows from the Hurwitz theorem that there exists a t' such that for all $t > t'$ each of the polynomials $W(z,t) \in \mathscr{W}_t$ has exactly k zeros in every sufficiently small neighbourhood of the point z_0.

4. Localization of roots of the right-hand side of the differential-functional equation in the limit case $M = +\infty$. In the method of proving the basic theorem, presented later on, the essential part will be played by the localization of roots of the right-hand side $N(z)$ of the differential-functional equation (9) in the limit case $M = +\infty$. In order to obtain this limit form of the function $N(z)$ we shall employ the lemmas proved in Section 3. And so, let F be an arbitrary function extremal with respect to the maximum of functional (6) in the family $S_R(M)$, that is, let $F \in \mathscr{F}_M$. The function $w = f(z) = (1/M)F(z)$ satisfies equation (9), where $M(w)$ and $N(z)$ are given by formulae (10). Let us recall the notation (cf. (19), (18))

$$F^m(z) = \sum_{n=m}^{\infty} A_{nF}^{(m)} z^n, \quad z \in K, \ m = 2, 3, \ldots,$$

$$f^m(z) = \sum_{n=m}^{\infty} a_{nf}^{(m)} z^n, \quad z \in K, \ m = 2, 3, \ldots$$

From the relationship $F(z) = Mf(z)$ it follows at once that

$$A_{nF}^{(m)} = M^m \cdot a_{nf}^{(m)}, \quad m = 2, 3, \ldots, \ n = m, m+1, \ldots$$

Multiplying both sides of equation (9) by M and taking account of the above, we obtain that the function $w = f(z)$ satisfies the equation

$$\left(\frac{zw'}{w}\right)^2 \tilde{M}(w) = \tilde{N}(z), \quad 0 < |z| < 1, \tag{20}$$

where

$$\tilde{M}(w) = \tilde{M}_F(w) = M \cdot M(w) = \sum_{j=2}^{N} \frac{A_{NF}^{(j)}}{M^{j-1}} \left(w^{j-1} + \frac{1}{w^{j-1}}\right) - P, \tag{21}$$

$$\tilde{N}(z) = \tilde{N}_F(z) = M \cdot N(z) = (N-1)A_{NF} + \sum_{j=2}^{N} (N-j+1) A_{N-j+1,F}\left(z^{j-1} + \frac{1}{z^{j-1}}\right) - P, \tag{22}$$

$$P = P_F = M \cdot \mathscr{P} = 2 \min_{0 \leqslant x \leqslant 2\pi} \left[\sum_{j=2}^{N} \frac{A_{NF}^{(j)}}{M^{j-1}} \cos(j-1)x\right]. \tag{23}$$

It follows from Corollary 3 that for every $\eta_1 > 0$ there exists a constant $\hat{M}'$ such that for all $M > \hat{M}'$, $\max_{2 \leqslant j \leqslant N} \left|\frac{A_{NF}^{(j)}}{M^{j-1}}\right| < \eta_1$, and consequently, for every $\varepsilon_1 > 0$ there exists an $M' > \hat{M}'$ such that for all $M > M'$, $|P| < \varepsilon_1$. In turn, from Corollary 2 it follows that for any $\eta_2 > 0$ there exists an M'' such that for every $M > M''$, $\max_{1 \leqslant j \leqslant N} |A_{jF} - j| < \eta_2$. From the above and

Lemma 5 we have that for any $\varepsilon_2 > 0$ and every closed and bounded set Δ there exists an $M''' \geqslant \max\{M', M''\}$ such that for $M > M'''$ and every $z \in \Delta$

$$\left|z^{N-1}\left(\tilde{N}(z) - N_0(z)\right)\right| < \varepsilon_2,$$

where $\tilde{N}(z)$ is given by (22), while $N_0(z)$ is the sought – for limit form of the right-hand side of equation (9) and is given by the formula:

$$(24)\qquad N_0(z) = (N-1)N + \sum_{j=2}^{N} (N-j+1)^2\left(z^{j-1} + \frac{1}{z^{j-1}}\right)$$
$$= \frac{1}{z^{N-1}}\left[z^{2N-2} + 2^2 z^{2N-3} + 3^2 z^{2N-4} + \ldots + (N-1)^2 z^N + \right.$$
$$\left. + (N-1)Nz^{N-1} + (N-1)^2 z^{N-2} + \ldots + 3^2 z^2 + 2^2 z + 1\right].$$

It turns out that the point $z = -1$ is a double root of function (24) and that

$$(25)\qquad N_0(z) = \frac{(z+1)^2}{z^{N-1}} L_0(z),$$

where

$$(26)\qquad L_0(z) = z^{2N-4} + 1\cdot 2z^{2N-5} + 2^2 z^{2N-6} + 2\cdot 3z^{2N-7} +$$
$$+ 3^2 z^{2N-8} + \ldots + (\tfrac{1}{2}N-1)^2 z^N + (\tfrac{1}{2}N-1)\tfrac{1}{2}N z^{N-1} +$$
$$+ (\tfrac{1}{2}N)^2 z^{N-2} + (\tfrac{1}{2}N-1)\tfrac{1}{2}N z^{N-3} + (\tfrac{1}{2}N-1)^2 z^{N-4} + \ldots$$
$$\ldots + 3^2 z^4 + 2\cdot 3z^3 + 2^2 z^3 + 1\cdot 2z + 1.$$

In view of (25), it is evident that the localization of roots of the function $N_0(z)$ reduces to examining the zeros of the polynomial $L_0(z)$. Perhaps it is worth observing that this examination seemed to be rather complicated. On the one hand, the symmetry of coefficients allows to confine our considerations to the interior and the boundary of the unit circle; on the other hand, however, it precludes the use of many general theorems concerning the localization of roots of polynomials (see e.g. [15]) It suffices just as much as to compare papers [16], [17]. While for $N = 6$ the examination of the polynomial $L_0(z)$ was almost immediate [16], for $N = 8$ [17] it required the use of laborious methods which, as it appears, cannot be applied in the case of an arbitrary even N. The simple proof of the lemma given below is therefore a consequence of rather long research.

LEMMA 6. *Let $L_0(z)$ be the polynomial defined by formula* (26), *where N is an arbitrary fixed natural even number. Then*

$$L_0(z) = \prod_{k=1}^{N/2-1} (z-z_k)(z-\bar{z}_k)\left(z - \frac{1}{z_k}\right)\left(z - \frac{1}{\bar{z}_k}\right),$$

where $|z_k| < 1$, $z_k \neq \bar{z}_k$, $= 1, 2, \ldots, \frac{1}{2}N-1$.

Proof. As observed $L_0(z)$ is a symmetric polynomial with real coefficients, and so, if $L_0(z_0) = 0$, then also $L_0(\bar{z}_0) = 0, L_0(1/z_0) = 0$ and $L_0(1/\bar{z}_0) = 0$.

We shall show that $L_0(z)$ has no roots on the circle $|z| = 1$. For the purpose, let us construct a function

$$L_1(z) = \frac{(1-z)^2}{z^{N-1}} L_0(z)$$
$$= \left(z^{N-1}+\frac{1}{z^{N-1}}\right)+\left(z^{N-3}+\frac{1}{z^{N-3}}\right)+ \ldots +\left(z^3+\frac{1}{z^3}\right)+\left(z+\frac{1}{z}\right)-N.$$

Since, by (26), $z = 1$ is not a root of the polynomial $L_0(z)$, its zeros coincide with those of the function $L_1(z)$, different from unity. Let us examine the function $L_1(z)$ on the circle $|z| = 1$. For $z = e^{ix}$, $0 \leqslant x < 2\pi$, we have

$$L_1(e^{ix}) = 2[\cos(N-1)x+\cos(N-3)x+ \ldots +\cos 3x+\cos x-N/2].$$

Of course, $L_1(e^{ix}) = 0$ if and only if

$$\cos(N-1)x+\cos(N-3)x+ \ldots +\cos 3x+\cos x = N/2. \tag{27}$$

The left-hand side of equation (27) has $N/2$ addends, each of them being not greater than unity. Thus, the equation is satisfied if and only if

$$\cos(N-1)x = \cos(N-3)x = \ldots = \cos 3x = \cos x = 0,$$

that is, if $x = 0$. So, the only root of the function $L_1(z)$ on the circle $|z| = 1$ is the point $z = 1$, and consequently, the polynomial $L_0(z)$ has no zeros on this circle.

Examining in the same way the function $L_1(z)$ for $z = r$, $-1 < r < 1$, we prove that the polynomial $L_0(z)$ has no real roots, either, which concludes the proof of the lemma.

Remark 5. It follows from Lemma 6 and formula (25) that $N_0(z)$ has only one double root on the circle $|z| = 1$. This fact is of essential importance in further considerations.

5. The basic theorem.

THEOREM. *Let N be an arbitrary fixed natural even number. Let $S_R(M)$, $M > 1$, be the family of functions $F(z) = z+\sum_{n=2}^{\infty} A_{nF} z^n$ holomorphic, univalent, with real coefficients, and such that $F \in S_R(M)$ implies $|F(z)| \leqslant M$ for $z \in K$. Then there exists a constant M_N, $M_N > 1$, such that for all $M > M_N$ and every function $F \in S_R(M)$ the estimation*

$$A_{NF} \leqslant P_N(M) \tag{28}$$

is true, where

$$P_N(M) = N+\sum_{k=2}^{N}\left[(-1)^{k+1}\frac{2^k}{M^{k-1}}\frac{1\cdot 3\cdot\ldots\cdot 2k-1}{(k+1)!}\times \times \sum_{\substack{(m_1,m_2,\ldots,m_k)\\ m_1+m_2+\ldots+m_k=N\\ 1\leqslant m_j\leqslant N,\, j=1,2,\ldots,k}} m_1\cdot m_2\cdot\ldots\cdot m_k\right].$$

The only function for which with a given $M, M > M_N$, equality holds in estimation (28) *is the Pick function $w = P(z, M), P(0, M) = 0$, given by the equation*

$$\frac{w}{(1-w/M)^2} = \frac{z}{(1-z)^2}.$$

Proof. Let $F\in\mathscr{F}_M$, $w = f(z) = (1/M)F(z)$. The function $w = f(z)$ satisfies the differential-functional equation (20), where $\tilde{M}(w)$, $\tilde{N}(z)$ and P are given by formulae (21), (22) and (23), respectively. In Section 4 we have proved that for any $\varepsilon > 0$ there exists an M''' such that for $M > M'''$ and for every $z\in\Delta$

$$\left|z^{N-1}\big(\tilde{N}(z)-N_0(z)\big)\right| < \varepsilon,$$

where $N_0(z)$ is given by formula (24). From (25) and Lemma 6 (Remark 5) we infer that the function $N_0(z)$ has on the circle $|z| = 1$ exactly one double root $z = -1$, and $N-2$ complex roots $z_k, \bar{z}_k$, $k = 1,\ldots,\frac{1}{2}N-1$, inside the circle. Let us surround all zeros of the function $N_0(z)$ by sufficiently small disjoint discs. It follows from Remark 4 that there exists an $M_N > M'''$ such that for all $M > M_N$ the zeros of the function $\tilde{N}(z)$ lie respectively in the selected neighbourhoods of the zeros of the function $N_0(z)$, in each of these neighbourhoods the number of zeros of both the functions (considering multiplicity) being the same. From this and from the properties of the function $\tilde{N}(z)$ [2] it follows, in turn, that this function has for $M > M_N$ the same factorization as the function $N_0(z)$, i.e.

$$\tilde{N}(z) = \frac{1}{z^{N-1}}(z+1)^2\prod_{k=1}^{(N/2)-1}(z-\tilde{z}_k)(z-\bar{\tilde{z}}_k)\left(z-\frac{1}{\tilde{z}_k}\right)\left(z-\frac{1}{\bar{\tilde{z}}_k}\right), \tag{29}$$

where $|\tilde{z}_k| < 1, \tilde{z}_k \neq \bar{\tilde{z}}_k$, $k = 1,2,\ldots,\frac{1}{2}N-1$.

In virtue of equation (20), the images $\tilde{w}_k = f(\tilde{z}_k)$ of the points $\tilde{z}_k$, $k = 1,2,\ldots,\frac{1}{2}N-1$, are the roots of the function $\tilde{M}(w)$ defined by formula (21), whereas from the very form of the function $\tilde{M}(w)$ it follows that also the points $\bar{\tilde{w}}_k, \dfrac{1}{\tilde{w}_k}, \dfrac{1}{\bar{\tilde{w}}_k}$, $k = 1,2,\ldots,\frac{1}{2}N-1$, are its roots. Moreover, it is known [2] that $\tilde{M}(w)$ has on the circle $|w| = 1$ at least one double root w_0, which, on account of the above, gives for $M > M_N$:

$$(30) \quad \tilde{M}(w) = \frac{1}{w^{N-1}}(w-\tilde{w}_0)^2 \prod_{k=1}^{(N/2)-1} (w-\tilde{w}_k)(w-\bar{\tilde{w}}_k)\left(w-\frac{1}{\tilde{w}_k}\right)\left(w-\frac{1}{\bar{\tilde{w}}_k}\right),$$

where $|\tilde{w}_k| < 1, \tilde{w}_k \neq \bar{\tilde{w}}_k$, $k = 1, 2, \ldots, \frac{1}{2}N-1$, and $\tilde{w}_0 = \pm 1$.

It is known ([8], cf. [2], [7], [14]) that every function $f(z) = (1/M)F(z)$, $F \in \mathscr{F}_M$, maps the disc $|z| < 1$ onto the disc $|w| < 1$ with cuts along a finite number of analytic arcs.

Making use of equation (20) and distributions (29) and (30), we prove, on the basis of the properties of the classes $S_R(M)$ considered (cf. [16]), that, for $M > M_N$, the image of the disc $|z| < 1$ under the mapping $w = f(z) = (1/M)F(z)$, $F \in \mathscr{F}_M$, is the disc $|w| < 1$ with one cut along an analytic arc, with the initial point at $\tilde{w}_0 = -1$. From the symmetry of functions of the classes $S_R(M)$ it follows that this arc is symmetric with respect to the real axis. Also, making use of the definition of an ordinary arc as a homeomorphic image of the segment $\langle 0, 1\rangle$, we prove conversely that this arc lies entirely on the real axis. Hence, and from the fact that $f(0) = 0$, it follows immediately that the above arc is a segment of the negative real half-axis, with the terminal points $\tilde{w}_0 = -1$ and $w_0 = \bar{w}_0 < 0$.

It follows from Schwarz's lemma that the only function having such an image, satisfying the condition that for a fixed $M, M > M_N$, the function $M \cdot f(z)$ belongs to the class $S_R(M)$, is the function $p_0(z) = (1/M)P(z, M)$, where $P(z, M)$ is the Pick function given by equation (4), with $P(0, M) = 0$. From Corollary 1 we obtain estimation (28), which completes the proof of the theorem.

It is evident that if $F \in S_R(M)$, $M > 1$, then the function G defined by the formula $G(z) = -F(-z)$, $z \in K$, also belongs to $S_R(M)$. Thus our theorem implies:

COROLLARY 4. *Let N be an arbitrary fixed natural even number. For all $M > M_N$ and any function $F \in S_R(M)$ the estimation*

$$A_{NF} \geqslant -P_N(M)$$

takes place, where $P_N(M)$ is given by formula (15). *The only function for which equality holds is the function $\tilde{P}(z) = -P(-z, M)$.*

The question of determining a minimal M_N such that for all $M > M_N$ the Pick function is an extremal function remains open.

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