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On the existence and uniqueness of solutions of a multipoint boundary value problem

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Abstract. A class of boundary value problems is shown for which the uniqueness of solutions implies the existence. The main theorem is stated for an arbitrary system of ordinary non-linear differential equations and then applied to a special case of two equations on the plane.

1. During the last decade it has been proved that for some boundary value problems related with systems of non-linear ordinary differential equations the uniqueness of solutions implies the existence. Typical results of this kind may be found in [1]–[3], [5] and [7]. The purpose of the present paper is to point out a new family of problems having the same property. The family contains as special cases the Nicoletti boundary value problem ([9], [10]) and the interpolation problem for an n-th order differential equation ([2], [12]). It is necessary, however, to stress that our results do not generalize the uniqueness-existence theorems for two-point and three-point boundary value problems which have been proved in [3], [5] and [6]. The reason is that in our statements the pure n-point problem $x(t_i) = r_i$ is always imbedded in a more general class of problems. On the other hand, applying our Theorem 1 to the system of two differential equations on the plane we obtain a corollary, which is closely related to the interesting results of E. Tutaj [6] and P. Waltman [13].

2. We consider a system of ordinary differential equations

$$x_i' = f_i(t, x_1, \ldots, x_m), \quad i = 1, \ldots, m \tag{1}$$

and the boundary value conditions

$$\sum_{j=1}^{m} a_{ij} x_j(t_i) = r_i, \quad i = 1, \ldots, m. \tag{2}$$

We assume that the functions $f_i(t, x_1, \ldots, x_m)$ are defined and continuous in the strip

$$D = \Delta \times R^m, \quad \Delta \subset R,$$

where Δ is an interval (bounded or not) of the real line, which contains all the points t_i. We assume, moreover, that f_i satisfy the following condition:

(C) For each point $(t_0, r_1, \ldots, r_m) \in D$ there exists exactly one solution $x_1, \ldots, x_m$ of system (1) defined on the whole interval Δ and such that $x_i(t_0) = r_i$ $(i = 1, \ldots, m)$.

Let $\mathscr{A}$ be a subset of the m^2-dimensional space R^{m^2}, which can be identified with the space of all $m \times m$-matrices. We shall consider the boundary value conditions (2) with all possible coefficients a_{ij} such that the matrix (a_{ij}) belongs to $\mathscr{A}$.

THEOREM 1. *Let a system* $(f_1, \ldots, f_m; t_1, \ldots, t_m; \mathscr{A})$ *be given, where* $f_i\colon D \to R$ *are continuous functions satisfying condition* (C), t_i *is an arbitrary sequence of points from* Δ *and* $\mathscr{A}$ *is an open subset of* $m \times m$*-matrices. Assume that for each matrix* $(a_{ij}) \in \mathscr{A}$ *and each vector* $(r_1, \ldots, r_m)$ *there exists at most one solution of* (1)–(2). *Then for each* $(r_1, \ldots, r_m)$ *and each* $(a_{ij}) \in \mathscr{A}$ *there exists exactly one solution of* (1)–(2).

Proof. Fix $t_0 \in \Delta$ and denote by $x(t, c)$ $(x = (x_1, \ldots, x_m),\ c = (c_1, \ldots, c_m))$ the solution of (1) satisfying the initial conditions $x_i(t_0) = c_i$. By assumption (C) such a solution exists and for each integer i $(i = 1, \ldots, m)$ and each $t \in \Delta$ the function $c \to x_i(t, c)$ is continuous. Let $(a_{ij}) \in \mathscr{A}$ be given. Define the mapping

$$R^m \ni c \to u(c) \in R^m, \qquad u = (u_1, \ldots, u_m),$$

by the formula

$$u_i(c) = \sum_{j=1}^{m} a_{ij} x_j(t_i, c), \qquad i = 1, \ldots, m.$$

The uniqueness of solutions of the boundary value problem (1)–(2) implies that u is an injection. Thus, according to the Brouwer open mapping theorem, the range of u is an open set. We are going to show that $u(R^m)$ is also closed.

Suppose the contrary. Then there exists a sequence $\{c^k\} \subset R^m$ such that $u(c^k)$ is convergent to a point $r \in u(R^m)$ and $\|c^k\| \to \infty$, where $\|\ \|$ denotes the Euclidean norm in R^m. We claim that

$$\lim_{k\to\infty} \sum_{j=1}^{m} |x_j(t_i, c^k)| = \infty, \qquad i = 1, \ldots, m. \tag{3}$$

Suppose not. Then there exists an integer i such that each sequence $\{x_j(t_i, c^k)\}$ $(j = 1, \ldots, m)$ is bounded. Passing to subsequences, if necessary, we may assume that each sequence $\{x_j(t_i, c^k)\}$ is convergent to a number s_j. Consider the solution $\bar{x}_j$ of (1) satisfying the initial conditions $\bar{x}_j(t_i) = s_j$.

From the continuous dependence on Cauchy problems it follows that

$$\lim_{k\to\infty} x_j(t_0, c^k) = \bar{x}_j(t_0), \quad j = 1, \ldots, m. \tag{4}$$

On the other hand, by the definition of $x(t, c)$ we have $x(t_0, c^k) = c^k$. Thus (4) cotradicts our assumption that $\|c^k\|\to\infty$ and proves the claim.

From (3) it follows that for each integer i there exists an integer $j(i)$ such that

$$\lim_{k\to\infty} |x_{j(i)}(t_i, c^k)| = \infty, \quad i = 1, \ldots, m.$$

Passing to subsequences, if necessary, we may assume that

$$|x_{j(i)}(t_i, c^k) - x_{j(i)}(t_i, c^{k+1})| > 1 \tag{5}$$

for $i = 1, \ldots, m$; $k = 1, 2, \ldots$ Now write

$$h_i^k = \frac{u_i(c^k) - u_i(c^{k+1})}{x_{j(i)}(t_i, c^k) - x_{j(i)}(t_i c^{k+1})}$$

and consider the matrix $\bar{a}_{ij} = a_{ij} + h_i^k \delta_{ij(i)}$, where δ_{ij} is the Kronecker symbol. Since $\{u_i(c^k)\}$ is convergent, $h_i^k\to 0$ as $k\to\infty$ and $(\bar{a}_{ij}) \in \mathscr{A}$ for k sufficiently large. Using the definition of $\bar{a}_{ij}$ it is easy to verify that

$$\sum_{j=1}^{m} \bar{a}_{ij} x_j(t_i, c^k) = \sum_{j=1}^{m} \bar{a}_{ij} x_j(t_i, c^{k+1}), \quad i = 1, \ldots, m.$$

The equality above together with (5) shows that $x(t, c^k)$ and $x(t, c^{k+1})$ are two different solutions of (1) satisfying the same boundary value conditions corresponding to the matrix $(\bar{a}_{ij}) \in \mathscr{A}$. This gives a contradiction and finishes the proof of the fact that $u(R^m)$ is a closed set. Since the range of u is both open and closed, the mapping $u\colon R^m \to R^m$ is onto and so, for every $r \in R^m$, the equation $u(c) = r$ has a (precisely one) solution $c = c(r)$. The corresponding function $x\big(t, c(r)\big)$ is the desired solution of the boundary value problem (1)–(2).

3. In the case of two differential equations on the plane

$$x_i' = f_i(t, x_1, x_2), \quad i = 1, 2, \tag{6}$$

the statement of Theorem 1 may be substantially simplified. Instead of (2), consider the boundary value conditions of the form

$$ax_1(0) + x_2(0) = r_1, \quad bx_1(1) + x_2(1) = r_2, \tag{7}$$

where (a, b) belongs to an open set $G \subset R$. We have the following

THEOREM 2. *Let a triple $(f_1, f_2; G)$ be given, where $f_i\colon [0, 1]\times R^2 \to R$ are continuous functions satisfying condition* (C) *(with $D = [0, 1]\times R^2$) and G is an open subset of R^2. Assume that for each choice $(a, b) \in G$ and*

$(r_1, r_2) \in R^2$ there exists at most one solution of the boundary value problem (6)–(7). Then for each $(a, b) \in G$ and $(r_1, r_2) \in R^2$ there exists exactly one solution of (6)–(7).

Proof. Fix $(a_0, b_0) \in G$ and choose an $\varepsilon > 0$ such that the square

$$S_\varepsilon = \{(a, b): \ |a-a_0| < \varepsilon, \ |b-b_0| < \varepsilon\}$$

is contained in G. Denote by $\mathscr{A}_\delta \subset R^4$ the set of matrices (a_{ij}) satisfying the inequalities

$$|a_{11}-a_0| < \delta, \quad |a_{12}-1| < \delta, \quad |a_{21}-b_0| < \delta, \quad |a_{22}-1| < \delta.$$

Now consider the boundary value conditions (2) for $m = 2$, that is,

$$a_{11}x_1(0)+a_{12}x_2(0) = r_1, \quad a_{21}x_1(1)+a_{22}x_2(1) = r_2. \tag{8}$$

It is easy to see that for sufficiently small δ problem (6)–(8) with $(a_{ij}) \in \mathscr{A}_\delta$ admits at most one solution. In fact, (8) is equivalent to

$$\frac{a_{11}}{a_{12}}x_1(0)+x_2(0) = \frac{r_1}{a_{12}}, \quad \frac{a_{21}}{a_{22}}x_1(1)+x_2(1) = \frac{r_2}{a_{22}},$$

and for each $(a_{ij}) \in \mathscr{A}_\delta$ with $\delta < \varepsilon/(2+\varepsilon+|a_0|+|b_0|)$ we have

$$\left|\frac{a_{11}}{a_{12}}-a_0\right| < \varepsilon, \quad \left|\frac{a_{21}}{a_{22}}-b_0\right| < \varepsilon,$$

which means that the point $(a_{11}/a_{12}, a_{21}/a_{22})$ belongs to S_ε. Thus, according to Theorem 1, problem (6)–(8) for $(a_{ij}) \in \mathscr{A}_\delta$ and $(r_1 r_2) \in R^2$ admits exactly one solution. In particular, setting $a_{11} = a_0, a_{12} = 1, a_{21} = b_0, a_{22} = 1$ we obtain the desired solution of problem (6)–(7) (with $a = a_0$ and $b = b_0$).

Remark. In the statement of Theorem 2 the boundary value conditions (7) may be, obviously, replaced by

$$x_{11}(0)+ax_{12}(0) = r_1, \quad x_{12}(1)+bx_{22}(1) = r_2 \tag{9}$$

or, which is more interesting, by

$$ax_{11}(0)+x_{12}(0) = r_1, \quad x_{12}(1)+bx_{22}(1) = r_2. \tag{10}$$

4. Now we shall examine Theorem 2 in some special cases. Our first application is related with the two-point boundary value problem for the second order differential equation and is interesting mainly from the methodological point of view. The same result may be proved by different methods (e.g. see [8]) but, in general, the proofs are much more complicated. The second application is stimulated by the classical results of Krasnosel'skiĭ [4].

Example 1. Consider the second order differential equation

$$x'' = f(t, x, x') \tag{11}$$

with the boundary value conditions

$$x(0)+ax'(0) = r_1 \quad x(1)+bx'(1) = r_2. \tag{12}$$

Assume that the function f is continuous in the strip $D = [0,1]\times R^2$ and satisfies the following

CONDITION (C). For each point $(t_0, r_0, r_1)\in D$ there exists exactly one solution x of equation (11) defined on [0,1] and such that $x(t_0) = r_0$, $x'(t_0) = r_1$.

From Theorem 2 it follows that problem (11)–(12) admits exactly one solution if $f(t, x_1, x_2)$ is increasing in x_1 and $a<0$, $b>0$. In fact, the uniqueness of solutions follows from the maximum principle (e.g. see [11]) and the set $G = \{(a,b)\colon a<0, b>0\}$ is evidently open.

EXAMPLE 2. Assume that the functions $f_i\colon D\to R$ ($D = [0,1]\times R^2$) are continuous, satisfy condition (C′) and admit continuous partial derivatives $f_{ij} = \partial f_i/\partial x_j$. Assume, moreover, that for each triple (t, x_1, x_2) the quadratic form

$$\omega(u_1, u_2) = f_{11}u_1^2+(f_{12}-f_{21})u_1u_2-f_{22}u_2^2$$

is semipositively defined ($\omega\geqslant 0$). Using Theorem 2 we shall show that for a, b satisfying $a^2<1$, $b^2>1$ the boundary value problem (6)–(7) admits exactly one solution. The set $G = \{(a,b)\colon a^2<1, b^2>1\}$ is open and it remains to prove only the uniqueness. Let x_i and y_i ($i = 1,2$) denote two solutions of (6)–(7). Write $u_i = y_i-x_i$ and $v = u_1^2-u_2^2$. From the boundary conditions (7) it follows that

$$v(0) = \big(u_1(0)\big)^2-\big(u_2(0)\big)^2 = (1-a^2)\big(u_1(0)\big)^2\geqslant 0, \tag{13}$$

$$v(1) = \big(u_1(1)\big)^2-\big(u_2(1)\big)^2 = (1-b^2)\big(u_1(1)\big)^2\leqslant 0. \tag{14}$$

On the other hand,

$$\begin{aligned}\tfrac{1}{2}v'(t) &= u_1u_1'-u_2u_2'\\ &= u_1[f_1(t,y_1,y_2)-f_1(t,x_1,x_2)]-u_2[f_2(t,y_1,y_2)-f_2(t,x_1,x_2)]\\ &= u_1\int_0^1[u_1f_{11}(s)+u_2f_{12}(s)]ds-u_2\int_0^1[u_1f_{21}(s)+u_2f_{22}(s)]ds,\end{aligned}$$

where

$$f_{ij}(s) = f_{ij}(t, x_1+su_1, x_2+su_2).$$

Consequently

$$v'(t) = 2\int_0^1\omega(u_1,u_2)ds\geqslant 0$$

and inequalities (13) and (14) cannot be strict. Therefore we have $u_1(0) = 0$ and $u_2(0) = -au_1(0) = 0$, which means that $x_i(0) = y_i(0)$. According to condition (C′), this implies that $x_i(t) = y_i(t)$ on the whole interval $[0,1]$ and finishes the proof of uniqueness.

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