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Some applications of functions of several complex variables to Toeplitz and subnormal operators

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Abstract. The paper consists of two parts. In the first part we shall prove a certain theorem on Toeplitz operators in a strongly pseudoconvex domain (this is a generalization of our previous result [6]), using the recent result of A. Aytuna and A. M. Chollet [1]. In the second part we shall prove a theorem of approximation type for the algebra $A(\Omega)$ (the Banach algebra of all functions continuous in $\bar{\Omega}$ and holomorphic in Ω), where Ω is a bounded and symmetric domain in C^n. Applications of this theorem to the case of an n-tuple of commuting subnormal operators and to Toeplitz operators are also given.

I. Let $D \subset C^n$ be a bounded strongly pseudoconvex domain. Denote by $H^\infty(\partial D)$ the Banach algebra of all functions on ∂D which are boundary values of bounded holomorphic functions in D. If $C(\partial D)$ denotes the Banach algebra of all continuous functions on ∂D, the recent result of A. Aytuna and A. M. Chollet says that $H^\infty(\partial D)+C(\partial D)$ is a Banach algebra with ess-sup norm. This a generalization of the previous result of W. Rudin, who proved the same theorem for the unit ball in C^n. The theorem of Rudin was applied in [6] to Toeplitz operators on odd spheres. Now we shall give an analogous application of the theorem of Aytuna and Chollet to Toeplitz operators in a strongly pseudoconvex domain.

First of all, we recall the definition of a Toeplitz operator. Let $L^2(\partial D)$ be the Hilbert space of all complex functions which are square integrable with respect to the surface Lebesgue measure μ on ∂D. Denote by $H^2(\partial D)$ the Hardy space of all functions in $L^2(\partial D)$ which are boundary values of functions holomorphic in D (see [10] for the definition and properties of $H^2(\partial D)$). Let P be the orthogonal projection from $L^2(\partial D)$ onto $H^2(\partial D)$. We define the Toeplitz operator T_φ on $H^2(\partial D)$ by putting, for any bounded measurable function φ on ∂D,

$$T_\varphi f = P(\varphi \cdot f).$$

Using the above-mentioned result of Aytuna–Chollet, we shall give a characterization of the Banach algebra generated by the family

$$\{T_\varphi,\ \varphi \in H^\infty(\partial D)+C(\partial D)\}.$$

The idea of proof is the same as in [6]. To begin with, note that combining the results of [4], § 5.4, and Venugopalkrishna's proof of [11], Theorem 2.1, one can prove the following lemma.

LEMMA 1. *If* $s \in C(\partial D)$, *then* $(I-P)\varphi\colon H^2(\partial D) \to L^2(\partial D)$ *is a compact operator.*

Remark 1. In the case of the unit ball the above lemma was proved in [6] with use of a result of Coburn [2].

Now we recall the definition and some properties of the Poisson–Szegö kernel,

$$P(z, \xi) = |S(z, \xi)| \cdot S(z, z)^{-1}, \quad z \in D,\ \xi \in \partial D,$$

where $S(z, \xi)$ is the Szegö kernel of D. In particular, $P(z, \xi)$ satisfies the following two conditions:

(i) For every $z \in D$, $\int\limits_{\partial D} P(z, \xi)\, d\mu(\xi) = 1$.

(ii) If $f \in C(\partial D)$, then for any $\mathring{\xi} \in \partial D$ and $D \ni z_n \to \mathring{\xi}$ we have

$$\int\limits_{\partial D} P(z_n, \xi) f(\xi)\, d\mu \xrightarrow[n\to\infty]{} f(\mathring{\xi}).$$

(See [10].)

We shall need a little more in order to adopt the proof from [6]. We have the following lemma.

LEMMA 2. *Let* D *be a strongly pseudoconvex domain in* C^n *with a Szegö kernel* $S(z, \xi)$. *If* $\mathring{\xi} \in \partial D$ *and* $D \in \lambda_m \to \mathring{\xi}$, *then the sequence*

$$f_m(\xi) = S(\lambda_m, \xi) S(\lambda_m, \lambda_m)^{-1/2}$$

has the properties:

1° $f_m \in H^2(\partial D)$ *and* $\|f_m\| = 1$,

2° $f_m \to 0$.

Proof. 1° follows immediately from property (i).

We shall prove 2°. First note that

$$(*) \qquad S(\lambda_m, \lambda_m) \to +\infty.$$

Suppose not; then for an arbitrary basis $\{\psi_i\} \subset H^2(D)$ (not the boundary values; see [10] for the definition) we can write

$$S(\lambda_m, \lambda_m) = \sum_{i=1}^{\infty} |\psi_i(\lambda_m)|^2 < M < +\infty \quad \text{for } m = 1, 2, \ldots$$

Therefore the sequence η_m of functionals on $H^2(D)$ given by

$$\eta_m(f) = f(\lambda_m)$$

satisfies the inequality $\|\eta_m\| \leqslant \sqrt{M}$.

Indeed, for any $h \in H^2(D)$

$$|\eta_m(h)| = \left|\eta_m\left(\sum_k c_k \psi_k\right)\right| = \left|\sum_k c_k \psi_k(\lambda_m)\right| \leqslant \left(\sum_k |c_k|^2\right)^{1/2}\left(\sum_k |\psi_k(\lambda_m)|^2\right)^{1/2} \leqslant M^{1/2} \cdot \|h\|.$$

But this is impossible because one can always find a sequence of functions $h_m \in H^2(D)$, $\|h_m\| = 1$, such that

$$|\eta_m(h_m)| = |h_m(\lambda_m)| \xrightarrow[m\to\infty]{} +\infty.$$

(It is sufficient to choose g_m holomorphic in a neighbourhood of D and such that $|g_m(\mathring{\xi})| > 1$, $|g_m| < 1$ in $\bar{D}\backslash U_m$, where U_m is a neighbourhood base at $\mathring{\xi}$. See [5], p. 275. Let $n_m \in N$ be such that $\|g_m^{n_m}\| \geqslant 1$ and define: $h_m = g_m^{n_m} \cdot \|g_m^{n_m}\|^{-1}$.)

Thus, for $H^2(\partial D) \ni g = \sum_k a_k \psi_k$, we can write

$$|(f_m, g)| = \left|\int f_m(\xi) g(\xi) d\mu(\xi)\right| = \left|\sum_k a_k \psi_k(\lambda_m) \cdot S(\lambda_m, \lambda_m)^{-1/2}\right|$$

$$\leqslant S(\lambda_m, \lambda_m)^{-1/2}\left[\sum_{k=1}^{N} |a_k \psi_k(\lambda_m)| + \sum_{k=N+1}^{\infty} |a_k \psi_k(\lambda_m)|\right]$$

$$\leqslant S(\lambda_m, \lambda_m)^{-1/2} \cdot \sum_{k=1}^{N} |a_k \psi_k(\lambda_m)| + \left(\sum_{k=N+1}^{\infty} |a_k|^2\right)^{1/2}.$$

Fix $\varepsilon > 0$ and take N such that $\left(\sum_{k=N+1}^{\infty} |a_k|^2\right)^{1/2} < \varepsilon/2$. Then from the above inequality and from (*) we get $|(f, g_m)| < \varepsilon$, for $m \geqslant m_0$. The proof is completed.

Now we are ready to formulate the above-mentioned characterization of the algebra $\{T_\varphi, \varphi \in H^\infty(\partial D) + C(\partial D)\}$.

THEOREM 1. *Let $\mathcal{K}$ be the ideal of all compact operators in $H^2(\partial D)$. Denote by $\mathcal{A}$ the Banach algebra generated by the family $\{T_\varphi, \varphi \in H^\infty(\partial D) + C(\partial D)\}$. The quotient algebra $\mathcal{A}/\mathcal{K}$ is isometrically isomorphic with $H^\infty + C$ by the map $\mathscr{C}: \varphi \to \{T_\varphi + K, K \in \mathcal{K}\}$.*

Proof. Using Lemmas 1 and 2, one can repeat our previous proof from [6] step by step (replacing the sequence h_m from [6], by the sequence f_m from Lemma 2).

Remark 2. In the case where D is the unit ball in C^n the above theorem was also proved by A. M. Davie and N. P. Jewell in [3], but their proof cannot be extended to an arbitrary strongly pseudoconvex domain.

We also have the following corollaries:

Corollary 1. *If $\varphi \in H^\infty + C$ and φ is invertible in this algebra, then the operator T_φ is Fredholm.*

Corollary 2. *If $\varphi \in H^\infty + C$, then*

$$\|T_\varphi\| = \|T_\varphi\|_s = \|\varphi\|_\infty \quad (\|T\|_s \text{ – the spectral norm of } T).$$

II. In order to formulate results of this part we shall recall some definitions.

Let $A \subset C(X)$ be a function algebra (X a compact Hausdorff space). The following definition was introduced by W. Mlak in [9].

Definition M. Let $p \geqslant 1$. We say that a function algebra $A \subset C(X)$ is *p-approximating in modulus* if the linear combinations $\sum c_i |u_i|^p$, $c_i \geqslant 0$ are dense in the cone $C^+(X)$ (non-negative, continuous functions on X).

Actually, we shall need a little weaker notion. Namely, we introduce the following technical condition. For a given finite non-negative Borel measure μ on X we shall call the function algebra $A \subset C(X)$ *(p, μ)-approximating in modulus* ($p \geqslant 1$) if the linear combinations $\sum_i c_i |u_i|^p$ are dense in the space $L^1_+(\mu)$ (of non-negative μ-integrable functions on X).

Let Ω be an arbitrary bounded domain in C^n, symmetric with respect to zero. If Aut Ω is the group of holomorphic automorphisms of Ω and K is the isotropy subgroup of Aut Ω, then there exists a K-invariant measure μ on the Bergman–Shilov boundary ∂_Ω of Ω. Moreover, K acts transitively on ∂_Ω. See [8]. Let $P(\xi, z)$ ($\xi \in \partial_\Omega$, $z \in \Omega$) be the Poisson–Szegö kernel of Ω. The following properties of $P(\xi, z)$ were proved in [8]:

(i) $P(\xi, z) \geqslant 0$;

(ii) $\int P(\xi, z)\, d\mu(\xi) = 1, z \in \Omega$;

(iii) $\forall \eta > 0 \lim\limits_{z \to \mathring{\xi}} \int\limits_{|\xi - \mathring{\xi}| > \eta} P(\xi, z)\, d\mu(\xi) = 0.$

Now we are ready to prove

Theorem 2. *Let Ω be as above. If $A(\Omega)$ is the Banach algebra of all continuous functions on $\bar{\Omega}$ and holomorphic in Ω, then for any finite, non-negative Borel measure v on ∂_Ω the algebra $A(\Omega)|_{\partial_\Omega}$ is $(2, v)$-approximating in modulus.*

Proof. We shall prove a little more. Namely, for an arbitrary function $v \in C^+(\partial_\Omega)$ there exists a sequence $h_{mi} \in A(\Omega)$ such that

(a) $\|\sum_i |h_{mi}|^2\|_\infty < M < +\infty$, $m = 1, 2, \ldots$

(b) $\forall_\xi \in \partial_\Omega, \sum_i |h_{mi}(\xi)|^2 \to v(\xi)$.

From (a) and (b) the assertion of the theorem follows easily.

Now let $v \in C^+(\partial_\Omega)$. We define the sequence

$$v_p(\xi) = \int\limits_{\partial_\Omega} P(\zeta, r_p \xi) v(\zeta)\, d\mu(\zeta),$$

where $0 < r_p < 1, r_p \xrightarrow[p\to\infty]{} 1$. Then for any fixed $\mathring{\xi} \in \partial_\Omega$

$$(*) \qquad v_p(\mathring{\xi}) \to v(\mathring{\xi}).$$

Indeed, by (ii) and (iii) we have for $\varepsilon > 0$ and for $\eta > 0$

$$\begin{aligned}
&\left| \int P(\zeta, r_p\mathring{\xi})v(\zeta)\,d\mu(\zeta) - v(\mathring{\xi})\right| \\
&\leqslant \int P(\zeta, r_p\mathring{\xi})|v(\zeta)-v(\mathring{\xi})|\,d\mu(\zeta) \\
&\leqslant 2\|v\|_\infty \cdot \int\limits_{|\zeta-\mathring{\xi}|>\eta} P(\zeta, r_p\mathring{\xi})\,d\mu(\zeta) + \int\limits_{|\zeta-\mathring{\xi}|\leqslant\eta} P(\zeta, r_p\mathring{\xi})|v(\zeta)-v(\mathring{\xi})|\,d\mu(\zeta) \\
&\leqslant \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon, \quad \text{if } p \geqslant p_0.
\end{aligned}$$

Now note that there exists a partition $\{\sigma_i\}$ of ∂_Ω such that

$$(**) \qquad \Big\| v_p(\xi) - \sum_i P(\zeta_i, r_p\xi)v(\zeta_i)\mu(\sigma_i)\Big\|_\infty < \varepsilon,$$

where $P(\zeta_i, r_p\xi) = |S(\zeta_i, r_p\xi)|^2 \cdot S(r_p\xi, r_p\xi)^{-1}$. But $S(r_p\xi, r_p\xi)$ does not depend on ξ! Indeed, since $S(z,z) = S(kz,kz)$ for every $z \in \Omega$ and every $k \in K$ and since K acts transitively on ∂_Ω, the above claim is true.

Therefore we can write

$$\begin{aligned}
P(\zeta_i, r_p\xi) &= S(r_p\xi, r_p\xi)^{-1} \cdot \Big| \sum_{s=1}^{\infty} \psi_s(\zeta_i)\overline{\psi_s(r_p\xi)}\Big|^2 \\
&= S(r_p\xi, r_p\xi)^{-1} \cdot \Big| \sum_{s=1}^{\infty} \overline{\psi_s(\zeta_i)}\psi_s(r_p\xi)\Big|^2,
\end{aligned}$$

where $\{\psi_s\}$ is an arbitrary basis in $H^2(\Omega)$. See [10]. Denoting $S(r_p\xi, r_p\xi)^{-1} = \alpha_p$ and $\sum_{s=1}^{\infty} \overline{\psi_s(\zeta_i)}\psi_s(r_p\xi) = g_{pi} \in A(\Omega)|_{\partial_\Omega}$, we have: $P(\zeta_i, r_p\xi) = \alpha_p |g_{pi}(\xi)|^2$.

Relations $(*)$, $(**)$ and the above equality prove that the sequence

$$h_{mi} = g_{pi} \cdot [v(\zeta_i)\mu(\sigma_i)\alpha_p]^{1/2}$$

satisfies (a) and (b). The proof is complete. In fact we have proved more that $A(\Omega)$ is p-approximating in modulus.

Now we shall show two applications of Theorem 2.

(a) Lifting of the commutant of an n-tuple of subnormal operators.

Let $T_1, \ldots, T_n$ be an n-tuple of commuting subnormal operators in a complex Hilbert space H, i.e., suppose there exists a larger Hilbert space $K \supset H$ and an n-tuple $N_1, \ldots, N_n$ of commuting normal operators such that

$$T_1^{k_1} \ldots T_n^{k_n} f = N_1^{k_1} \ldots N_n^{k_n} f, \quad f \in H,$$

where k_i are non-negative integers. We say that a normal extension $(N_1, \ldots, N_n)$ is minimal, if

$$K = \bigvee_{(k_1, \ldots, k_n) \in N^n} N_1^{*k_1} \ldots N_n^{*k_n} H.$$

We are interested in the following problem: whether an operator S commuting with T_i $(i = 1, \ldots, n)$ extends uniquely to an operator S commuting with a minimal normal extension $(N_1, \ldots, N_n)$: $\tilde{S}N_i = N_i\tilde{S}$, $i = 1, \ldots, n$.

Let X be a compact set in C^n such that $X = \overline{\operatorname{Int} X}$ and $\operatorname{Int} X$ is a bounded symmetric domain. Assume that $T\colon A(X)|_{\partial_{\operatorname{Int}X}} \to L(H)$ is a representation of $A(X)$ induced by the n-tuple $(T_1, \ldots, T_n)$, i.e., $T(z_i) = T_i$, $i = 1, \ldots, n$, T is a homomorphism and $\|T(u)\| \leqslant \|u\|_{\partial_{\operatorname{Int}X}}$ (1). (Note that $A(X) = P(X)$, where $P(X)$ is the Banach algebra of all functions on X which are uniformly approximated by polynomials on X.)

Example. As an example of such a set X it is enough to take $X = \sigma(T_1, \ldots, T_n)$ – an arbitrary joint spectrum of the n-tuple $T_1, \ldots, T_n$ for which the spectral mapping theorem holds true, i.e., such that $p(\sigma(T_1, \ldots, T_n)) = \sigma(p(T_1, \ldots, T_n))$, p denoting any polynomial. The conditions

(a) $\overline{\operatorname{Int} \sigma(T_1, \ldots, T_n)} = \sigma(T_1, \ldots, T_n)$,

(b) $\operatorname{Int} \sigma(T_1, \ldots, T_n)$ is symmetric,

are to be assumed additionally.

Using the inequality $\|p(T_1, \ldots, T_n)\| \leqslant \|p\|_{\partial_{\operatorname{Int}X}}$ we obtain, by continuity, a subnormal representation of the function algebra $P(X)|_{\partial_{\operatorname{Int}X}} = A(X)|_{\partial_{\operatorname{Int}X}}$.

The answer to the problem of lifting of the commutant of an n-tuple $(T_1, \ldots, T_n)$ is positive, if we assume additionally that $\sigma(N_1, \ldots, N_n) \subset \partial_{\operatorname{Int}X}$. Then, applying Theorem 2 and repeating the reasoning of our previous paper [7], we obtain the following corollary.

Corollary 1. *Let X be as above. Suppose that $T\colon A(X)|_{\partial_{\operatorname{Int}X}} \to L(H)$ is a representation induced by an n-tuple $(T_1, \ldots, T_n)$ of commuting subnormal operators with a minimal normal extension $(N_1, \ldots, N_n)$. Assume that $\sigma(N_1, \ldots, N_n) \subset \partial_{\operatorname{Int}X}$. Then every $S \in \{T_1, \ldots, T_n\}'$ (the commutant) extends uniquely to $\tilde{S} \in \{N_1, \ldots, N_n\}'$ and $\|\tilde{S}\| = \|S\|$.*

Remark 2. In the case of X being the unit ball in C^n the above corollary was proved in [7].

(b) Spectral inclusion theorem for Toeplitz operators.

This is another application of Theorem 2.

We shall prove a generalization of the spectral inclusion theorem for Toeplitz operators in Hardy space over a bounded symmetric domain.

Let Ω be a bounded symmetric domain in C^n. For any non-negative finite Borel measure ν on $\partial_{\operatorname{Int}\Omega}$ we define the Hardy space $H^2(\nu)$ as the $L^2(\nu)$-closure of the algebra $A(\Omega)|_{\partial_{\operatorname{Int}\Omega}}$. Note that if $\nu = \mu$, where μ is the K-invariant measure mentioned before Theorem 2, then $H^2(\mu)$ is the classical

(1) See [6] for a general definition.

Hardy space over a symmetric domain. The definition of the Toeplitz operator is obvious. For $\varphi \in L^\infty(v)$ we define the operator T_φ on $H^2(v)$ by $T_\varphi f = P(\varphi \cdot f)$, where $P\colon L^2(v) \to H^2(v)$ is the orthogonal projection. Denote by L_φ the operator of multiplication by φ in $L^2(v)$.

COROLLARY 2. *If $\varphi \in L^\infty(v)$, then $\sigma_\pi(T_\varphi) \supset \sigma(L_\varphi)$, where $\sigma_\pi(T_\varphi)$ is the approximate point spectrum.*

Proof. The proof is based on Theorem 2. Denote, for simplicity, $\partial_{\mathrm{Int}\,\Omega} = Z$. Assume that $0 \notin \sigma_\pi(T_\varphi)$; then

$$\int |\varphi|^2 |f|^2 dv \geqslant \|T_\varphi f\|^2 \geqslant \varepsilon \cdot \|f\|^2, \tag{1}$$

for a certain $\varepsilon > 0$ and every $f \in A(\Omega)$.

According to the proof of Theorem 2, for any function $v \in C^+(Z)$ we can find a sequence $\{u_{im}\} \subset A(\Omega)$ such that

$$\sum_i |u_{im}|^2 \leqslant M, \quad m = 1, 2, \ldots, \tag{i}$$

$$\sum_i |u_{im}(x)|^2 \xrightarrow[m\to\infty]{} v(x) \quad \text{for } x \in Z. \tag{ii}$$

By (1) we get

$$\int |\varphi|^2 \sum_i |u_{im}|^2 dv = \sum_i \int |\varphi|^2 |u_{im}|^2 dv \geqslant \varepsilon \sum_i \int |u_{im}|^2 dv$$
$$= \varepsilon \int \sum_i |u_{im}|^2 dv.$$

Thus by the Lebesgue dominated convergence theorem

$$\int |\varphi|^2 v dv \geqslant \varepsilon \int v dv.$$

Since v is arbitrary, we have $|\varphi|^2 \geqslant \varepsilon$ (v – almost everywhere). The proof is complete.

Remark 3. As in the classical case of the unit disc, Corollary 2 has some interesting consequences (for example $\|T_\varphi\| = \|\varphi\|_\infty$).

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