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Some extremal problems for certain families of analytic functions I

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Abstract. Let Ω be the class of functions $\vartheta(z)$, $\vartheta(0) = 0$, $|\vartheta(z)| < 1$ regular in the disc $K = \{z: |z| < 1\}$, A and B — arbitrary fixed numbers, $A \in (-1, 1]$, $B \in [-1, A)$, $\wp(A, B)$ — the class of functions $P(z)$, $P(0) = 1$, regular in K such that $P(z) \in \wp(A, B)$ if and only if $P(z) = (1+A\vartheta(z))(1+B\cdot\vartheta(z))^{-1}$ for some function $\vartheta(z) \in \Omega$ and every $z \in K$, and $S^*(A, B)$ — the class of functions $f(z)$, $f(0) = 0$, $f'(0) = 1$, regular in K satisfying the condition: $f(z) \in S^*(A, B)$ if and only if $zf'(z)(f(z))^{-1} = P(z)$ for some $P(z) \in \wp(A, B)$ and all z in K.

In the present paper the author determines the bounds for $\operatorname{re}\big(P(z)+zP'(z)(P(z))^{-1}\big)$ and $\operatorname{re}\big(zP'(z)(P(z))^{-1}\big)$ on $|z| = r < 1$ within $\wp(A, B)$, the bounds of $|f(z)|$ and $|f'(z)|$ in $S^*(A, B)$ and the exact value of the radius of convexity for $S^*(A, B)$.

1. Introduction. Let Ω be the family of functions $\vartheta(z)$ regular in the disc $K = \{z: |z| < 1\}$ and satisfying the conditions $\vartheta(0) = 0$, $|\vartheta(z)| < 1$ for $z \in K$.

Next, for arbitrary fixed numbers A, B, $-1 < A \leqslant 1$, $-1 \leqslant B < A$, denote by $\wp(A, B)$ the family of functions

$$P(z) = 1+b_1z+\ldots \tag{1.1}$$

regular in K and such that $P(z)$ is in $\wp(A, B)$ if and only if

$$P(z) = \frac{1+A\vartheta(z)}{1+B\vartheta(z)}$$

for some function $\vartheta(z) \in \Omega$ and every $z \in K$.

Moreover, let $S^*(A, B)$ denote the family of functions

$$f(z) = z+a_2z^2+\ldots \tag{1.2}$$

regular in K and such that $f(z)$ is in $S^*(A, B)$ if and only if

$$\frac{zf'(z)}{f(z)} = P(z) \tag{1.3}$$

for some $P(z)$ in $\wp(A, B)$ and all z in K.

Finally, we consider the following classes of functions defined in K (the first five of them consisting of functions of form (1.1) the remaining ones — of form (1.2)): $\wp$ — the class of Carathéodory functions, i. e. of functions $P(z)$ for which $\operatorname{re} P(z) > 0$ in K; $\wp_\alpha$ — the class of Carathéodory functions of order α, $0 \leqslant \alpha < 1$, i. e. such that $\operatorname{re} P(z) > \alpha$ for $z \in K$; $\wp(M)$, $M > \frac{1}{2}$, $\wp^{(\beta)}$ and $\wp_{(\beta)}$, $0 < \beta \leqslant 1$ — the classes of functions satisfying the conditions

$$|P(z) - M| < M \ [1], \qquad \left|\frac{P(z)-1}{P(z)+1}\right| < \beta, \qquad |P(z)-1| < \beta$$

for $z \in K$, respectively and S^* — the class of functions starlike w. r. t. the origin; S^*_α — the class of functions starlike of order α [7]; $S^*(M)$, $S^{*(\beta)}$ and $S^*_{(\beta)}$ — the classes of functions satisfy (1.3), where $P(z)$ belong to $\wp(M)$, $\wp^{(\beta)}$ and $\wp_{(\beta)}$ for $z \in K$, respectively.

The classes $S^*(M)$, $S^{*(\beta)}$ and $S^*_{(\beta)}$ have been introduced in [1], [6] and [4].

It is easy to prove that

$$\wp(A, B) \subseteq \wp_{\frac{1-A}{1-B}}, \qquad \wp(A, B) \subseteq \wp\left(\frac{1}{1+B}\right),$$

$$\wp(A, -1) \equiv \wp_{\frac{1-A}{2}}, \qquad \wp(1, B) \equiv \wp\left(\frac{1}{1+B}\right), \qquad \wp(1, 1) \equiv \wp,$$

and

$$\wp(A, -A) \equiv \wp^{(A)}, \qquad \wp(A, 0) \equiv \wp_{(A)}.$$

Analogous relations hold for the corresponding classes of starlike functions.

In this paper we give the greatest lower bound and the smallest upper bound for $\operatorname{re}\left[P(z) + \dfrac{zP'(z)}{P(z)}\right]$ and $\operatorname{re}\dfrac{zP'(z)}{P(z)}$ on $|z| = r < 1$ within $\wp(A, B)$, the bounds of $|f(z)|$ and $|f'(z)|$ in $S^*(A, B)$ and the exact value of the radius of convexity for $S^*(A, B)$ for every admissible A and B. As corollaries we obtain certain results given by the present author [1], Libera [2], Mac Gregor [3], [4], Nevanlinna [5], Padmanabhan [6], Robertson [7], [8] and Zmorovič [9].

2. Auxiliary lemmas. From the definitions of the classes $\wp$ and $\wp(A, B)$ we easily obtain the following

LEMMA 1. *If $P(z) \in \wp(A, B)$, then*

$$P(z) = \frac{(1+A)p(z)+1-A}{(1+B)p(z)+1-B} \tag{2.1}$$

for some $p(z) \in \wp$ and conversely.

Let ζ be an arbitrary fixed point of K. We consider the functional

$$F(P) = P(\zeta), \quad P(z) \in \wp(A, B). \tag{2.2}$$

LEMMA 2. *The set of values of the functional* (2.2) *is the closed disc with centre* c *and radius* ϱ, *where*

$$c = c(r) = \frac{1-ABr^2}{1-B^2r^2}, \quad \varrho = \varrho(r) = \frac{(A-B)r}{1-B^2r^2}, \quad r = |\zeta|. \tag{2.3}$$

Proof. Every boundary function $P_0(z)$ of $\wp(A,B)$ w.r.t. the functional (2.2) is of form (2.1), where

$$p(z) = \frac{1+\varepsilon z}{1-\varepsilon z}, \quad |\varepsilon| = 1$$

[8]. Hence

$$P_0(z) = \frac{1+A\varepsilon z}{1+B\varepsilon z}. \tag{2.4}$$

Since for $z = re^{i\varphi}$, $0 \leqslant \varphi \leqslant 2\pi$,

$$P_0(z) = c+\varrho\eta_0, \tag{2.5}$$

where

$$\eta_0 = \varepsilon e^{i\varphi}\frac{1+Br\bar{\varepsilon}e^{-i\varphi}}{1+Br\varepsilon e^{i\varphi}}, \tag{2.6}$$

the lemma has been proved.

Denote by $\wp_2(A, B)$ the subclass of $\wp(A, B)$ containing all functions of form (2.1), where

$$p(z) = \frac{1+\lambda}{2}p_1(z)+\frac{1-\lambda}{2}p_2(z), \tag{2.7}$$

$$p_k(z) = \frac{1+\varepsilon_k z}{1-\varepsilon_k z} \quad \text{for } k = 1, 2 \tag{2.8}$$

and

$$|\varepsilon_k| = 1, \quad -1 \leqslant \lambda \leqslant 1. \tag{2.9}$$

Next let $F(u, v)$ be an analytical function in the v-plane and in the half-plane $\operatorname{re} u > 0$, such that

$$|F'_u|^2+|F'_v|^2 > 0$$

at every point (u, v).

Since every boundary function of $\wp$ w.r.t. the functional $F\big(p(z), zp'(z)\big)$, $|z| = r$, is of form (2.7) [8], every boundary function of $\wp(A, B)$ w.r.t. the functional $F\big(P(z), zP'(z)\big)$, $|z| = r$, belongs to $\wp_2(A, B)$. Thus, the extremal problem for $\operatorname{re} F\big(P(z), zP'(z)\big)$, $|z| = r$, in $\wp(A, B)$ can be replaced by an analogous problem for this functional in the class $\wp_2(A, B)$.

LEMMA 3. *If $P(z) \in \wp_2(A, B)$, then for $z = re^{i\varphi}$, $0 \leqslant r < 1$, $0 \leqslant \varphi \leqslant 2\pi$, we have*

$$P(z) = c + \varkappa\psi, \tag{2.10}$$

where

$$\varkappa = \varrho |(1+\lambda)h_1\eta_1 + (1-\lambda)h_2\eta_2|, \qquad \psi = \frac{(1+\lambda)h_1\eta_1 + (1-\lambda)h_2\eta_2}{|(1+\lambda)h_1\eta_1 + (1-\lambda)h_2\eta_2|},$$

$$h_k = \frac{g + (1+B)\varrho\eta_{3-k}}{v}, \qquad v = 2g + (1+B)[(1+\lambda)\eta_2 + (1-\lambda)\eta_1]\varrho, \tag{2.11}$$

$$g = (1+B)c - 1 - A, \qquad \eta_k = \varepsilon_k e^{i\varphi}\frac{1 + B\bar{\varepsilon}_k re^{-i\varphi}}{1 + B\varepsilon_k re^{i\varphi}} \quad \text{for } k = 1, 2,$$

c and ϱ are given by (2.3) *and* $0 \leqslant \varkappa \leqslant \varrho$.

Proof. Assume

$$x\big(p(z); \mu\big) = (1+\mu)p(z) + 1 - \mu$$

for every function $p(z)$ of $\wp$ and every number μ. Next let $P(z) \in \wp_2(A, B)$. Then

$$P(z) = \frac{(1+\lambda)x\big(p_1(z); A\big) + (1-\lambda)x\big(p_2(z); A\big)}{(1+\lambda)x\big(p_1(z); B\big) + (1-\lambda)x\big(p_2(z); B\big)} \tag{2.12}$$

for some functions $p_k(z)$ of form (2.8).

Let

$$\begin{aligned} &P_k(z) = \frac{x\big(p_k(z); A\big)}{x\big(p_k(z); B\big)}, \qquad Q_k(z) = (1+B)P_k(z) - 1 - A, \\ &V(z) = (1+\lambda)Q_2(z) + (1-\lambda)Q_1(z), \qquad H_k(z) = \frac{Q_{3-k}(z)}{V(z)}, \quad k = 1, 2. \end{aligned} \tag{2.13}$$

Since

$$x\big(p_k(z); B\big) \equiv 2(B-A)Q_k^{-1}(z) \quad \text{and} \quad (1+\lambda)H_1(z) + (1-\lambda)H_2(z) \equiv 1,$$

we find after some calculation that $P(z)$ can be represented in the form

$$P(z) = (1+\lambda)P_1(z)H_1(z) + (1-\lambda)P_2(z)H_2(z). \tag{2.14}$$

Since

$$P_k(re^{i\varphi}) = c + \varrho\eta_k \tag{2.15}$$

(comp. (2.4)-(2.6)), we have

$$Q_k(re^{i\varphi}) = g + (1+B)\varrho\eta_k, \qquad V(re^{i\varphi}) = v$$

and

$$H_k(re^{i\varphi}) = h_k. \tag{2.16}$$

Therefore

$$P(re^{i\varphi}) = c[(1+\lambda)h_1+(1-\lambda)h_2]+\varrho[(1+\lambda)h_1\eta_1+(1-\lambda)h_2\eta_2].$$

The first term of the last sum is equal to c, and thus $P(re^{i\varphi})$ is of form (2.10), where

$$\varkappa\psi = \varrho[(1+\lambda)h_1\eta_1+(1-\lambda)h_2\eta_2]. \tag{2.17}$$

Equality (2.17) implies

$$\varkappa^2 = \varrho^2|(1+\lambda)h_1\eta_1+(1-\lambda)h_2\eta_2|^2.$$

Assuming $\eta_k = e^{i\gamma_k}$, $k = 1, 2$, we find hence the relationship

$$\varkappa^2 = \varrho^2[(1+\lambda)^2h_1\bar{h}_1+(1-\lambda)^2h_2\bar{h}_2+(1-\lambda^2)(h_1\bar{h}_2\eta_1\bar{\eta}_2+\bar{h}_1h_2\bar{\eta}_1\eta_2)]$$

and because of

$$(1+\lambda)\bar{h}_1 = 1-(1-\lambda)\bar{h}_2, \quad (1-\lambda)\bar{h}_2 = 1-(1+\lambda)\bar{h}_1 \tag{2.18}$$

we get

$$\varkappa^2 = \varrho^2\{1-(1-\lambda^2)[(1-\eta_1\bar{\eta}_2)h_1\bar{h}_2+(1-\bar{\eta}_1\eta_2)\bar{h}_1h_2]\}.$$

Finally we obtain

$$\varkappa^2 = \varrho^2\left[1-4(1-\lambda^2)\frac{g^2-(1+B)^2\varrho^2}{|v|^2}\sin^2\frac{\gamma_1-\gamma_2}{2}\right]. \tag{2.19}$$

Since

$$g^2-(1+B)^2\cdot\varrho^2 = \frac{(A-B)^2(1-r^2)}{1-B^2r^2} > 0, \tag{2.20}$$

we have $\varkappa \leqslant \varrho$, which ends the proof.

LEMMA 4. *If $P(z)\in\wp_2(A, B)$, then on $|z| = r < 1$*

$$zP'(z) = \frac{-BP^2(z)+(A+B)P(z)-A}{A-B}-\frac{1}{2}\frac{\varrho^*}{\varrho}[\varrho^2-|P(z)-c|^2]\eta^*, \tag{2.21}$$

where c, ϱ are given by (2.3),

$$\varrho^* = \frac{2r}{1-r^2} \quad \textit{and} \quad |\eta^*| = 1. \tag{2.21'}$$

Proof. The differentiation in (2.14) yields

$$zP'(z) = U(z)+W(z), \tag{2.22}$$

where

$$U(z) = z[(1+\lambda)P_1(z)H_1'(z)+(1-\lambda)P_2(z)H_2'(z)],$$

and

$$W(z) = z[(1+\lambda)H_1(z)P_1'(z)+(1-\lambda)H_2(z)P_2'(z)].$$

Using (2.14), we obtain after simplification

$$(2.23)\qquad U(z) = \frac{(1-\lambda^2)(1+B)}{(A-B)\,V^2(z)}\,[P_1(z)-P_2(z)]^2 N(z),$$

where

$$(2.24)\quad N(z) = A^2+B+B(1+B)P_1(z)P_2(z)-B(1+A)[P_1(z)+P_2(z)],$$

and

$$(2.25)\qquad W(z) = \frac{-A+(A+B)P(z)-BT(z)}{A-B},$$

where

$$T(z) = (1+\lambda)P_1^2(z)\cdot H_1(z)+(1-\lambda)P_2^2(z)H_2(z).$$

Because of (2.18) we have

$$[(1+\lambda)P_1(z)H_1(z)+(1-\lambda)P_2(z)H_2(z)]^2$$
$$= (1+\lambda)P_1^2(z)H_1(z)+(1-\lambda)P_2^2(z)H_2(z)-(1-\lambda^2)H_1(z)H_2(z)[P_1(z)-\\ -P_2(z)]^2;$$

thus

$$(2.26)\qquad T(z) = P^2(z)+(1-\lambda^2)[P_1(z)-P_2(z)]^2H_1(z)H_2(z).$$

From (2.22)-(2.26) we conclude that (2.21) may be represented in the form

$$zP'(z) = \frac{-BP^2(z)+(A+B)P(z)-A}{A-B}+\frac{(1-\lambda^2)(A-B)}{V^2(z)}[P_1(z)-P_2(z)]^2$$

for every $P(z)\in\wp_2(A,B)$ and $z\in K$.

Let $z = re^{i\varphi}$, $0\leqslant r<1$, $0\leqslant\varphi\leqslant 2\pi$. Then from (2.15) it follows that

$$[P_1(re^{i\varphi})-P_2(re^{i\varphi})]^2 = -4\varrho^2\eta\sin^2\frac{\gamma_1-\gamma_2}{2},$$

where

$$(2.27)\qquad \eta = \eta_1\cdot\eta_2.$$

Hence in view of (2.19) and because of

$$\frac{A-B}{g^2-(1+B)^2\varrho^2} = \frac{1}{2}\,\frac{\varrho^*}{\varrho}$$

we obtain

$$\frac{(1-\lambda^2)(A-B)}{V^2(re^{i\varphi})}[P_1(re^{i\varphi})-P_2(re^{i\varphi})]^2 = -\frac{1}{2}\,\frac{\varrho^*}{\varrho}\,(\varrho^2-\varkappa^2)\eta^*,$$

where

$$\eta^* = \frac{\overline{V}(re^{i\varphi})}{V(re^{i\varphi})}\,\eta, \tag{2.28}$$

which ends the proof.

COROLLARY. *If* $P(z) \in \wp_2(1, -1)$, *then on* $|z| = r < 1$

$$zP'(z) = \tfrac{1}{2}[P^2(z)-1] - \tfrac{1}{2}[\varrho^{*2} - |P(z)-c^*|^2]\eta,$$

where

$$c^* = \frac{1+r^2}{1-r^2},$$

cf. [9].

3. An extremum problem over $\wp(A, B)$.

I. Let $P(z) \in \wp_2(A, B)$. Thus, because of (2.1), (2.7)-(2.9), where $\varepsilon_k = e^{i\vartheta_k}$ ($k = 1, 2$), and in view of Lemma 4, the expression

$$\omega(r) = \min\left\{\operatorname{re}\left[P(z) + \frac{zP'(z)}{P(z)}\right]:\ |z| = r < 1,\, P \in \wp_2(A, B)\right\}$$

may be represented for $z = re^{i\varphi}$, $0 \leqslant r < 1$, $0 \leqslant \varphi \leqslant 2\pi$, as follows:

$$\omega(r) = \min_{\lambda, \vartheta_1, \vartheta_2} L\big(P(re^{i\varphi})\big),$$

where

$$L(w) = \frac{(A-2B)w^2 + (A+B)w - A}{(A-B)w} - \frac{\varrho^*}{2\varrho}\,[\varrho^2 - |w-c|^2]w^{-1}\eta^*, \tag{3.1}$$

$-1 \leqslant \lambda \leqslant 1$, $0 \leqslant \vartheta_k \leqslant 2\pi$ for $k = 1, 2$ and c, ϱ, ϱ^*, η^* are given by (2.3), (2.21′) and (2.28), respectively.

Let

$$P(re^{i\varphi}) = se^{it}, \quad s > 0,\ \operatorname{im} t = 0. \tag{3.2}$$

Since

$$\operatorname{re}\eta^* e^{-it} \leqslant 1, \tag{3.3}$$

we obtain because of (3.1)

$$\omega(r) \geqslant \tau(r), \tag{3.4}$$

where

$$\tau(r) = \min_{s,t} \Phi(s, t) \tag{3.5}$$

and

$$\Phi(s, t) = \Phi(s, t;\, r) = (E_1 s + E_2 + E_3 s^{-1})\cos t + E_4 s + E_5 + E_6 s^{-1} \tag{3.6}$$

with

$$
(3.7)\qquad
\begin{aligned}
E_1 &= \frac{A-2B}{A-B}, & E_2 &= -c\frac{\varrho^*}{\varrho} = -2\,\frac{1-ABr^2}{(A-B)(1-r^2)},\\
E_3 &= -\frac{A}{A-B}, & E_4 &= \frac{\varrho^*}{2\varrho} = \frac{1-B^2r^2}{(A-B)(1-r^2)},\\
E_5 &= \frac{A+B}{A-B}, & E_6 &= \frac{\varrho^*(c^2-\varrho^2)}{2\varrho} = \frac{1-A^2r^2}{(A-B)(1-r^2)}.
\end{aligned}
$$

In view of Lemmas 2 and 3 the function $\Phi(s,t)$ is defined in the region

$$(3.8)\qquad D = \{(s,t)\colon\ c-\varrho < s < c+\varrho,\ -\psi(s) < t < \psi(s)\}$$

and on its boundary ∂D, where

$$(3.9)\qquad \psi(s) = \arccos\frac{s^2+c^2-\varrho^2}{2cs}, \qquad 0 \leqslant \psi(s) \leqslant \psi(s_0)$$

with $s_0 = \sqrt{c^2-\varrho^2}$ ([1]).

If, at some point (s_1, t_1) of the region D, $\Phi(s,t)$ attains its minimum, then s_1 and t_1 are the solutions of the system of equations

$$\frac{\partial\Phi(s,t)}{\partial s} = 0, \qquad \frac{\partial\Phi(s,t)}{\partial t} = 0$$

with the unknowns s and t, i. e. of the system

$$(3.10)\qquad (E_1-E_3s^{-2})\cos t+E_4-E_6s^{-2} = 0, \qquad \sin t = 0$$

or

$$(3.11)\qquad (E_1-E_3s^{-2})\cos t+E_4-E_6s^{-2} = 0, \qquad E_1s+E_2+E_3s^{-1} = 0,$$

where $|\cos t| \neq 1$.

The numbers s_1 and t_1 do not satisfy (3.11) ([2]); thus, in view of (3.8)-(3.10), the minimum problem for $\Phi(s,t)$ if $(s,t)\in D$ is equivalent to an analogous problem for

$$(3.12)\qquad \tilde{\Phi}(s) = \tilde{\Phi}(s;r) = \Phi(s,0,r),$$

where $\tilde{\Phi}(s) = \tilde{\Phi}(s;r)$ is defined in the interval $I = \{s\colon c-\varrho < s < c+\varrho\}$.

([1]) In the sequel $|\sqrt{a}|$, for $a > 0$ will be denoted by $\sqrt{a}$.

([2]) In fact, putting

$$\delta(s,t) = \frac{\partial^2\Phi(s,t)}{\partial s^2}\,\frac{\partial^2\Phi(s,t)}{\partial t^2} - \left(\frac{\partial^2\Phi(s,t)}{\partial s\partial t}\right)^2,$$

we obtain $\delta(s_1,t_1) = -(E_1-E_3s_1^{-2})\sin^2 t_1$. Supposing $\delta(s_1,t_1) = 0$ for $\sin t_1 \neq 0$, we would have, because of (3.11), $E_1E_6 = E_3E_4$, whence in view of (3.7) and (2.3) we would obtain $2-A(A-B)r^2 = 0$, which is impossible (cf. (1.2)).

LEMMA 5. *The function* $\tilde{\Phi}(s) = \tilde{\Phi}(s; r)$ *attains its minimum at the point*

$$s_1 = s_1(r) = \sqrt{\frac{(1-A)(1+Ar^2)}{A-2B+1-(A-2B+B^2)r^2}} \tag{3.13}$$

of I *only for* $r^* < r < 1$, *where* $r^* = r^*(A, B)$ *is the unique root of the polynomial*

$$g(r; A, B) = A(A-B)r^4 - 2A(1-B)r^3 - \\ -(A^2-AB+2A+2B-2)r^2 + 2(1+A)r - 2 \tag{3.14}$$

in the interval $(0, 1]$.

Proof. Differentiating (3.12) w.r.t. s we obtain

$$\tilde{\Phi}'(s) = E_1 + E_4 - (E_3 + E_6)s^{-2},$$

where

$$E_1 + E_4 = \frac{A-2B+1-(A-2B+B^2)r^2}{(A-B)(1-r^2)}, \tag{3.15}$$

$$E_3 + E_6 = \frac{(1-A)(1+Ar^2)}{(A-B)(1-r^2)}.$$

Since $E_1 + E_4 > 0$ and $E_3 + E_6 > 0$ for every admissible A, B, r, the function $\tilde{\Phi}(s)$ attains its minimum in s_1 if $s_1 \in I$.

For $A \neq 1$ put

$$k(r) = [c(r) - \varrho(r)]^2, \quad l(r) = s_1^2(r), \quad n(r) = [c(r) + \varrho(r)]^2. \tag{3.16}$$

It is easy to verify that the function $k(r)$ decreases and $l(r)$ increases for $0 < r < 1$. Since $k(0) > l(0)$ and $k(1) < l(1)$, we have $s_1 > c - \varrho$ for $r^* < r < 1$, where r^*, $0 < r^* \leqslant 1$, is the root of the equation $k(r) - l(r) = 0$, i. e. the root of the polynomial $g(r; A, B)$.

At the same time $n(r) - l(r) > 0$ for $0 < r < 1$. In fact, if $A + B \geqslant 0$, then $l(1) \leqslant n(0)$; hence $l(r) < n(r)$ for $0 < r < 1$. If $A + B < 0$ and $A > 0$, then $B < 0$. Thus

$$l(r) - n(r) < \frac{(1+Ar)^2}{1-B^2r^2}\left[(1-A)\frac{1+Ar^2}{(1+Ar)^2} - \frac{1-Br}{1+Br}\right]$$

$$< \frac{(1+Ar)^2}{1-B^2r^2}\left(1 - \frac{1-Br}{1+Br}\right) < 0.$$

Finally, for $A + B < 0$ and $A \leqslant 0$, because of $n(r) > c(r) + \varrho(r)$, we have

$$l(r) - n(r) < \frac{(B-A)\chi(r)}{(1+Br)[A-2B+1-(A-2B+B^2)r^2]},$$

where

$$\chi(r) = -A(1-B)r^3-(2-A-B)r^2+(1+A)r+2 > 0.$$

Hence we always have $s_1 < c+\varrho$ for $r^* < r < 1$, which ends the proof of the lemma.

COROLLARY. *If $0 < r \leqslant r^*$, then $\Phi(s, t)$ attains its minimum at a point of ∂D.*

Remark. If $A = 1$ and only in this case we have $r^* = 1$.

Therefore, if $A = 1$, $\tilde{\Phi}(s)$ does not attain its minimum in I.

Assuming

$$s_2 = s_2(r) = c(r) - \varrho(r) \tag{3.17}$$

and $\tilde{\Phi}(s_2) = \Phi(s_2, 0)$, we have

$$\tilde{\Phi}(s_1) < \tilde{\Phi}(s_2) \quad \text{for } r^* < r < 1. \tag{3.18}$$

Let $(s, t) \in \partial D$. Then, because of $\Phi(s, t) = \Phi(s, -t)$, we have

$$\Phi\big(s, \psi(s)\big) = \Phi\big(s, -\psi(s)\big) = \hat{\Phi}(s),$$

where $\psi(s)$ is given by (3.9) and $s \in J$, where $J = \{s\colon c-\varrho \leqslant s \leqslant c+\varrho\}$.

Thus

$$\hat{\Phi}(s) = (E_1 s + E_3 s^{-1})\cos\psi(s) + E_5 \tag{3.19}$$

with

$$\cos\psi(s) = -\frac{E_4 s + E_6 s^{-1}}{E_2}. \tag{3.20}$$

LEMMA 6. *Let*

$$Z_1 = \{(A, B)\colon\ -1 < A < 0,\ -1 \leqslant B < A\},$$

$$Z_2 = \left\{(A, B)\colon\ 0 \leqslant A \leqslant 1,\ -1 \leqslant B < \frac{A}{2}\right\},$$

$$Z_3 = \left\{(A, B)\colon\ 0 < A \leqslant 1,\ \frac{A}{2} \leqslant B < A\right\},$$

$$s' = s'(r) = \sqrt[4]{\frac{E_3\cdot E_6}{E_1\cdot E_4}} \quad \textit{for } (A, B) \in Z_1 \tag{3.21}$$

and $I = \{s\colon c-\varrho < s < c+\varrho\}$.

Then

$$\min_{(s,t)\in\partial D} \Phi(s, t) = \min_{s\in I} \hat{\Phi}(s) = \begin{cases} \hat{\Phi}(c-\varrho), & \textit{if } (A,B)\in Z_2\cup Z_3 \text{ or } (A,B)\in Z_1,\ s'\notin I, \\ \hat{\Phi}(s'), & \textit{if } (A, B)\in Z_1,\ s'\in I. \end{cases}$$

Proof. Differentiating (3.19) w.r.t. s, we find by (3.20) that

$$\hat{\Phi}'(s) = \frac{-1}{E_2}[(E_1-E_3s^{-2})(E_4s+E_6s^{-1})+(E_1s+E_3s^{-1})(E_4-E_6s^{-2})], \tag{3.22}$$

i. e.

$$\hat{\Phi}'(s) = -\frac{2(E_1E_4s^4-E_3E_6)}{E_2s^3}.$$

For any admissible A, B and r we find from (3.7) that $E_2 < 0$, $E_4 > 0$ and $E_6 > 0$. If $(A, B)\in Z_1$, then $E_1 > 0$, $E_3 > 0$; for $(A, B)\in Z_2$ we have $E_1 > 0$, $E_3 \leqslant 0$ and the condition $(A, B)\in Z_3$ implies $E_1 \leqslant 0$ and $E_3 < 0$. Thus, in view of $\hat{\Phi}(c-\varrho) < \hat{\Phi}(c+\varrho)$, the lemma has been proved.

LEMMA 7. *If $s'\in I$, where s' is given by* (3.21), *then*

$$\tilde{\Phi}(s') < \hat{\Phi}(s'). \tag{3.23}$$

Proof. Since $\tilde{\Phi}(s) = \Phi(s, 0)$, then, in view of (3.6) and because of (3.19), we obtain

$$\tilde{\Phi}(s)-\hat{\Phi}(s) = U(s)\big(1-\cos\psi(s)\big)+V(s), \tag{3.24}$$

where

$$U(s) = E_1s+E_3s^{-1}, \qquad V(s) = E_4s+E_2+E_6s^{-1}. \tag{3.25}$$

Since

$$1-\cos\psi(s) = -\frac{V(s)}{E_2}, \tag{3.26}$$

equality (3.24) now becomes

$$\tilde{\Phi}(s)-\hat{\Phi}(s) = -\frac{V(s)}{E_2}[U(s)+E_2]. \tag{3.27}$$

Since $E_2 < 0$, we find from (3.26) that $V(s) < 0$. Because of (3.22) and (3.25) we obtain for $s = s'$

$$U(s')+E_2 = E_2\left[\frac{E_1s'^2-E_3}{E_4s'^2-E_6}\cos\psi(s')+1\right]. \tag{3.28}$$

From (3.7) we get $E_3 < E_1$, and thus

$$E_1s'^2-E_3 > E_1(s'^2-1). \tag{3.29}$$

On the other hand, basing ourselves on Lemma 6, we conclude that $E_3 > 0$ in (3.28); thus we obtain $A^2 < B^2$. Hence $E_6 > E_4$. Therefore

$$E_4s'^2-E_6 < E_4(s'^2-1). \tag{3.30}$$

From (3.29) and (3.30) because of $E_2 < 0$ we have $U(s')+E_2 < 0$. Thus, in view of (3.27), inequality (3.23) is true.

Let Φ_* be the minimal value of $\Phi(s, t)$ in $D\cup\partial D$. In view of Lemmas 5–7 and inequality (3.18) $\Phi_* = \Phi(s_2, 0)$ for $0 < r \leqslant r^*$ and $\Phi_* = \Phi(s_1, 0)$ for $r^* < r < 1$.

We shall now prove the following

THEOREM 1. *For all $P(z)$ in $\wp(A, B)$ and $|z| = r$, $0 < r < 1$*

$$\mathrm{re}\left[P(z)+\frac{zP'(z)}{P(z)}\right] \geqslant \begin{cases} X_1(r;\ A, B) & \text{for } 0 < r\leqslant r^*, \\ X_2(r;\ A, B) & \text{for } r^* < r < 1, \end{cases} \tag{3.31}$$

where

$$X_1(r;\ A, B) = \frac{A^2r^2-(3A-B)r+1}{(1-Ar)(1-Br)}, \tag{3.32}$$

$$X_2(r; A, B) = 2\frac{\sqrt{\mathfrak{A}\mathfrak{B}}-(1-ABr^2)}{(A-B)(1-r^2)}+\frac{A+B}{A-B}, \tag{3.33}$$

$$\mathfrak{A} = \mathfrak{A}(r;\ A, B) = A-2B+1-(A-2B+B^2)r^2, \tag{3.34}$$

$$\mathfrak{B} = \mathfrak{B}(r;\ A, B) = (1-A)(1+Ar^2) \tag{3.35}$$

and $r^* = r^*(A, B)$ is the unique root the polynomial

$$\begin{aligned} g(r;\ A, B) = {} & A(A-B)r^4-2A(1-B)r^3- \\ & -(A^2-AB+2A+2B-2)r^2+2(1+A)r-2 \end{aligned} \tag{3.36}$$

in the interval $(0, 1]$.

These bounds are sharp, being attained at the point $z = re^{i\varphi}$, $0\leqslant\varphi\leqslant 2\pi$, by

$$P^*(z;\ A, B) = \frac{1-Ae^{-i\varphi}z}{1-Be^{-i\varphi}z} \quad \text{for } 0 < r\leqslant r^* \tag{3.37}$$

and by

$$P^{**}(z;\ A, B) = \frac{1-(1-A)de^{-i\varphi}z-Ae^{-2i\varphi}z^2}{1-(1-B)de^{-i\varphi}z-Be^{-2i\varphi}z^2} \quad \text{for } r^* < r < 1 \tag{3.38}$$

respectively, where

$$d = d(r; A, B) = \frac{1}{r}\,\frac{(1-Br^2)s_1-(1-Ar^2)}{(1-B)s_1-(1-A)}, \quad s_1 = \sqrt{\mathfrak{B}\mathfrak{A}^{-1}}. \tag{3.39}$$

Proof. For s_2 and s_1 given by (3.17) and (3.13), respectively, we obtain $\Phi(s_2, 0) = X_1(r)$ and $\Phi(s_1, 0) = X_2(r)$; thus, in view of (2.9), (3.4), (3.5) and Lemmas 5–7, inequality (3.31) is true. We shall prove that this estimation is sharp.

To this end we observe first that if a function $P^*(z)$ of the family $\wp_2(A, B)$ satisfies condition (3.2) at some point $re^{i\varphi}$, $0 < r\leqslant r^*$, $0\leqslant\varphi\leqslant 2\pi$, with $s = s_2$ and $t = 0$, then

$$P^*(re^{i\varphi}) = s_2. \tag{3.40}$$

To make notation simpler, we denote the values of the parameters appearing in Lemma 3 by the same letters as the parameters themselves.

Since $s_2 = c-\varrho$, from (2.10) we obtain $\varkappa = \varrho$ and $\psi = -1$. Therefore from (2.19) it follows that $\lambda^2 = 1$ or $\gamma_1 = \gamma_2$. If $\lambda^2 = 1$, then because of (2.7), (2.8) and (2.1) we get

$$P^*(z) = \frac{1+A\varepsilon z}{1+B\varepsilon z} \tag{3.41}$$

for some $|\varepsilon| = 1$.

If $\gamma_1 = \gamma_2$, then in view of $\eta_k = e^{i\gamma_k}$ $(k = 1, 2)$ and because of (2.11) we obtain $\varepsilon_1 = \varepsilon_2$. Thus, from (2.7) and (2.8) and because of (2.1) we infer that $P^*(z)$ is also of form (3.41).

We find ε. For $z = re^{i\varphi}$ we have

$$P^*(re^{i\varphi}) = \frac{1+A\varepsilon re^{i\varphi}}{1+B\varepsilon re^{i\varphi}}. \tag{3.42}$$

Equating (3.40) and (3.42) we obtain

$$\varepsilon = \frac{1}{r}\,\frac{s_2-1}{A-B}\,e^{-i\varphi}$$

and because of (3.17) we obtain $\varepsilon = -e^{-i\varphi}$. Thus $P^*(z)$ is of form (3.37). Evidently $P^*(z)\in\wp(A, B)$. It is easy to verify that for $z = re^{i\varphi}$

$$P^*(z)+\frac{zP^{*\prime}(z)}{P^*(z)} = X_1(r;\ A, B).$$

Next, if a function $P^{**}(z)$ of $\wp_2(A, B)$ satisfies condition (3.2) at some point $re^{i\varphi}$, $r^* < r < 1$, $0\leqslant\varphi\leqslant 2\pi$, with $s = s_1$ and $t = 0$, then

$$P^{**}(re^{i\varphi}) = s_1. \tag{3.43}$$

We accept the foregoing agreement concerning the notation of values of the parameters corresponding to the function $P^{**}(z)$.

Since $t = 0$ (comp. (3.2) and (3.12)), by (3.3)-(3.6) we have $\eta^* = 1$. Therefore, in view of (2.27)

$$\frac{\overline{V}(re^{i\varphi})}{V(re^{i\varphi})}\eta_1\eta_2 = 1,$$

i. e.

$$\frac{\bar{v}}{v}\eta_1\eta_2 = 1$$

(cf. (2.16)). Hence

$$\frac{2g+(1+B)[(1+\lambda)\bar{\eta}_2+(1-\lambda)\bar{\eta}_1]\varrho}{2g+(1+B)[(1+\lambda)\eta_2+(1-\lambda)\eta_1]\varrho}\eta_1\eta_2 = 1 \tag{3.44}$$

(cf. (2.11)).

We conclude from (3.44) that

$$g(\eta_1\eta_2-1)+\lambda(1+B)\varrho(\eta_1-\eta_2) = 0. \tag{3.45}$$

Moreover, since $P^{**}(re^{i\varphi})$ is real, by (2.10) and (2.11) we have $\psi = \bar{\psi}$, i. e.

$$(1+\lambda)h_1\eta_1+(1-\lambda)h_2\eta_2 = (1+\lambda)\bar{h}_1\bar{\eta}_1+(1-\lambda)\bar{h}_2\bar{\eta}_2. \tag{3.46}$$

By (3.46), in view of (2.11), an easy calculation yields

$$\lambda g(\eta_1-\eta_2)+(1+B)\varrho(\eta_1\eta_2-1) = 0. \tag{3.47}$$

We shall solve the system of equations (3.45) and (3.47) with the unknows λ, η_1 and η_2.

Supposing that for $\eta_1 = \eta_2$ we would have $\varepsilon_1 = \varepsilon_2$; then because of (2.13), (2.7), (2.8) and (2.16) we would obtain $h_1 = h_2 = \frac{1}{2}$; hence, in view of (2.17), we would get $\varkappa\psi = \varrho\cdot\eta$, where $\eta = \eta_1 = \eta_2$. Therefore, because of (2.12) we would obtain $P^{**}(re^{i\varphi}) = c+\varrho\eta$ and because of the equalities $P^{**}(re^{i\varphi}) = \overline{P^{**}(re^{i\varphi})}$ and (3.43) we would find that $s_1 = c-\varrho$ or $s_1 = c+\varrho$, which is impossible. Thus, $\eta_1 \neq \eta_2$. From (3.47) we find

$$\lambda = -\frac{(1+B)\varrho(\eta_1\eta_2-1)}{g(\eta_1-\eta_2)}. \tag{3.48}$$

Substituting λ from (3.48) into (3.45), we obtain

$$(\eta_1\eta_2-1)[g^2-(1+B)^2\cdot\varrho^2] = 0.$$

Since $g^2-(1+B)^2\cdot\varrho^2 \neq 0$ (cf. (2.20)), we have

$$\eta_1\eta_2 = 1. \tag{3.49}$$

It follows from (3.49) and (3.47) that

$$\lambda = 0. \tag{3.50}$$

Because of (3.49) we find from the equality

$$\eta_k = \varepsilon_k e^{i\varphi}\frac{1+B\bar{\varepsilon}_k re^{-i\varphi}}{1+B\varepsilon_k re^{i\varphi}} \quad (k = 1, 2)$$

(cf. (2.11)) that

$$\varepsilon_1\varepsilon_2 = e^{-2i\varphi}. \tag{3.51}$$

Thus, because of (3.50), (2.12) and (2.8),

$$P^{**}(z) = \frac{1-\frac{1}{2}(1-A)(\varepsilon_1+\varepsilon_2)z - A\varepsilon_1\varepsilon_2 z^2}{1-\frac{1}{2}(1-B)(\varepsilon_1+\varepsilon_2)z - B\varepsilon_1\varepsilon_2 z^2}.$$

Let

$$2d = \varepsilon_1 e^{i\varphi} + \bar{\varepsilon}_1 e^{-i\varphi}. \tag{3.52}$$

From (3.51) and (3.52) we obtain $\varepsilon_1+\varepsilon_2 = 2e^{-i\varphi}\cdot d$; thus

$$P^{**}(z) = \frac{1-(1-A)de^{-i\varphi}z - Ae^{-2i\varphi}z^2}{1-(1-B)de^{-i\varphi}z - Be^{-2i\varphi}z^2}. \tag{3.53}$$

It follows from (3.43) and (3.53) that

$$s_1 = \frac{1-(1-A)dr - Ar^2}{1-(1-B)dr - Br^2}.$$

Therefore $P^{**}(z)$ is of form (3.38) with d given by (3.39). Evidently $P^{**}(z) \in \wp(A, B)$.

Finally we prove that, for $z = re^{i\varphi}$,

$$P^{**}(z) + \frac{zP^{**\prime}(z)}{P^{**}(z)} = X_2(r;\ A, B).$$

Differentiating the function $P^{**}(z)$, we obtain

$$P^{**\prime}(z) = (A-B)e^{-i\varphi}\frac{d-2e^{-i\varphi}z+e^{-2i\varphi}dz^2}{[1-(1-B)de^{-i\varphi}z - Be^{-2i\varphi}z^2]^2}.$$

Therefore, for $z = re^{i\varphi}$,

$$P^{**}(z) + \frac{zP^{**\prime}(z)}{P^{**}(z)} = s_1 + \frac{A-B}{s_1}\,\frac{dr(1+r^2)-2r^2}{[1-(1-B)dr-Br^2]^2},$$

and by (3.39) we get

$$P^{**}(z) + \frac{zP^{**\prime}(z)}{P^{**}(z)}$$
$$= \frac{\mathfrak{A}(r;A,B)s_1^2 - [2-A-B+(B-2AB+A)r^2]s_1 + \mathfrak{B}(r;\ A,B)}{(A-B)(1-r^2)s_1}, \quad z = re^{i\varphi}.$$

Since $s_1 = \sqrt{\mathfrak{B}\mathfrak{A}^{-1}}$, we have, for $z = re^{i\varphi}$,

$$P^{**}(z) + \frac{zP^{**\prime}(z)}{P^{**}(z)} = 2\,\frac{\mathfrak{B}(r)-(1-ABr^2)s_1}{(A-B)(1-r^2)s_1} + \frac{A+B}{A-B} = X_2(r;\ A, B),$$

which ends the proof of Theorem 1.

II. Let

$$\omega_1(r) = \max_{\substack{|z|=r<1 \\ P(z)\in \mathcal{P}(A,B)}} \operatorname{re}\left[P(z)+\frac{zP'(z)}{P(z)}\right].$$

Proceeding as in part I of this section and preserving the same notation, we obtain first $\omega_1(r) \leqslant \tau_1(r)$, where

$$\tau_1(r) = \max_{(s,t)\in D\cup\partial D} \Phi_1(s, t)$$

and

$$\Phi_1(s, t) = \Phi_1(s, t; r) = (E_1 s - E_2 + E_3 s^{-1})\cos t - E_4 s + E_5 - E_6 s^{-1}.$$

Next we prove that if $\Phi_1(s, t)$ attains its maximum at a point $(\hat{s}_1, \hat{t}_1) \in D$, then $\hat{t}_1 = 0$. Let

$$\tilde{\Phi}_1(s) = \tilde{\Phi}_1(s; r) = \Phi_1(s, 0, r).$$

Since

$$\tilde{\Phi}_1'(s) = E_1 - E_4 + (E_6 - E_3)s^{-2}$$

and $E_6 - E_3 > 0$, the point $\hat{s}_1$ exists only if $E_1 < E_4$.

Hence

$$\hat{s}_1 = \hat{s}_1(r) = \sqrt{\frac{E_6 - E_3}{E_4 - E_1}},$$

if 1° $E_1 < E_4$ and 2° $[(E_6 - E_3)(E_4 - E_1)^{-1}]^{1/2} \in I$.

$E_1 < E_4$ only if 1° $A - 2B - 1 \leqslant 0$, $0 < r < 1$ or 2° $A - 2B - 1 > 0$, $r_0 < r < 1$, where $r_0 = \sqrt{(A - 2B - 1)/(A - 2B - B^2)}$.

Putting $\hat{l}(r) = \hat{s}_1^2(r)$ in these cases, we find that $\hat{l}(r)$ decreases for $0 < r < 1$ and for $r_0 < r < 1$, respectively. The function $k(r)$ defined by (3.16) decreases for $0 < r < 1$ and $k(0) = 1$. Next, we obtain $\hat{l}(r) > k(r)$ for $0 < r < 1$ and for $r_0 < r < 1$, respectively (3) and $\hat{l}(r) < n(r)$ (cf. (3.16)) for $r^{**} < r < 1$, where r^{**} is the unique root of the polynomial $g(r; -A, -B)$, in the intervals $(0, 1)$ and $(r_0, 1)$, respectively.

Summing, we obtain

LEMMA 5'. *The function $\tilde{\Phi}_1(s) = \tilde{\Phi}_1(s; r)$ attains its maximum at the point*

$$\hat{s}_1 = \hat{s}_1(r) = \sqrt{\frac{(1+A)(1-Ar^2)}{(A-2B-B^2)r^2-(A-2B-1)}}$$

(3) In fact, if $A + B \leqslant 0$, then $\hat{l}(1) \geqslant k(0)$; hence $\hat{l}(r) > k(r)$ in this case. If $A + B > 0$, then $A > 0$ and $\hat{l}(r) - k(r) > (A - B)\hat{\chi}(r)/(1 - Br)[(A - 2B - B^2)r^2 - (A - 2B - 1)]$, where $\hat{\chi}(r) = A(1+B)r^3 - (A+B+2)r^2 + (1-A)r + 2 > 0$ for $0 < r < 1$.

of I only for $r^{**} < r < 1$, *where* $r^{**} = r^{**}(A, B)$ *is the unique root of the polynomial*

$$A(A-B)r^4+2A(1+B)r^3-(A^2-AB-2A-2B-2)r^2+2(1-A)r-2$$

in the interval $(0, 1]$.

COROLLARY. *If* $0 < r \leqslant r^{**}$, *then* $\Phi_1(s, t)$ *attains its maximum at a point of* ∂D.

Remark. If $B = -1$ and only in this case we have $r^{**} = 1$. Therefore, if $B = -1$, then $\tilde{\Phi}_1(s)$ does not attain its maximum in I.

We see that for $r^{**} < r < 1$

$$\tilde{\Phi}_1(\hat{s}_1) > \Phi_1(c+\varrho).$$

As in part I we obtain

LEMMA 6'. *Let*

$$\Phi_1\big(s, \psi(s)\big) = \hat{\Phi}_1(s),$$

where $\psi(s)$ *is given by* (3.7), $s \in J$, $J = \{s\colon c-\varrho \leqslant s \leqslant c+\varrho\}$ *and let*

$$s' = s'(r) = \sqrt[4]{\frac{E_3E_6}{E_1E_4}} \quad \textit{for } (A, B) \in Z_3$$

(cf. Lemma 6). *Then*

$$\max_{(s,t)\in\partial D} \Phi_1(s, t) = \max_{s\in J} \hat{\Phi}_1(s) = \begin{cases} \hat{\Phi}_1(c+\varrho), & \textit{if } (A, B) \in Z_1 \cup Z_2 \textit{ or } \\ & \quad (A, B) \in Z_3,\ s' \notin I; \\ \Phi_1(s'), & \textit{if } (A, B) \in Z_3,\ s' \in I. \end{cases}$$

Remark. If $(A, B) \in Z_3$, then $A-2B-1 < 0$.

We prove the following

LEMMA 7'. *If* $s' \in I$, *then*

$$\tilde{\Phi}_1(s') > \hat{\Phi}_1(s').$$

Proof. Preserving the notation adopted in Lemma 7, we easy obtain the equality

$$\tilde{\Phi}_1(s') - \hat{\Phi}_1(s') = V(s')[T(s')\cos\psi(s') - 1],$$

where

$$T(s') = \frac{E_1 s'^2 - E_3}{E_4 s'^2 - E_6}.$$

Since $(A, B) \in Z_3$, we have $E_3 < E_1 < 0$ and $E_4 > E_6 > 0$.

Putting $h(r) = E_6E_4 - E_1E_3$, we get

$$h(r) = \frac{h_1(r)}{(A-B)^2(1-r^2)^2},$$

where

$$h_1(r) = A(AB^2-2B+A)r^4+(A-B)(B-3A)r^2+A^2-2AB+1.$$

If $h_1(r) > 0$ for $0 < r < 1$, then $h(r) > 0$ and because of $E_1E_6-E_3E_4 > 0$ we obtain after some calculation the inequality

$$s'^2 > \frac{E_6-E_3}{E_4-E_1}.$$

Hence $T(s') < 1$; thus, because of $\cos\psi(s') > 0$ and $V(s') < 0$, in case of $h_1(r) > 0$, the lemma is proved.

The relation $(A, B) \in Z_3$ implies $0 < A \leqslant 2B$. Thus $h_1(0) > 0$; moreover, $h_1(1) \geqslant 0$. Therefore $h_1(r) > 0$ for $0 < r < 1$ if $AB^2-2B+A < 0$. Similarly for $AB^2-2B+A \geqslant 0$ this inequality is true, which ends the proof.

Basing ourselves on Lemmas $5'$–$7'$, we obtain

THEOREM 2. *For all $P(z)$ in $\wp(A, B)$ and $|z| = r$, $0 < r < 1$*

$$\text{(3.54)}\qquad \operatorname{re}\left[P(z)+\frac{zP'(z)}{P(z)}\right] \leqslant \begin{cases} X_1(r;\, -A,\, -B) & \text{for } 0 < r \leqslant r^{**}; \\ X_2(r;\, -A,\, -B) & \text{for } r^{**} < r < 1, \end{cases}$$

equality holding in $z = re^{i\varphi}$ for $P^*(z;\, -A,\, -B)$ if $0 < r \leqslant r^{**}$ and for $P^{**}(z;\, -A,\, -B)$ if $r^{**} < r < 1$, respectively, where $r^{**} = r^{**}(A, B)$ is the unique root of the polynomial $g(r;\, -A,\, -B)$ in the interval $(0, 1]$ (cf. Theorem 1).

III. Similarly we prove the following

THEOREM 3. *For all $P(z)$ in $\wp(A, B)$ and $|z| = r$, $0 < r < 1$*

$$1^\circ\qquad \operatorname{re}\frac{zP'(z)}{P(z)} \geqslant \begin{cases} Y_1(r;\, A, B) & \text{for } 0 < r \leqslant \hat{r}^*, \\ Y_2(r;\, A, B) & \text{for } \hat{r}^* < r < 1, \end{cases}$$

where

$$Y_1(r;\, A, B) = -\frac{(A-B)r}{(1-Ar)(1-Br)},$$

$$Y_2(r;\, A, B) = 2\frac{\sqrt{\hat{\mathfrak{A}}\hat{\mathfrak{B}}}-(1-ABr^2)}{(A-B)(1-r^2)}+\frac{A+B}{A-B},$$

$$\hat{\mathfrak{A}} = \hat{\mathfrak{A}}(r;\, A, B) = (1-B)(1+Br^2), \qquad \hat{\mathfrak{B}} = \mathfrak{B}$$

(cf. Theorem 1) *and $\hat{r}^* = \hat{r}^*(A, B)$ is the unique root of the polynomial $g(r, A, B) \equiv ABr^4-2ABr^3+(2A+2B-AB-1)r^2-2r+1$ in the interval* $(0, 1]$. *Functions* (3.37) *and* (3.38), *where d is given by* (3.39) *and $s_1 = \sqrt{\hat{\mathfrak{B}}\hat{\mathfrak{A}}^{-1}}$ shows this result to be sharp.*

$$2^\circ\qquad \operatorname{re}\frac{zP'(z)}{P(z)} \leqslant \begin{cases} Y_1(r;\, -A,\, -B) & \text{for } 0 < r \leqslant \hat{r}^{**}, \\ Y_2(r;\, -A,\, -B) & \text{for } r^{**} < r < 1, \end{cases}$$

equality holding for functions $P^*(z; -A, -B)$ *and* $P^{**}(z; -A, -B)$, *respectively*; $\hat{r}^{**}$ *is the unique root of the polynomial* $\hat{g}(r; -A, -B)$ *in the interval* $(0, 1]$.

Remark. If $A = 1$ and only in this case we have $\hat{r}^* = 1$, $\hat{r}^{**} = 1$ only for $B = -1$.

Applying Theorem 3 to the special cases where $A = 1-2\alpha$ and $B = -1$, $A = 1$ and $B = \frac{1}{M}-1$, $A = \beta$ and $B = -\beta$, $A = \beta$ and $B = 0$, we obtain the corresponding theorems on $\operatorname{re}\frac{zP'(z)}{P(z)}$ in the families $\wp_\alpha$, $\wp(M)$, $\wp^{(\beta)}$ and $\wp_{(\beta)}$, respectively. If $A = 1$ and $B = -1$, then we obtain a result of Libera [2].

4. The estimations of $|f(z)|$ and $|f'(z)|$ in $S^*(A, B)$.

THEOREM 4. *If* $f(z) \in S^*(A, B)$, *then for* $|z| = r$, $0 \leqslant r < 1$

$$C(r; -A, -B) \leqslant |f(z)| \leqslant C(r; A, B), \tag{4.1}$$

where

$$C(r; A, B) = \begin{cases} r(1+Br)^{(A-B)/B} & \text{for } B \neq 0, \\ re^{Ar} & \text{for } B = 0. \end{cases}$$

These bounds are sharp, being attained at the point $z = re^{i\varphi}$, $0 \leqslant \varphi \leqslant 2\pi$, *by*

$$f_*(z) = z\cdot f_0(z; -A, -B) \tag{4.2}$$

and

$$f^*(z) = z\cdot f_0(z; A, B), \tag{4.3}$$

respectively, where

$$f_0(z; A, B) = \begin{cases} (1+Be^{-i\varphi}z)^{(A-B)/B} & \text{for } B \neq 0, \\ e^{Ae^{-i\varphi}z} & \text{for } B = 0. \end{cases}$$

Proof. Since $f(z) \in S^*(A, B)$, we have

$$f(z) = z\cdot\exp\left(\int_0^z \frac{P(\zeta)-1}{\zeta}\,d\zeta\right), \quad P(z) \in \wp(A, B).$$

Therefore

$$|f(z)| = |z|\exp\left(\operatorname{re}\int_0^z \frac{P(\zeta)-1}{\zeta}\,d\zeta\right).$$

Substituting $\zeta = zt$, we obtain

$$|f(z)| = |z|\exp\left(\operatorname{re}\int_0^1 \frac{P(zt)-1}{t}\,dt\right).$$

Hence

$$|f(z)| \leqslant |z| \exp\left(\int_0^1 \max_{|zt|=rt}\left(\operatorname{re}\frac{P(zt)-1}{t}\right)dt\right).$$

From Lemma 2 it follows that

$$\max_{|zt|=rt} \operatorname{re}\frac{P(zt)-1}{t} = \frac{(A-B)r}{1+Brt};$$

then, after integration, we obtain the upper bounds in (4.1). Similarly, we obtain the bounds an the left-hand side of (4.1), which ends the proof.

From Theorem 4 follows immediately the corresponding theorems on $|f(z)|$ in the families S_a^* [7], $S^*(M)$ [1], $S^{*(\beta)}$ [6] and $S_{(\beta)}^*$ [4].

THEOREM 5. *If $f(z) \in S^*(A, B)$, then for $|z| = r$, $0 \leqslant r < 1$,*

$$\tilde{L}(r) \leqslant |f'(z)| \leqslant L(r),$$

where

$$L(r) = \begin{cases} D(r), & \text{if } 0 < r \leqslant r^{**}, \\ D(r^{**})\dfrac{\exp H(r)}{\exp H(r^{**})}, & \text{if } r^{**} < r < 1, \end{cases}$$

(4.4)

$$\tilde{L}(r) = \begin{cases} \tilde{D}(r), & \text{if } 0 < r \leqslant r^{*}, \\ \tilde{D}(r^{*})\dfrac{\exp \tilde{H}(r)}{\exp \tilde{H}(r^{*})}, & \text{if } r^{*} < r < 1, \end{cases}$$

r^* and r^{**} are the roots of the polynomial $g(r; A, B)$ and $g(r; -A, -B)$, respectively (cf. Theorems 2 and 1),

$$D(r) = D(r; A, B) = \begin{cases} (1+Ar)(1+Br)^{(A-2B)/B}, & \text{if } B \neq 0, \\ (1+Ar)e^{Ar}, & \text{if } B = 0, \end{cases}$$

$$H(r) = H(r; A, B)$$

$$= \frac{2}{A-B}\int \frac{1+B-B(1+A)r^2-\sqrt{(1+A)(1-Ar^2)(a_1-a_2r^2)}}{r(1-r^2)}\,dr \text{ (4)},$$

$$r^{**} \leqslant r < 1,$$

---

(4) After integration we obtain $H(r) = 2\dfrac{1+B}{A-B}\log r - \dfrac{1-AB}{A-B}\log(1-r^2) + \sum_{k=1}^{3} J_k + \text{const}$, where

$$J_1 = \begin{cases} 0 & \text{for } A = 0 \text{ or } a_2 = 0, \\ -2b_1 \arctan t_1^{-1} & \text{for } a_2 < 0 \text{ and } A > 0, \\ b_1 \log\left|\dfrac{1-t_1}{1+t_1}\right| & \text{for } a_2 > 0 \text{ or } a_2 < 0 \text{ and } A < 0, \end{cases}$$

$$a_1 = a_1(A, B) = -A+2B+1, \quad a_2 = a_2(A, B) = -A+2B+B^2$$

and

$$\tilde{D}(r) = D(r; -A, -B), \quad \tilde{H}(r) = H(r; -A, -B).$$

The upper bound $L(r)$ for $0 < r \leqslant r^{**}$ and the lower bound $\tilde{L}(r)$ for $0 < r \leqslant r^{*}$ are sharp, being attained by functions (4.3) *and* (4.2), *respectively.*

Proof. If $f(z) \in S^*(A, B)$, then because of (1.3) an easy calculation yields

$$1+\frac{zf''(z)}{f'(z)} = P(z)+\frac{zP'(z)}{P(z)} \tag{4.5}$$

for some $P(z)$ in $\wp(A, B)$. On the other hand, we have

$$\operatorname{re}\frac{zf''(z)}{f'(z)} = r\frac{\partial}{\partial r}\log|f'(z)|, \qquad |z| = r$$

then, using (3.54), (3.32) and (3.33), we obtain

$$\frac{\partial}{\partial r}\log|f'(z)| \leqslant \frac{(A-B)(Ar+2)}{(1+Ar)(1+Br)} \tag{4.6}$$

for $0 < r \leqslant r^{**}$

$$\frac{\partial}{\partial r}\log|f'(z)| \leqslant 2\,\frac{1+B-B(1+A)r^2-\sqrt{(1+A)(1-Ar^2)(a_1-a_2r^2)}}{(A-B)r(1-r^2)} \tag{4.7}$$

for $r^{**} < r < 1$.

Integrating both sides of inequality (4.6) from 0 to r, we obtain

$$|f'(z)| \leqslant D(r), \tag{4.8}$$

where $D(r)$ is given by (4.4).

$$J_2 = \begin{cases} 0 & \text{for } a_1 = 0, \\ 2b_2\arctan t_2^{-1} & \text{for } a_1 < 0, \\ -b_2\log\left(\dfrac{1-t_2}{1+t_2}\right) & \text{for } a_1 > 0, \end{cases} \qquad J_3 = \begin{cases} 0 & \text{for } A = 1, \\ b_3\log\left|\dfrac{1-t_3}{1+t_3}\right| & \text{for } A \neq 1, \end{cases}$$

$$b_1 = A\,\frac{\sqrt{(1+A)|A|^{-1}|a_2|}}{A-B}, \quad b_2 = \frac{\sqrt{(1+A)|a_1|}}{A-B}, \quad b_3 = \frac{\sqrt{(1-A^2)(1-B^2)}}{A-B},$$

$$t_1 = \sqrt{\left|\frac{a_2}{A}\right|}\cdot t \quad \text{for } A \neq 0, \quad t_2 = \sqrt{|a_1|}\cdot t, \quad t_3 = \sqrt{\frac{1-B^2}{1-A}}\,t \quad \text{for } A \neq 1$$

and

$$t = \sqrt{\frac{1-Ar^2}{a_1-a_2r^2}}.$$

Let $r^{**} < r < 1$. Denoting by $I_1(r)$ and $I_2(r)$ the right-hand sides of inequalities (4.6) and (4.7), respectively, we get the inequalities

$$\log|f'(z)| \leqslant \int_0^{r^{**}} I_1(r)\,dr + \int_{r^{**}}^{r} I_2(r)\,dr. \tag{4.9}$$

We easily obtain

$$\int_0^{r^{**}} I_1(r)\,dr = \log D(r^{**}) \tag{4.10}$$

and

$$\int_{r^{**}}^{r} I_2(r)\,dr = H(r) - H(r^{**}), \tag{4.11}$$

where $H(r)$ is given by (4.4).

By (4.8)-(4.11) the first part of the theorem on the upper estimation of $|f'(z)|$ has been proved. Similarly, the second part of the theorem on the lower estimation of $|f'(z)|$ can be proved.

The lower bound of $|f'(z)|$ in the classes S_α^*, $S^*(M)$, $S^{*(\beta)}$ and $S^*_{(\beta)}$ is sharp in the following intervals of r: $(0, r_\alpha]$; $(0, 1)$ [1]; $(0, r_1^{(\beta)}]$; $(0, r'_{(\beta)}]$, respectively, where $r_\alpha = r^*(1-2\alpha, -1)$, $r_1^{(\beta)} = r^*(\beta, -\beta)$, $r'_{(\beta)} = r^*(\beta, 0)$ and the upper bound — in the intervals: $(0, 1)$ [7]; $(0, 1)$ for $M \geqslant 1$ [1] and in $(0, R(M)]$ for $M < 1$; $(0, 1)$ for $\beta \geqslant \frac{1}{3}$ and in $(0, r_2^{(\beta)}]$ for $\beta < \frac{1}{3}$; $(0, r''_{(\beta)}]$, respectively, where $R(M) = r^{**}\left(1, \dfrac{1}{M} - 1\right)$, $r_2^{(\beta)} = r^{**}(\beta, -\beta)$, $r''_{(\beta)} = r^{**}(\beta, 0)$.

5. The radius of convexity for the family $S^*(A, B)$. Let S be the family of functions (1.2) regular and univalent in K and T an arbitrary subclass of S. If f is in T, then r.c. $\{f\}$, the radius of convexity of f, is

$$\text{r. c. } \{f\} = \sup\left[r\colon \operatorname{re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ |z| < r\right]$$

and r. c. T, the radius of convexity of T, is

$$\text{r. c. } T = \inf_{f \in T}[\text{r. c. } \{f\}].$$

If T is compact, then the problem of finding r. c. T is reduced to finding the greatest value of r, $0 < r \leqslant 1$, for which

$$\operatorname{re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geqslant 0$$

for every $|z| \leqslant r$ and every function $f(z) \in T$.

Since $S^*(A, B)$ is compact, it follows immediately that r. c. $S^*(A, B)$ equals the smallest root r_0, $0 < r_0 \leqslant 1$, of the equation $\omega(r) = 0$, where

$$\omega(r) = \min\left\{\mathrm{re}\left(1+\frac{zf''(z)}{f'(z)}\right):\ |z| = r < 1, f \in S^*(A, B)\right\}.$$

Let $f(z)$ be an arbitrary function of $S^*(A, B)$. Then, in view of (1.3), (4.5) and because of Theorem 1,

$$\omega(r) = \begin{cases} u(r)/u_1(r) & \text{for } 0 < r \leqslant r^*, \\ v(r)/v_1(r) & \text{for } r^* < r < 1, \end{cases} \tag{5.1}$$

where

$$u(r) = A^2r^2-(3A-B)r+1, \quad u_1(r) = (1-Ar)(1-Br) > 0, \tag{5.2}$$

$$v(r) = c_1r^4-2c_2r^2+c_3, \tag{5.3}$$

$$v_1(r) = (A-B)(1-r^2)[2\sqrt{\mathfrak{A}\mathfrak{B}}+2(1-ABr^2)-(A+B)(1-r^2)] > 0,$$

$$c_1 = 4A^2-5A+B, \quad c_2 = 2A^2-3A+2-B, \quad c_3 = 4-5A+B,$$

$\mathfrak{A}$ and $\mathfrak{B}$ are given by (3.34)-(3.35) and r^* is the unique root of the polynomial (3.36) in the interval $(0, 1]$.

Let

$$B_1 = B_1(A) = -A^2+3A-1 \quad \text{for } 0 \leqslant A < 1, \tag{5.4}$$

$$B_2 = B_2(A) = 5A-4 \quad \text{for } \tfrac{3}{5} \leqslant A < 1, \tag{5.5}$$

$$G_1 = \{(A, B):\ (-1 < A \leqslant 0,\ -1 \leqslant B < A) \cup \cup (0 < A < 1, B_1 \leqslant B < A)\}, \tag{5.6}$$

$$G_2 = \{(A, B):\ (0 < A \leqslant \tfrac{3}{5},\ -1 \leqslant B < B_1) \cup \cup (\tfrac{3}{5} < A < 1, B_2 \leqslant B < B_1)\}, \tag{5.7}$$

$$G_3 = \{(A, B):\ \tfrac{3}{5} < A < 1,\ -1 \leqslant B < B_2\}. \tag{5.8}$$

It can easily be verified that $u(r) > 0$ for $0 < r < 1$ if $(A, B) \in G_1$; $u(r)$ has one root r_1 in the interval $(0, 1)$ if $(A, B) \in G_2 \cup G_3$, $u(r) > 0$ for $0 < r < r_1$; hence $u(r) < 0$ for $r_1 < r < 1$ in this case; $v(r)$ has one root r_2 in the interval $(0, 1)$ if $(A, B) \in G_1 \cup G_2$ and at the same time $v(r) > 0$ for $0 < r < r_2$; thus $v(r) < 0$ for $r_2 < r < 1$; finally $v(r) < 0$ for $0 < r < 1$ when $(A, B) \in G_3$.

Hence, because of (5.1)-(5.3) and the fact that $u(r^*)$ and $v(r^*)$ must have the same sign, we obtain the following

LEMMA 8. *If*: 1° $(A, B) \in G_1$ *or* 2° $(A, B) \in G_2$ *and* $r_1 \geqslant r^*$, *then* $\omega(r) > 0$ *for* $0 < r < r_2$, $\omega(r_2) = 0$ *and* $\omega(r) < 0$ *for* $r_2 < r < 1$. *If*: 3° $(A, B) \in G_2$ *and* $r_1 < r^*$ *or* 4° $(A, B) \in G_3$, *then* $\omega(r) > 0$ *for* $0 < r < r_1$, $\omega(r_1) = 0$ *and* $\omega(r) < 0$ *for* $r_1 < r < 1$.

Proof. The first or the second assumption implies immediately the assertion. If the third condition is satisfied, then, because of $u(r^*) < 0$, we have $v(r^*) < 0$, thus the lemma is true in this case. Finally if $(A, B) \in G_3$, then $v(r^*) < 0$; thus $u(r^*) < 0$ and $r_1 < r^*$, which ends the proof.

LEMMA 9. *The root r_1 of the polynomial $u(r)$ satisfies the condition $r_1 < r^*$ if and only if*

$$y(A, B) \equiv B^3 + k_1(A)B^2 - k_2(A)B + k_3(A) < 0, \tag{5.9}$$

where

$$\begin{aligned} k_1(A) &= 2A^2 - 11A + 2, \\ k_2(A) &= A^4 + 12A^3 - 41A^2 + 12A + 1, \\ k_3(A) &= 5A^5 + 10A^4 - 39A^3 + 10A^2 + 5A. \end{aligned} \tag{5.10}$$

Proof. If $\Phi(s, t)$ defined by (3.6) attains its minimum equal to zero for $r = r_1$, then because of (3.6) we get for $r = r_1$

$$(E_1 + E_4)(c - \varrho) + E_2 + E_5 + \frac{E_3 + E_6}{c - \varrho} = 0.$$

Thus

$$E_3 + E_6 + (E_1 + E_4)(c - \varrho)^2 + (E_2 + E_5)(c - \varrho) = 0, \qquad r = r_1. \tag{5.11}$$

On the other hand, in view of (3.14)-(3.15) and because of the definition of r^*, we obtain for $r = r_1$

$$E_3 + E_6 - (E_1 + E_4)(c - \varrho)^2 = \frac{g(r)}{(1 - r^2)(1 - Br)^2}. \tag{5.12}$$

Equalities (5.11) and (5.12) imply

$$2(E_3 + E_6) + (E_2 + E_5)(c - \varrho) = \frac{g(r)}{(1 - r^2)(1 - Br)^2}, \qquad r = r_1.$$

Hence, in view of (3.7) and (3.15),

$$2(1 - A)(1 + Ar_1^2) + [(2AB - A - B)r_1^2 + A + B - 2]\frac{1 - Ar_1}{1 - Br_1} = \frac{(A - B)g(r_1)}{(1 - Br_1)^2}.$$

Thus

$$Ar_1^3 + (1 - 2A)r_1^2 - (A - 2)r_1 - 1 = \frac{g(r_1)}{1 - Br_1}. \tag{5.13}$$

Since $u(r_1) = 0$, we have

$$A^2 r_1^2 - (3A - B)r_1 + 1 = 0. \tag{5.14}$$

From (5.13) and (5.14) we obtain

$$A^2 r_1^2 + A(1 - A)^2 r_1 - A(4A - B - 2) = A\frac{g(r_1)}{r_1(1 - Br_1)}. \tag{5.15}$$

By (5.15) and (5.14) we get

$$(A^3-2A^2+4A-B)r_1-A(4A-B-2)-1 = A\frac{g(r_1)}{r_1(1-Br_1)}.$$

The polynomial $g(r)$ increases in the interval $(0,1)$ and $g(0)<0$, $g(r^*)=0$. Thus $r_1<r^*$ if and only if $g(r_1)<0$. Since the root r_1, $0<r_1<1$ of $u(r)$ exists if and only if $(A,B)\in G_2\cup G_3$, we have $A>0$.

Therefore $r_1<r^*$ if

$$(A^3-2A^2+4A-B)r_1<4A^2-AB-2A+1.$$

Putting

$$r_0=\frac{4A^2-AB-2A+1}{A^3-2A^2+4A-B},$$

we easily find that $0<r_0<1$. It follows immediately that $r_1<r^*$ if and only if $u(r_0)<0$, i. e. when

$$A^2r_0^2-(3A-B)r_0+1<0.$$

Hence we obtain after some calculations inequality (5.9).

Let $0<A<\frac{3}{5}$. For $B=-1$ we obtain

$$y(A,-1)=(1+A)\cdot\hat{y}(A), \tag{5.16}$$

where

$$\hat{y}(A)=5A^4+6A^3-33A^2+4A+2. \tag{5.17}$$

Since the derivative $\hat{y}'(A)$ decreases as A increases in the interval $(0,\frac{3}{5})$ and $\hat{y}'(0)>0$, $\hat{y}'(\frac{3}{5})<0$, $\hat{y}'(A)$ has a root $\hat{A}$ in this interval. Hence, in view of (5.17), the polynomial (5.16) has exactly one root A_0 in the interval $(0,\frac{3}{5})$.

LEMMA 10. *For every A of the interval $A_0<A<1$ the equation $y(A,B)=0$ with the unknown B* (cf. (5.9)) *has exactly one solution $B=B(A)$ in the interval $(-1,B_1(A))$ for every $A\in(A_0,\frac{3}{5})$ and in the interval $(B_2(A),B_1(A))$ for every $A\in[\frac{3}{5},1)$, $B_1(A)$ and $B_2(A)$ being given by* (5.4) *and* (5.5), *respectively.*

Proof. For $A_0<A<\frac{3}{5}$ we have $y(A,-1)<0$. If $0<A<1$, then

$$y(A,B_1)=2(1-A)^2(A^4+2A^3+2A+1)>0. \tag{5.18}$$

Thus, for $A_0<A<\frac{3}{5}$ the equation $y(A,B)=0$ has at least one solution in the interval $(-1,B_1)$.

Now, differentiating the function $y(A,B)$ twice w. r. t. B, we obtain

$$y'_B(A,B)=3B^2+2k_1(A)B-k_2(A) \tag{5.19}$$

and

$$y''_{BB}(A,B)=2[3B+k_1(A)] \tag{5.20}$$

(cf. (5.9) and (5.10)).

Since $y''_{BB}(A, B)$ is negative for $B < B_1$, $y'_B(A, B)$ decreases in the interval $(-1, B_1)$. Next we have

$$y'_B(A, B_1) = -2[A^2+(\sqrt{3}-1)A+1][A^2-(\sqrt{3}+1)A+1]. \tag{5.21}$$

It can easily be verified that

$$A^2+1 < (1+\sqrt{3})A \quad \text{for } A_0 < A < 1. \tag{5.22}$$

From (5.21) and (5.22) it follows that $y(A, B)$ increases in the interval $-1 < B < B_1$ for $A_0 < A < \frac{3}{5}$. Hence, the lemma is true in this case.

Let $\frac{3}{5} \leqslant A < 1$. Since

$$y(A, B_2) = 4(A-1)(A^3-3A^2-A+7) < 0,$$

by (5.18) the equation $y(A, B) = 0$ has at least one solution in the interval (B_1, B_2). Next we have

$$y'_B(A, B_2) = -A^4+8A^3-10A^2-24A+31$$

and

$$y''_{BA}(A, B_2) = -4A^3+24A^2-20A-24 < 0;$$

thus, because of $y'_B(1, B_2) > 0$, the function $y(A, B)$ increases in the interval (B_2, B_1), which completes the proof.

Corollary. *If $(A, B) \in G_2$, then $y(A, B) < 0$ if and only if $B < B(A)$ and $y(A, B) > 0$ for $B > B(A)$.*

Basing ourselves on Lemmas 8–10, we obtain

Theorem 6. *Let*

$$\begin{aligned} D_1 &= \{(A, B)\colon A_0 < A \leqslant 1,\ -1 < B < B(A)\}, \\ D_2 &= \{(A, B)\colon (-1 < A \leqslant A_0,\ -1 \leqslant B < A) \cup \\ &\qquad \cup (A_0 < A < 1,\ B(A) < B < A)\}, \end{aligned} \tag{5.23}$$

where $B(A)$ is the unique solution of the equation $y(A, B) = 0$ in the interval $(-1, B_1(A))$ for $A \in (A_0, \frac{3}{5})$ and in the interval $(B_2(A), B_1(A))$ for $A \in [\frac{3}{5}, 1)$, where A_0 is the unique root of the equation $y(A, -1) = 0$ in the interval $(0, \frac{3}{5})$ (cf. (5.9), (5.4), (5.5)).

Then the radius of convexity for the family $S^*(A, B)$ is

$$\text{r.c.} S^*(A, B) = \begin{cases} r_1, & \text{if } (A, B) \in D_1, \\ r_2, & \text{if } (A, B) \in D_2, \end{cases} \tag{5.24}$$

where

$$r_1 = r_1(A, B) = 2[3A-B+\sqrt{(A-B)(5A-B)}]^{-1}, \tag{5.25}$$

$$\begin{aligned} r_2 &= r_2(A, B) \\ &= \sqrt{(4-5A+B)[2A^2-3A+2-B+2(1-A)\sqrt{A^2+4A+1-2B}]^{-1}}. \end{aligned} \tag{5.26}$$

The equality r. c. $\{f\} = r_1$ *holds for the functions*

$$f^*(z) = \begin{cases} z\cdot\exp\left[\dfrac{A-B}{B}\log(1-B\varepsilon z)\right] & \textit{if } B \neq 0, \\ z\cdot\exp(-A\varepsilon z) & \textit{if } B = 0 \end{cases} \tag{5.27}$$

(cf. Theorem 4) *and* r. c. $\{f\} = r_2$ *— for the function*

$$f^{**}(z) = \begin{cases} z\cdot\exp(-\frac{1}{2}A\varepsilon^2 z^2) & \textit{if } B = 0,\ A = A^*, \\ z\cdot\exp\left\{A\left[\dfrac{\varepsilon}{d}z+\dfrac{1-d^2}{d^2}\log(1-d\varepsilon z)\right]\right\} & \\ & \textit{if } B = 0,\ A \neq A^*, \\ z\cdot\exp\left\{(A-B)\left[\dfrac{2(1+B)\varepsilon z_0 z}{d(1-B)^2\cdot(z-z_0)}+\dfrac{1}{B}\log\left(1-\dfrac{z}{z_0}\right)\right]\right\} & \\ & \textit{if } B \neq 0,\ \Delta = 0, \\ z\cdot\exp\left\{\dfrac{A-B}{2B}\left[\log W(z)+\dfrac{(1+B)d}{\sqrt{\Delta}}\log\dfrac{z_2(z_1-z)}{z_1(z_2-z)}\right]\right\} & \\ & \textit{if } B \neq 0,\ \Delta \neq 0, \end{cases} \tag{5.28}$$

where

$$\log 1 = 0, \quad \varepsilon = e^{-i\varphi}, \quad 0 \leqslant \varphi \leqslant 2\pi, \quad A^* = (14-5\sqrt{3})11^{-1},$$

$$d = \frac{Ar_2^4+3(1-A)r_2^2-1}{(1-A)(1+r_2^2)r_2}, \quad \Delta = (1-B)^2 d^2+4B, \tag{5.29}$$

$$\sqrt{\Delta} = \begin{cases} |\sqrt{\Delta}| & \textit{if } \Delta > 0, \\ i|\sqrt{-\Delta}| & \textit{if } \Delta < 0, \end{cases}$$

$$z_0 = -\frac{(1-B)d\bar{\varepsilon}}{2B}, \quad z_k = \frac{-(1-B)d+(-1)^k\sqrt{\Delta}}{2B}\bar{\varepsilon}, \quad k = 1, 2,$$

$$W(z) = -B\varepsilon^2 z^2-(1-B)d\varepsilon z+1.$$

Proof. In view of Theorem 1 and Lemma 8

$$\text{r. c. } S^*(A, B) = \begin{cases} r_1 & \text{if } (A, B)\in G_3 \text{ or } (A, B)\in G_2,\ r_1 < r^*, \\ r_2 & \text{if } (A, B)\in G_1 \text{ or } (A, B)\in G_2,\ r_1 \geqslant r^*, \end{cases}$$

where G_k ($k = 1, 2, 3$) are given by (5.6)-(5.8), r_j ($j = 1, 2$) are the roots of polynomials $u(r)$ and $v(r)$ (cf. (5.2), (5.3)), i. e. are the numbers (5.25) and (5.26), respectively, and finally r^* is the root of equation (3.36). Because of Lemma 9 the condition $r_1 < r^*$ is satisfied if inequality (5.9) is satisfied, and this is equivalent to $B < B(A)$ (Lemma 10). Hence, we obtain (5.24). For $B = B(A)$ we have $r_1 = r_2$.

Let $f^*(z)$ be a function of $\hat{S}^*(A, B)$ such that

$$\frac{zf^{*\prime}(z)}{f^*(z)} = P^*(z), \tag{5.30}$$

where $P^*(z)$ is given by (3.37). Then, from (5.30) we find

$$\frac{f^{*\prime}(z)}{f^*(z)} - \frac{1}{z} = -\frac{(A-B)\varepsilon}{1-B\varepsilon z}. \tag{5.31}$$

The functions of the variable z which appears on the right-hand side and the left-hand side of equation (5.31) are regular in the disc K; hence the integrals of these functions exist along any regular curve $\Gamma \subset K$ with the origin and the end-point at 0 and z, respectively, where $z \in K$. Thus we conclude that $f^*(z)$ is of the form (5.27).

Evidently

$$\operatorname{re}\left(1+\frac{zf^{*\prime\prime}(z)}{f^{*\prime}(z)}\right) \geqslant 0$$

for $|z| \leqslant r_1$ with equality if and only if $z = r_1\bar{\varepsilon}$. Thus $f^*(z)$ is not convex in the disc $|z| < r$ for $r > r_1$, i. e. r. c. $\{f^*\} = r_1$.

Next, let $f^{**}(z)$ be a function of $S^*(A, B)$ for which

$$\frac{zf^{**\prime}(z)}{f^{**}(z)} = P^{**}(z),$$

where $P^{**}(z)$ is given by (3.38).

Thus

$$\frac{f^{**\prime}(z)}{f^{**}(z)} - \frac{1}{z} = J(z) \tag{5.32}$$

where

$$J(z) = (A-B)\varepsilon\frac{d-\varepsilon z}{W(z)}. \tag{5.33}$$

We distinguish four cases.

1. $B = 0$, $d = 0$. Integrating (5.32) we obtain the first formula in (5.28). Since $B = 0$, we have $X_2(r_2; A, 0) = 0$. Thus

$$(4A^2-5A)r_2^4 - 2(2A^2-3A+2)r_2^2 + 4 - 5A = 0 \tag{5.34}$$

and in view of $d = 0$ we have

$$Ar_2^4 + 3(1-A)r_2^2 - 1 = 0 \tag{5.35}$$

(cf. (5.29)).

Eliminating r_2 from (5.34) and (5.35), we obtain $A = A^*$. It can easily be verified that $(A^*, 0) \in D_2$.

2. $B = 0$, $d \neq 0$. We have $W(z) = 1-d\varepsilon z$, thus because of $|d| \leqslant 1$ the function (5.33) is regular in K. Integrating (5.32), we obtain the second formula in (5.28).

3. $B \neq 0$. $\Delta = 0$. In this case $W(z) = -B\varepsilon^2(z-z_0)^2$, where $B<0$ and $z_0 \neq 0$. Next, we obtain $|z_0| = \sqrt{-B^{-1}} \geqslant 1$. Thus

$$J(z) = -\frac{(A-B)\bar{\varepsilon}}{Bz_0^2}\frac{d-\varepsilon z}{\left(1-\dfrac{z}{z_0}\right)^2}$$

is a regular function in K. Integrating (5.32) we obtain the third formula in (5.28).

4. $B \neq 0$, $\Delta \neq 0$. The polynomial $W(z)$ can be represented in the form

$$W(z) = \left(1-\frac{z}{z_1}\right)\left(1-\frac{z}{z_2}\right).$$

We state that $|z_k| \geqslant 1$ for $k = 1, 2$. If $\Delta > 0$ and $B > 0$, then $\varepsilon z_1 \leqslant -1$. Supposing the contrary, we would have $\Delta < 2B-(1-B)d$ and hence $(1-B)(1+d) < 0$, which is impossible. Similarly we prove that $\varepsilon z_2 \geqslant 1$. As in the case just considered, we find that, for $\Delta > 0$ and $B < 0$, $\varepsilon z_1 \geqslant 1$ and $\varepsilon z_2 \leqslant -1$. If $\Delta < 0$, then $B < 0$ and $|z_k|^2 = -B^{-1} \geqslant 1$. Thus $|z_k| \geqslant 1$ for $k = 1, 2$ in every case. Hence, $J(z)$ is regular in K. Integrating (5.32), we obtain the fourth formula in (5.27).

Evidently, in each of the four cases considered above, we have

$$\mathrm{re}\left(1+\frac{zf^{**''}(z)}{f^{**'}(z)}\right) \geqslant 0$$

for $|z| \leqslant r_2$, with equality if and only if $z = r_2\bar{\varepsilon}$. Thus r. c. $\{f^{**}\} = r_2$, and this completes the proof.

Applying Theorem 6 to the case where $A = 1-2a$ and $B = -1$, we obtain the result for the class S_a^* given by Zmorovič [9]. The problems of the radius of convexity for S^* and $S_{\frac{1}{2}}^*$ have first been solved by Nevanlinna [5] and Mac Gregor [3], respectively. If $A = 1$, $B = 1/M-1$ or $A = \beta$, $B = -\beta$, then we obtain the corresponding theorems on r. c. $S^*(M)$ [1] and r. c. $S^{*(\beta)}$ [6], respectively. For the class $S_{(\beta)}^*$ we have

$$\text{r. c. } S_{(\beta)}^* = \begin{cases} r_1, & \text{if } \beta_0 < \beta \leqslant 1, \\ r_2, & \text{if } 0 < \beta \leqslant \beta_0, \end{cases}$$

where $r_1 = r_1(\beta, 0)$, $r_2 = r_2(\beta, 0)$ and

$$\beta_0 = \frac{(3-\sqrt{5})(1+\sqrt{6})}{2\sqrt{5}}.$$

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