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Continuous solutions of some functional equations in the indeterminate case

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1. In the present paper we shall deal with continuous solutions of the functional equations

$$\varphi(x) = h\big(x, \varphi[f(x)]\big) \tag{1}$$

and

$$\varphi[f(x)] = g\big(x, \varphi(x)\big), \tag{2}$$

where φ is the unknown function. The theory of continuous solutions of equations (1) and (2) has been developed in [2], [3], [4] (cf. also [5], Chapter III) under the condition that

$$\left|\frac{\partial h}{\partial y}(\xi, \eta)\right| \neq 1, \quad \text{resp.} \quad \left|\frac{\partial g}{\partial y}(\xi, \eta)\right| \neq 1,$$

where the point (ξ, η) is characterized by the property that $f(\xi) = \xi$ and $h(\xi, \eta) = \eta$, resp. $g(\xi, \eta) = \eta$. The indeterminate case

$$\left|\frac{\partial h}{\partial y}(\xi, \eta)\right| = 1, \quad \text{resp.} \quad \left|\frac{\partial g}{\partial y}(\xi, \eta)\right| = 1 \tag{3}$$

has been dealt with [1] only in the case of the linear equation

$$\varphi[f(x)] = g(x)\varphi(x) + F(x). \tag{4}$$

In the present paper we are going to extend some of those results to the general case (1) and (2). Instead of (3), we shall assume that the functions g, h fulfil the Lipschitz condition with respect to y:

$$|h(x, y_1) - h(x, y_2)| \leqslant \gamma(x)\,|y_1 - y_2|, \tag{5}$$

resp.

$$|g(x, y_1) - g(x, y_2)| \leqslant \gamma(x)\,|y_1 - y_2|, \tag{6}$$

in a neighbourhood

$$U\colon\ |x - \xi| < c, \quad |y - \eta| < d, \quad c > 0, d > 0, \tag{7}$$

of the point (ξ, η). The indeterminate case is that where $\lim_{x\to\xi}\gamma(x) = 1$; nevertheless, most of our results are valid also in other cases. The behaviour of continuous solutions of the equations considered will depend essentially on the behaviour of the sequence

$$G_n(x) = \prod_{i=0}^{n-1} \gamma[f^i(x)], \quad n = 1, 2, \ldots, \quad G_0(x) \equiv 1. \tag{8}$$

(Here $f^i(x)$ denotes the i-th iterate of the function $f(x)$: $f^0(x) \equiv x$, $f^{n+1}(x) = f(f^n(x))$, $n = 0, 1, 2, \ldots$) The sequence $G_n(x)$ fulfils the recurrence formula

$$G_{n+1}(x) = \gamma(x) G_n[f(x)], \quad n = 0, 1, 2, \ldots \tag{9}$$

2. First we consider equation (1). The given functions $f(x)$ and $h(x, y)$ will be subjected to certain conditions.

(i) The function $f(x)$ is defined and continuous in an interval I and, for a certain $\xi \in I$, it fulfils the inequalities

$$\big(f(x)-x\big)(\xi - x) > 0 \quad \text{for } x \in I,\ x \neq \xi,$$
$$\big(f(x)-\xi\big)(\xi - x) < 0 \quad \text{for } x \in I,\ x \neq \xi.$$

Let us note that the above conditions imply that $f(x) \in I$ for every $x \in I$, $f(\xi) = \xi$, the sequence $f^n(x)$ is strictly decreasing for $x > \xi$, $x \in I$, and strictly increasing for $x < \xi$, $x \in I$, and $\lim_{n\to\infty} f^n(x) = \xi$ for every $x \in I$ ([5], p. 21, Lemmas 0.6, 0.7 and Theorem 0.4). Setting $x = \xi$ in (1) we obtain hence for $\eta = \varphi(\xi)$ the condition

$$\eta = h(\xi, \eta). \tag{10}$$

This justifies the next assumption.

(ii) The function $h(x, y)$ is defined and continuous in an open region Ω containing the point (ξ, η), where η is a solution of (10). Moreover, h fulfils the Lipschitz condition (5) in $U \cap \Omega$, where U is given by (7).

For every fixed x, let Ω_x denote the x-section of Ω (1):

$$\Omega_x = \{y\colon\ (x, y) \in \Omega\}, \tag{11}$$

and let Λ_x be the set of the values (the range) of the function $h(x, y)$ for $y \in \Omega_x$. Our next assumption reads as follows.

(1) Let us note that in the present section Ω and Ω_x correspond to what has been denoted in [5], p. 68, by Ω^* and Ω_x^*, respectively. Since we do not consider equations (1) and (2) simultaneously here, the relation between Ω_x and Ω_x^* occurring in [5] is irrelevant in the present case and we may simply use the same letter to denote the domain of definition of h and of g. All theorems in [5] concerning only equation (1.2) (i.e., equation (1) according to the notation of the present paper) are valid whenever in Hypothesis 3.1 in [5] the set Ω_x is replaced by Ω_x^*.

(iii) For every $x \in I$ the set Ω_x is an interval and $\Lambda_{f(x)} \subset \Omega_x$.

We shall be interested in solutions $\varphi(x)$ of equation (1) in I with the following properties:

(a) φ is defined and continuous in I.

(b) $\varphi(\xi) = \eta$.

(c) For every $x \in I$ we have $\varphi[f(x)] \in \Omega_x$.

The class of functions with properties (a), (b), (c) (not necessarily satisfying equation (1)) will be denoted by Φ.

THEOREM 1. *Suppose that hypotheses* (i), (ii) *and* (iii) *are fulfilled and that sequence* (8) *is bounded in a neighbourhood of* ξ:

$$G_n(x) \leqslant M, \qquad n = 0, 1, 2, \ldots; \; x \in I \cap (\xi - \delta, \xi + \delta). \tag{12}$$

Then equation (1) *may have at most one solution* $\varphi \in \Phi$ *in* I.

Proof. Suppose that equation (1) has solutions $\varphi_1 \in \Phi$ and $\varphi_2 \in \Phi$ in I. We choose $\varepsilon > 0$ so small that $\varepsilon < \min(c, \delta)$ and for $|x - \xi| < \varepsilon$, $x \in I$, we have $|\varphi_1[f(x)] - \eta| < d$ and $|\varphi_2[f(x)] - \eta| < d$. Since $|x - \xi| < \varepsilon$, $x \in I$, implies $|f^n(x) - \xi| < \varepsilon < c$ for every n, we have

$$|\varphi_1[f^n(x)] - \eta| < d \quad \text{and} \quad |\varphi_2[f^n(x)] - \eta| < d$$
$$\text{for } |x - \xi| < \varepsilon, x \in I \text{ and } n = 1, 2, \ldots$$

In virtue of (5) and (8) we derive hence by induction that

$$|\varphi_1(x) - \varphi_2(x)| \leqslant G_n(x) \, |\varphi_1[f^n(x)] - \varphi_2[f^n(x)]|$$
$$\text{for } |x - \xi| < \varepsilon, x \in I \text{ and } n = 0, 1, 2, \ldots$$

Since $\lim_{n \to \infty} \varphi_1[f^n(x)] = \lim_{n \to \infty} \varphi_2[f^n(x)] = \eta$, we get by (12),

$$\varphi_1(x) = \varphi_2(x) \quad \text{for } |x - \xi| < \varepsilon, \; x \in I.$$

Hence it follows that $\varphi_1(x) \equiv \varphi_2(x)$ in I ([5], p. 70, Theorem 3.2), which was to be proved.

Let us take an arbitrary function $\varphi_0 \in \Phi$ and let us define the sequence $\varphi_n(x)$ by the relation

$$\varphi_{n+1}(x) = h\big(x, \varphi_n[f(x)]\big), \qquad n = 0, 1, 2, \ldots \tag{13}$$

It follows from (i)-(iii) that $\varphi_n \in \Phi$ for every n. Moreover, we have the following result (cf. [5], p. 72, Theorem 3.3):

LEMMA. *If the sequence* $\varphi_n(x)$ *converges in a neighbourhood* I_0 *of* ξ *to a function* φ *fulfilling conditions* (a), (b), (c) *with* I *replaced by* I_0, *then it converges in the whole of* I *and its limit provides a solution of equation* (1) *in the class* Φ.

Now, we have the following

THEOREM 2. *Suppose that hypotheses* (i), (ii) *and* (iii) *are fulfilled and condition* (12) *holds. If* $\varphi \in \Phi$ *is a solution of equation* (1) *in* I, *then*

$$\varphi(x) = \lim_{n\to\infty} \varphi_n(x) \tag{14}$$

in I, *where the sequence* φ_n *is defined by* (13) *and* φ_0 *is an arbitrary function belonging to* Φ.

Proof. We choose $\varepsilon > 0$ so small that $\varepsilon < \min(c, \delta)$ and for $|x-\xi| < \varepsilon$, $x \in I$, we have

$$|\varphi(x)-\eta| < d/(M+1) \quad \text{and} \quad |\varphi(x)-\varphi_0(x)| < d/(M+1). \tag{15}$$

We shall prove that for $|x-\xi| < \varepsilon$, $x\in I$, and for $n = 0, 1, 2, \ldots$, we have

$$|\varphi_n(x)-\eta| < d \tag{16}$$

and

$$|\varphi(x)-\varphi_n(x)| \leqslant G_n(x)|\varphi[f^n(x)]-\varphi_0[f^n(x)]|. \tag{17}$$

For $n = 0$ (17) is obvious and (16) results from (15) in view of the fact that $M \geqslant G_0(x) \equiv 1$ (cf. (12) and (8)). Suppose that they hold for a certain $n \geqslant 0$. We have by (1), (13), (16) for n, (15), (5), (17) for n, and (9), for $|x-\xi| < \varepsilon$, $x \in I$,

$$\begin{aligned}|\varphi(x)-\varphi_{n+1}(x)| &= \big|h\big(x, \varphi[f(x)]\big)-h\big(x, \varphi_n[f(x)]\big)\big| \\ &\leqslant \gamma(x)\,|\varphi[f(x)]-\varphi_n[f(x)]| \\ &\leqslant \gamma(x)G_n[f(x)]|\varphi[f^{n+1}(x)]-\varphi_0[f^{n+1}(x)]| \\ &= G_{n+1}(x)|\varphi[f^{n+1}(x)]-\varphi_0[f^{n+1}(x)]|,\end{aligned}$$

i.e. (17) for $n+1$. Hence we get further by (15) and (12), for $|x-\xi| < \varepsilon$, $x\in I$,

$$\begin{aligned}|\varphi_{n+1}(x)-\eta| &\leqslant |\varphi(x)-\eta|+|\varphi_{n+1}(x)-\varphi(x)| \\ &\leqslant |\varphi(x)-\eta|+G_{n+1}(x)|\varphi[f^{n+1}(x)]-\varphi_0[f^{n+1}(x)]| \\ &< \frac{d}{M+1}+\frac{Md}{M+1} = d,\end{aligned}$$

i.e. (16) for $n+1$. Thus (16) and (17) are generally valid.

The convergence $\varphi_n(x) \to \varphi(x)$ results for $|x-\xi| < \varepsilon$, $x\in I$, from (17), and then for all $x \in I$ in virtue of the Lemma and Theorem 1.

Remark. In [2] results analogous to those contained in the above theorems have been obtained, essentially under assumptions (i), (iii) and the condition that $h(x, y)$ is continuous in Ω and has a continuous derivative $\partial h/\partial y$ such that $|\partial h/\partial y| \leqslant 1$ in a neighbourhood of (ξ, η).

This implies, of course, (ii) and (12). On the other hand, our present assumptions are weaker, as may be seen from the following example:

EXAMPLE. Let $I = (-\infty, +\infty)$, $\Omega = (-\infty, +\infty)\times(-\infty, +\infty)$, and consider the equation

$$\varphi(x) = (1+x)\arctan\varphi(\tfrac{1}{2}x)+x. \tag{18}$$

Here $f(x) = \frac{1}{2}x$, $h(x, y) = (1+x)\arctan y+x$, $\Omega_x = (-\infty, +\infty)$,

$$\Lambda_x = \left(-\frac{\pi}{2}\,|x+1|+x,\ +\frac{\pi}{2}\,|x+1|+x\right).$$

Assumptions (i), (ii) and (iii) are fulfilled, and we have $\xi = \eta = 0$ and $\gamma(x) = (1+x)$. The sequence

$$G_n(x) = \prod_{i=0}^{n-1}\left(1+\frac{x}{2^i}\right)$$

converges almost uniformly in $(-\infty, +\infty)$ and thus is bounded in every bounded neighbourhood of zero. Consequently Theorems 1 and 2 apply to equation (18).

On the other hand, $\partial h/\partial y = (1+x)/(1+y^2)$ and thus $\partial h/\partial y > 1$ inside the parabola $x = y^2$.

Now we shall find some conditions for the existence of solutions $\varphi\in\Phi$ of equation (1). We put

$$H(x) = |h(x, \eta)-\eta|.$$

THEOREM 3. *Suppose that hypotheses* (i), (ii) *and* (iii) *are fulfilled. If $\gamma(x)$ is bounded in a neighbourhood of ξ and, for a certain $\delta > 0$, the series*

$$\sum_{n=0}^{\infty} G_n(x)H[f^n(x)] \tag{19}$$

converges uniformly in $I\cap(\xi-\delta, \xi+\delta)$, then equation (1) *has a solution $\varphi\in\Phi$ in I.*

Proof. Let us take a $\varphi_0\in\Phi$ such that $\varphi_0(x) \equiv \eta$ in a neighbourhood of ξ, and define the sequence φ_n by (13). The sum of series (19) is continuous at ξ and vanishes for $x = \xi$. Consequently we can find an $\varepsilon > 0$ so small that $\varepsilon < \min(c, \delta)$ and for $|x-\xi| < \varepsilon$, $x\in I$, we have $\varphi_0(x) = \eta$ and

$$\sum_{n=0}^{\infty} G_n(x)H[f^n(x)] < d.$$

We shall prove that for $|x-\xi| < \varepsilon$, $x\in I$, we have

$$|\varphi_{n+1}(x)-\varphi_n(x)| \leqslant G_n(x)H[f^n(x)], \quad n = 0, 1, 2, \ldots \tag{20}$$

For $n = 0$ (20) is obvious. Suppose that (20) holds for $n \leqslant N$, $N \geqslant 0$. Then we have for $n \leqslant N$ and for $|x-\xi| < \varepsilon$, $x \epsilon I$,

$$|\varphi_{n+1}(x)-\eta| = |\varphi_{n+1}(x)-\varphi_0(x)| \leqslant \sum_{i=0}^{n} |\varphi_{i+1}(x)-\varphi_i(x)| \leqslant \sum_{i=0}^{n} G_i(x) H[f^i(x)] \leqslant \sum_{i=0}^{\infty} G_i(x) H[f^i(x)] < d.$$

Hence by (13), (5), (20) for N, and (9) we obtain

$$|\varphi_{N+2}(x)-\varphi_{N+1}(x)| = \big|h\big(x, \varphi_{N+1}[f(x)]\big)-h\big(x, \varphi_N[f(x)]\big)\big| \leqslant \gamma(x)|\varphi_{N+1}[f(x)]-\varphi_N[f(x)]| \leqslant \gamma(x) G_N[f(x)] H[f^{N+1}(x)] = G_{N+1}(x) H[f^{N+1}(x)],$$

i.e. (20) for $N+1$.

Relation (20) and the uniform convergence of series (19) imply that the sequence $\varphi_n(x)$ uniformly converges in $I_0 = I \cap (\xi-\varepsilon, \xi+\varepsilon)$ to a function $\varphi(x)$. This function $\varphi(x)$ is continuous in I_0 (since all $\varphi_n(x)$ are continuous in I), $\varphi(\xi) = \eta$ (since $\varphi_n(\xi) = \eta$ for all n), and $|\mathring{\varphi}(x)| < d$ in I_0. This last condition implies that $\varphi[f(x)] \epsilon \Omega_x$ for $x \epsilon I_0$ provided d and ε have been chosen sufficiently small. Our theorem follows in virtue of the Lemma.

As an immediate consequence of Theorems 1, 2 and 3 we obtain the following

THEOREM 4. *Suppose that hypotheses* (i), (ii) *and* (iii) *are fulfilled, condition* (12) *holds and, for a certain* $\delta' > 0$, *the series* $\sum_{n=0}^{\infty} H[f^n(x)]$ *converges uniformly in* $I \cap (\xi-\delta', \xi+\delta')$. *Then equation* (1) *has in* I *a unique solution* $\varphi \epsilon \Phi$. *This solution is given by formula* (14), *where the sequence* $\varphi_n(x)$ *is defined by* (13), *and* $\varphi_0(x)$ *is an arbitrary function belonging to the class* Φ.

Let us note also the following

THEOREM 5. *Suppose that hypotheses* (i), (ii) *and* (iii) *are fulfilled and that there exist positive constants* $A, B, \varkappa, \mu, \delta$ *and* ϑ, $0 < \vartheta < 1$, *such that the inequalities*

$$|\gamma(x)-1| \leqslant A|x-\xi|^{\varkappa}, \quad |h(x, \eta)-\eta| \leqslant B|x-\xi|^{\mu}, \quad |f(x)-\xi| \leqslant \vartheta|x-\xi|$$

hold for $|x-\xi| < \delta$, $x \epsilon I$. *Then equation* (1) *has in* I *a unique solution* $\varphi \epsilon \Phi$. *This solution is given by formula* (14), *where the sequence* $\varphi_n(x)$ *is defined by* (13), *and* $\varphi_0(x)$ *is an arbitrary function belonging to the class* Φ.

Proof. It is enough to verify that the conditions of Theorem 4 are fulfilled. We have for $|x-\xi| < \delta$, $x \in I$,

$$G_n(x) \leqslant \tilde{G}_n(x) = \prod_{i=0}^{n-1} \tilde{\gamma}[f^i(x)],$$

where $\tilde{\gamma}(x) = 1+A|x-\xi|^{\varkappa}$. The proof given in [1], p. 166, or in [5], p. 52, serves to show that the sequence $\tilde{G}_n(x)$ converges almost uniformly in I to a continuous limit; consequently (12) holds. On the other hand, we have for $x \in I \cap (\xi-\delta, \xi+\delta)$

$$H[f^n(x)] \leqslant B|f^n(x)-\xi|^{\mu};$$

but $|f^n(x)-\xi| \leqslant \vartheta^n|x-\xi|$ (induction), whence

$$H[f^n(x)] \leqslant B\vartheta^{\mu n}|x-\xi|^{\mu},$$

which proves that the series $\sum_{n=0}^{\infty} H[f^n(x)]$ converges uniformly in $I \cap (\xi-\delta, \xi+\delta)$. This completes the proof.

In particular, it follows from Theorem 5 that equation (18) has a unique continuous solution in $(-\infty, +\infty)$.

3. Now we turn to equation (2). In this case it will be necessary to make somewhat stronger assumptions. Regarding $f(x)$ we shall assume that it fulfils (i) and, moreover,

(iv) The function $f(x)$ is strictly increasing in I.

The function $g(x, y)$ will be subjected to the following conditions:

(v) The function $g(x, y)$ is defined and continuous in an open region Ω containing the point (ξ, η), where η is a solution of $\eta = g(\xi, \eta)$. For every fixed $x \in I$ the function $g(x, y)$ as a function of y is invertible. Moreover, g fulfils the Lipschitz condition (6) in $U \cap \Omega$, where U is given by (7) and $\gamma(x)$ has a positive lower bound in I.

Ω_x being defined by (11), we denote by Γ_x the set of the values (the range) of the function $g(x, y)$ for $y \in \Omega_x$.

(vi) For every $x \in I$ the set Ω_x is an interval and $\Gamma_x = \Omega_{f(x)}$.

We now replace the class Φ by the class Ψ of functions $\varphi(x)$ fulfilling conditions (a), (b) and

(c′) For every $x \in I$ we have $\varphi(x) \in \Omega_x$.

THEOREM 6. *Suppose that hypotheses* (i), (iv), (v) *and* (vi) *are fulfilled and that there exists an interval* $J \subset I$ *such that* $\lim_{n\to\infty} G_n(x) = 0$ *uniformly in* J. *Then equation* (2) *has in* I *either no solution* $\varphi \in \Psi$, *or a solution* $\varphi \in \Psi$ *depending on an arbitrary function.*

Proof. Let us suppose that equation (2) has a solution $\varphi_0 \in \Psi$ in I. We shall show that the general solution $\varphi \in \Psi$ of equation (2) in I depends on an arbitrary function. For the sake of simplicity we shall assume that ξ is the left end-point of the interval I. In other cases the proof runs similarly.

It follows from (9) that, for every k, the sequence $G_n(x)$ converges to zero uniformly in $f^k(J)$. Therefore we may choose an $x_0 \in I$ and an interval $\langle a, b\rangle \subset \langle f(x_0), x_0\rangle$ so that $x_0 - \xi < c$, $|\varphi_0(x) - \eta| < \frac{1}{2}d$ for $x \in \langle \xi, x_0\rangle$, and $\lim\limits_{n\to\infty} G_n(x) = 0$ uniformly in $\langle a, b\rangle$. Hence it follows by a simple argument that there exists a positive constant M such that

$$G_n(x) \leqslant M \quad \text{for } x \in \langle a, b\rangle \text{ and } n = 0, 1, 2, \ldots \tag{21}$$

Let $\psi(x)$ be an arbitrary function defined and continuous in $\langle f(x_0), x_0\rangle$, and fulfilling the following conditions:

$$\psi[f(x_0)] = g\big(x_0, \psi(x_0)\big), \tag{22}$$

$$|\psi(x) - \varphi_0(x)| < d/2M \quad \text{for } x \in \langle a, b\rangle, \tag{23}$$

$$\psi(x) = \varphi_0(x) \quad \text{for } x \in \langle f(x_0), x_0\rangle \setminus \langle a, b\rangle, \tag{24}$$

$$\psi(x) \in \Omega_x \quad \text{for } x \in \langle f(x_0), x_0\rangle. \tag{25}$$

It follows from (22) and (25) that there exists a unique function $\varphi(x)$ defined in $I \setminus \{\xi\}$, satisfying equation (2) in $I \setminus \{\xi\}$ and such that $\varphi(x) \in \Omega_x$ for $x \in I \setminus \{\xi\}$ and

$$\varphi(x) = \psi(x) \quad \text{in } \langle f(x_0), x_0\rangle. \tag{26}$$

This function is continuous in $I \setminus \{\xi\}$. ([5], p. 70, Theorem 3.1). Putting $\varphi(\xi) = \eta$, we extend φ to a solution of equation (2) in I and in order to prove that $\varphi \in \Psi$ it is enough to show that φ is continuous at ξ, i.e. that

$$\lim_{x\to\xi} \varphi(x) = \eta. \tag{27}$$

We shall show that

$$|\varphi[f^n(x)] - \varphi_0[f^n(x)]| \leqslant G_n(x)\,|\psi(x) - \varphi_0(x)| \quad \text{for } x \in \langle f(x_0), x_0\rangle,\ n = 0, 1, 2, \ldots \tag{28}$$

For $n = 0$ (28) results from (26). Supposing (28) true for an $n \geqslant 0$, we have by (21), (23) and (24)

$$\begin{aligned} |\varphi[f^n(x)] - \eta| &\leqslant |\varphi[f^n(x)] - \varphi_0[f^n(x)]| + |\varphi_0[f^n(x)] - \eta| \\ &\leqslant G_n(x)\,|\psi(x) - \varphi_0(x)| + |\varphi_0[f^n(x)] - \eta| \\ &< M\frac{d}{2M} + \frac{d}{2} = d. \end{aligned}$$

Consequently

$$\begin{aligned}|\varphi[f^{n+1}(x)]-\varphi_0[f^{n+1}(x)]| &= \big|g\big(f^n(x), \varphi[f^n(x)]\big)-g\big(f^n(x), \varphi_0[f^n(x)]\big)\big| \\ &\leqslant \gamma[f^n(x)]|\varphi[f^n(x)]-\varphi_0[f^n(x)]| \\ &\leqslant \gamma[f^n(x)]G_n(x)|\psi(x)-\varphi_0(x)| \\ &= G_{n+1}(x)|\psi(x)-\varphi_0(x)|,\end{aligned}$$

which proves (28).

Given an $\varepsilon > 0$, we can find an N such that

$$G_n(x) < \frac{2M}{d}\varepsilon \quad \text{for } x \in \langle a, b\rangle \text{ and } n \geqslant N. \tag{29}$$

For every $x \in \big(\xi, f^N(x_0)\big)$ we can find an $x^* \in \langle f(x_0), x_0\rangle$ and an $n \geqslant N$ such that $x = f^n(x^*)$. Hence in view of (28)

$$|\varphi(x)-\varphi_0(x)| = |\varphi[f^n(x^*)]-\varphi_0[f^n(x^*)]| \leqslant G_n(x^*)|\psi(x^*)-\varphi_0(x^*)|.$$

If $x^* \in \langle a, b\rangle$, we get hence by (29) and (23)

$$|\varphi(x)-\varphi_0(x)| < \varepsilon. \tag{30}$$

If $x^* \langle f(x_0), x_0\rangle \setminus \langle a, b\rangle$, then (30) is also valid, since, according to (24), $|\psi(x^*)-\varphi_0(x^*)| = 0$. (30) implies that $\lim_{x\to\xi}\big(\varphi(x)-\varphi_0(x)\big) = 0$, whence (27) results in view of the fact that $\lim_{x\to\xi}\varphi_0(x) = \eta$.

As we see, the solution $\varphi \in \Psi$ of equation (2) may be prescribed to a great extent arbitrarily on an interval $\langle a, b\rangle$, i.e., it depends on an arbitrary function (cf. also [1], and [5], p. 45). This completes the proof.

Let us note that both cases (lack of a solution and a solution depending on an arbitrary function) can actually occur even for the linear equation (4); cf. [1], Examples 3 and 4.

Now, let us write

$$F(x) = |g(x, \eta)-\eta|,$$

and

$$H_n(x) = \sum_{i=0}^{n-2}\Big\{\prod_{j=i+1}^{n-1}\gamma[f^j(x)]\Big\}F[f^i(x)] = \sum_{i=0}^{n-2}\frac{F[f^i(x)]}{G_{i+1}(x)}G_n(x), \qquad n = 2, 3, \ldots$$

The sequence $H_n(x)$ fulfils the following recurrences:

$$H_{n+1}(x) = \gamma[f^n(x)]H_n(x)+\gamma[f^n(x)]F[f^{n-1}(x)] \tag{31}$$

and

$$H_n[f(x)] = H_{n+1}(x)-\frac{F(x)}{\gamma(x)}G_{n+1}(x). \tag{32}$$

Relation (32) implies that $H_n[f(x)] \leqslant H_{n+1}(x)$, whence

$$\sup_{\langle f^{k+1}(x_0), f^k(x_0)\rangle} H_n(x) \leqslant \sup_{\langle f(x_0), x_0\rangle} H_{n+k}(x), \qquad n \geqslant 2, k \geqslant 0. \tag{33}$$

In the next theorem we assume again that ξ is the left end-point of the interval I. If ξ is the right end-point, then the interval $\langle f(x_0), x_0\rangle$ should be replaced by $\langle x_0, f(x_0)\rangle$, and if ξ is an inner point of I, then the interval $\langle f(x_0), x_0\rangle$ should be replaced by $\langle x', f(x')\rangle \cup \langle f(x''), x''\rangle$ with $x' < \xi < x''$, $x', x'' \in I$.

THEOREM 7. *Suppose that hypotheses* (i), (iv), (v) *and* (vi) *are fulfilled and let ξ be the left end-point of the interval I. Further suppose that there exists an $x_0 \in I$, $x_0 \neq \xi$, such that $\lim_{n\to\infty} G_n(x) = \lim_{n\to\infty} H_n(x) = 0$ uniformly in $\langle f(x_0), x_0\rangle$. Then equation* (2) *has in I a solution $\varphi \in \Psi$ depending on an arbitrary function.*

Proof. In view of (9) and (33) we may replace the interval $\langle f(x_0), x_0\rangle$ by $\langle f^{k+1}(x_0), f^k(x_0)\rangle$ with k arbitrarily large. Consequently, we may assume that x_0 is so close to ξ that

$$H_n(x) < \tfrac{1}{3}d \quad \text{for } x \in \langle f(x_0), x_0\rangle, n = 2, 3, \ldots, \tag{34}$$

(cf. (33)) and

$$F(x) < \tfrac{1}{3}d \quad \text{for } x \in (\xi, x_0\rangle. \tag{35}$$

Further, it is easy to see that there exists a positive constant M such that

$$G_n(x) < M \quad \text{for } x \in \langle f(x_0), x_0\rangle, \; n = 0, 1, 2, \ldots \tag{36}$$

Let $\psi(x)$ be an arbitrary function defined and continuous in $\langle f(x_0), x_0\rangle$, fulfilling conditions (22) and (25) and such that

$$|\psi(x) - \eta| < d/3M \quad \text{for } x \in \langle f(x_0), x_0\rangle. \tag{37}$$

Then there exists a unique function $\varphi(x)$ defined and continuous in $I \setminus \{\xi\}$, satisfying equation (2) in $I \setminus \{\xi\}$, fulfilling condition (26) and such that $\varphi(x) \in \Omega_x$ for $x \in I \setminus \{\xi\}$ ([5], p. 70, Theorem 3.1). Putting $\varphi(\xi) = \eta$ we extend φ to a solution of equation (2) in I and it is enough to show that condition (27) is fulfilled. For this purpose we shall prove the estimation

$$|\varphi[f^n(x)] - \eta| \leqslant H_n(x) + F[f^{n-1}(x)] + G_n(x)|\psi(x) - \eta| \tag{38}$$

valid for $x \in \langle f(x_0), x_0\rangle$ and $n = 2, 3, \ldots$ In fact, we have

$$\begin{aligned} |\varphi[f(x)] - \eta| &= |g(x, \varphi(x)) - \eta| \\ &\leqslant |g(x, \varphi(x)) - g(x, \eta)| + |g(x, \eta) - \eta|, \end{aligned}$$

whence in view of (6), (26), (37), (36) and (35) we get for $x \in \langle f(x_0), x_0 \rangle$ (note that, by (8), $\gamma(x) = G_1(x)$)

$$|\varphi[f(x)]-\eta| \leqslant \gamma(x)|\psi(x)-\eta|+F(x) < \tfrac{2}{3}d < d,$$

and similarly

$$\begin{aligned}|\varphi[f^2(x)]-\eta| &\leqslant \gamma[f(x)]|\varphi[f(x)]-\eta|+F[f(x)]\\ &\leqslant \gamma[f(x)]\gamma(x)|\psi(x)-\eta|+\gamma[f(x)]F(x)+F[f(x)]\\ &= G_2(x)|\psi(x)-\eta|+H_2(x)+F[f(x)],\end{aligned}$$

which proves (38) for $n = 2$. Assuming (38) true for an $n \geqslant 2$, we have by (34), (35), (36) and (37), $|\varphi[f^n(x)]-\eta| < d$, whence it follows by (6) and by (38) for n

$$\begin{aligned}|\varphi[f^{n+1}(x)]-\eta| &\leqslant \big|g\big(f^n(x), \varphi[f^n(x)]\big)-g\big(f^n(x), \eta\big)\big|+\big|g\big(f^n(x), \eta\big)-\eta\big|\\ &\leqslant \gamma[f^n(x)]|\varphi[f^n(x)]-\eta|+F[f^n(x)]\\ &\leqslant \gamma[f^n(x)]H_n(x)+\gamma[f^n(x)]F[f^{n-1}(x)]+\\ &\quad+\gamma[f^n(x)]G_n(x)|\psi(x)-\eta|+F[f^n(x)],\end{aligned}$$

and by (8) and (31) we obtain finally

$$|\varphi[f^{n+1}(x)]-\eta| \leqslant H_{n+1}(x)+G_{n+1}(x)|\psi(x)-\eta|+F[f^n(x)],$$

i.e. (38) for $n+1$.

Given an $\varepsilon > 0$, we can find an N such that

(39) $$G_n(x) < \frac{M}{d}\varepsilon \quad \text{for } x \in \langle f(x_0), x_0 \rangle \text{ and } n \geqslant N,$$

(40) $$H_n(x) < \tfrac{1}{3}\varepsilon \quad \text{for } x \in \langle f(x_0), x_0 \rangle \text{ and } n \geqslant N,$$

(41) $$F(x) < \tfrac{1}{3}\varepsilon \quad \text{for } x \in \big(\xi, f^{N-1}(x_0)\big\rangle.$$

Condition (41) implies that

(42) $$F[f^{n-1}(x)] < \tfrac{1}{3}\varepsilon \quad \text{for } x \in \langle f(x_0), x_0 \rangle \text{ and } n \geqslant N.$$

For every $x \in \big(\xi, f^N(x_0)\big)$ we can find an $x^* \in \langle f(x_0), x_0 \rangle$ and an $n \geqslant N$ such that $x = f^n(x^*)$. Hence it follows by (38), (39), (40), (42) and (37) that

$$\begin{aligned}|\varphi(x)-\eta| &= |\varphi[f^n(x^*)]-\eta|\\ &\leqslant H_n(x^*)+F[f^{n-1}(x^*)]+G_n(x^*)|\psi(x^*)-\eta| < \varepsilon.\end{aligned}$$

This proves relation (27) and completes the proof.

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References

[1] B. Choczewski and M. Kuczma, *On the "indeterminate case" in the theory of a linear functional equation*, Fund. Math. 58 (1966), p. 163-175.
[2] J. Kordylewski and M. Kuczma, *On the functional equation* $F(x, \varphi(x), \varphi[f(x)]) = 0$, Ann. Polon. Math. 7 (1959), p. 21-32.
[3] M. Kuczma, *On continuous solutions of a functional equation*, ibidem 8 (1960), p. 209-214.
[4] — *On the form of solutions of some functional equations*, ibidem 9 (1960), p. 55-63.
[5] — *Functional equations in a single variable*, Monografie Mat. 46, Warszawa 1968.

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