ANNALES
POLONICI MATHEMATICI
XLIX (1988)

On a Mocanu type generalization of the Kaplan class $K(\alpha, \beta)$ of analytic functions

by K. S. PADMANABHAN, R. PARVATHAM and S. RADHA (Madras)

Abstract. In this paper, we introduce a new class $K_\lambda(\alpha, \beta)$ of functions and discuss certain properties of this class, in particular, the closure property for this class under the well-known integral operator

$$\left\{\frac{c+1/\lambda}{z^c}\int_0^z t^{c-1} f^{1/\lambda}(t)\,dt\right\}^\lambda$$

for suitable choice of α, β, c and λ by making use of the corresponding results for the classes $\pi_{\alpha-\beta}$ and $K(\alpha, \beta)$ (for definition, see Sheil-Small [6]) which are also established here.

Let E be the open unit disc in C and let $H(E)$ denote the class of functions holomorphic in E. The class $K(\alpha, \beta)$ was first defined by Sheil-Small [6] as consisting of functions f in $H(E)$ such that $f(z) = 1 + a_1 z + \ldots$, $z \in E$ with $f(z) \neq 0$ in E and satisfying

$$-\alpha\pi \leqslant \int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})} - \frac{(\alpha-\beta)}{2}\right\} d\theta \leqslant \beta\pi$$

for $\alpha \geqslant 0$, $\beta \geqslant 0$, $0 < |z| = r < 1$ and $\theta_1 < \theta_2 < \theta_1 + 2\pi$. A characterization of this class of functions was obtained [6] in terms of functions belonging to the class π_λ defined as follows:

DEFINITION. For λ real, $g \in \pi_\lambda$ if and only if

$$\mathrm{Re}\left\{\frac{zg'(z)}{g(z)}\right\} \begin{matrix} < \frac{1}{2}\lambda & (\lambda > 0) \\ > \frac{1}{2}\lambda & (\lambda < 0) \end{matrix} \quad \text{in } E.$$

π_0 consistss of the single function $f(z) \equiv 1$.

Sheil-Small [6] established that $f \in K(\alpha, \beta)$ if and only if $f(z) = g(z)h(z)$, where $g \in \pi_{\alpha-\beta}$ and $|\arg e^{i\mu} h(z)| \leqslant \frac{1}{2}\pi \min(\alpha, \beta)$ for a suitable real μ. Here we first prove that the classes $\pi_{\alpha-\beta}$ and $K(\alpha, \beta)$ are closed under the Rusche-

AMS classification number: 30 C 45.
Key words: *Ruscheweyh integral operator.*

weyh operator

$$F(z) = \left\{\frac{c}{z^c}\int_0^z t^{c-1} f^{1/\lambda}(t)\,dt\right\}^\lambda \quad \text{for } \alpha \leqslant \beta$$

and suitable c and λ. Then we introduce a new class $K_\lambda(\alpha, \beta)$ and investigate its properties. We need the following result due to Eenigenburg, Miller, Mocanu and Reade [2], for our discussion.

THEOREM A. *Let* $\beta, \gamma \in C$ *and* $h \in H(E)$ *be convex univalent in* E *with* $h(0) = 1$ *and* $\mathrm{Re}(\beta h(z)+\gamma) > 0$ *in* E. *Let* $p(z) = 1+p_1 z+ \dots$ *Then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z) \Rightarrow p(z) \prec h(z) \quad \text{in } E.$$

We start by proving the following

THEOREM 1. *Let* G *be defined for* $z \in E$ *by*

$$G(z) = \left\{\frac{c}{z^c}\int_0^z t^{c-1} g^{1/\lambda}(t)\,dt\right\}^\lambda .$$

Then $G \in \pi_{\alpha-\beta}$ *whenever* $g \in \pi_{\alpha-\beta}$ *for* $\lambda > 0$, $\alpha \leqslant \beta$ *and* $\mathrm{Re}\, c \geqslant (\beta-\alpha)/2\lambda$.

Proof. If $g \in \pi_{\alpha-\beta}$, then there exists a $\varphi \in S^*$ such that $g(z) = \{\varphi(z)/z\}^{-(\alpha-\beta)/2}$, $g(z) \neq 0$ in E ([7], Lemma 3.2). It is clear that $G(z)$ defined above is holomorphic in a neighbourhood of $z = 0$ and $G(0) = 1$. Thus there exists a $R > 0$ such that $G(z) \neq 0$ for $0 \leqslant |z| < R$. We begin by showing that $G(z)$ is in $\pi_{\alpha-\beta}$ in $|z| < R$. Then there exists $\varphi \in S^*$ such that $G(z) = \{\varphi(z)/z\}^{-(\alpha-\beta)/2}$ in $|z| < R$, $\varphi(z) \neq 0$ in $E-\{0\}$. The proof will be complete if we show that $R \geqslant 1$. Indeed, if $G(z_0) = 0$, $|z_0| = R < 1$, then, for any given $\varepsilon > 0$, there exists a neighbourhood of z_0 in which $|G(z)| < \varepsilon$. But $G(z) = \{\varphi(z)/z\}^{-(\alpha-\beta)/2}$, where $\varphi \in S^*$. For $\beta > \alpha$, we have

$$|G(z)| \geqslant \left(\frac{1}{1+|z|}\right)^{(\beta-\alpha)/2} \quad \text{in } |z| < R.$$

This contradiction shows that $G(z) \neq 0$ in E.

Since $g \in \pi_{\alpha-\beta}$, where $\beta \geqslant \alpha$,

$$\mathrm{Re}\frac{zg'(z)}{g(z)} > \frac{\alpha-\beta}{2} \quad \text{for } \alpha < \beta$$

and for $\alpha = \beta$, $g(z) \equiv 1$. Now, consider

$$G(z) = \left\{\frac{c}{z^c}\int_0^z t^{c-1} g^{1/\lambda}(t)\,dt\right\}^\lambda \quad \text{with } \lambda > 0.$$

This on differentiation with respect to z gives

$$\frac{z}{\lambda} G^{1/\lambda - 1}(z) G'(z) + c G^{1/\lambda}(z) = c g^{1/\lambda}(z).$$

Let $p(z) = zG'(z)/G(z) + 1$; so that $p(0) = 1$ and $G^{1/\lambda}(z)(p(z) + c\lambda - 1) = c\lambda g^{1/\lambda}(z)$. By logarithmic differentiation with respect to z, we get

$$\frac{1}{\lambda}\frac{zG'(z)}{G(z)} + \frac{zp'(z)}{p(z) + c\lambda - 1} = \frac{1}{\lambda}\frac{zg'(z)}{g(z)};$$

$$p(z) + \frac{zp'(z)}{\frac{p(z)}{\lambda} + c - \frac{1}{\lambda}} = \frac{zg'(z)}{g(z)} + 1 \prec h(z) = \frac{1 + (\alpha - \beta + 1)z}{1 + z}.$$

$h(z)$ maps E onto the half plane $\operatorname{Re}\omega > (\alpha - \beta)/2 + 1$ and $h(0) = 1$. The hypotheses on α, β, λ and c imply $\operatorname{Re}(h(z)/\lambda + c - 1/\lambda) > 0$. Hence an application of Theorem A shows $p(z) \prec h(z)$ in E. When $\alpha = \beta$, $G(z) = 1$. Thus $G \in \pi_{\alpha-\beta}$ whenever $g \in \pi_{\alpha-\beta}$ under the given conditions on α, β, c and λ.

THEOREM 2. *Let f be in $K(\alpha, \beta)$. If F is defined by*

$$F(z) = \left\{\frac{c}{z^c}\int_0^z t^{c-1} f^{1/\lambda}(t)\,dt\right\}^{\lambda},$$

then $F \in K(\alpha, \beta)$ for $\beta \geqslant \alpha > 0$, $\lambda \geqslant \alpha$, $\operatorname{Re} c \geqslant (\beta - \alpha)/2\lambda$ and $\lambda = \mu/2\pi n$, where μ is determinated by f in $K(\alpha, \beta)$ and n is a nonzero integer of the same sign as μ.

Proof. Let $f \in K(\alpha, \beta)$. Then $f(0) = 1$ and $f(z)$ can be written as $f(z) = g(z)h(z)$ in E, where $g \in \pi_{\alpha-\beta}$ and $|\arg e^{i\mu} h(z)| \leqslant \frac{1}{2}\pi\alpha$ for a suitable real μ. Let

$$G(z) = \left\{\frac{c}{z^c}\int_0^z t^{c-1} g^{1/\lambda}(t)\,dt\right\}^{\lambda}.$$

By Theorem 1, $G \in \pi_{\alpha-\beta}$ if $\beta \geqslant \alpha \geqslant 0$, $\lambda > 0$ and $\operatorname{Re} c \geqslant (\beta - \alpha)/2\lambda$. Now consider

$$F(z) = \left\{\frac{c}{z^c}\int_0^z t^{c-1} f^{1/\lambda}(t)\,dt\right\}^{\lambda}.$$

Then $F(0) = 1$. If we write $F(z) = G(z)H(z)$, then $H(0) = 1$, and hence $H(z) \neq 0$ in a neighbourhood of $z = 0$. Hence there exist a $R > 0$ such that $H(z) \neq 0$ in $|z| < R$. We begin by showing that $H(z)$ satisfies $|\arg e^{i\mu} H(z)| < \frac{1}{2}\pi\alpha$ in $|z| < R$. Then $|\arg e^{i\mu} H(z)| < \frac{1}{2}\pi\lambda$ if $\lambda \geqslant \alpha$; that is, $|\arg P(z)| < \frac{1}{2}\pi$, where $P(z) = e^{i\mu/\lambda_1} H^{1/\lambda_1}(z)$ for $\lambda_1 > \lambda$, in $|z| < R < 1$ and $P(0) = 1$ for our choice of λ_1. Hence $|P(z)| > (R - |z|)/(R + |z|)$ in $|z| < R$; or $|H(z)| > ((R - |z|)/(R + |z|))^{\lambda_1}$ in $|z| < R$. If possible let $H(z_0) = 0$ with $|z_0| = R < 1$.

Then, for any $\varepsilon > 0$, there exists a neighbourhood of z_0 in which $|H(z)| < \varepsilon$. This contradicts the fact that $|H(z)| > ((R-|z|)/(R+|z|))^{\lambda}$ for all z in $|z| < R$. Hence $H(z) \neq 0$ in E.

Now, we have

$$F^{1/\lambda}(z) = G^{1/\lambda}(z) H^{1/\lambda}(z) = \left\{\frac{c}{z^c}\int_0^z t^{c-1} f^{1/\lambda}(t)\,dt\right\}^{\lambda} \quad \text{in } E.$$

This on differentiation with respect to z yields the following:

$$\frac{z}{\lambda}H'(z)H^{1/\lambda-1}(z)G^{1/\lambda}(z)+\frac{z}{\lambda}G^{1/\lambda-1}(z)G'(z)H^{1/\lambda}(z)+cG^{1/\lambda}(z)H^{1/\lambda}(z) = cf^{1/\lambda}(z);$$

$$\frac{z(H^{1/\lambda}(z))'}{\dfrac{1}{\lambda}\dfrac{zG'(z)}{G(z)}+c}+H^{1/\lambda}(z) = \frac{cf^{1/\lambda}(z)}{\dfrac{z}{\lambda}G^{1/\lambda-1}(z)G'(z)+cG^{1/\lambda}(z)} = \frac{f^{1/\lambda}(z)}{g^{1/\lambda}(z)} = h^{1/\lambda}(z) \quad \text{in } E.$$

Let $P(z) = e^{i\mu/\lambda}H^{1/\lambda}(z)$. Then, if $\lambda = \mu/2\pi n$, where n is a nonzero integer of the same sign as μ, then $P(0) = 1$. Also

$$\frac{zP'(z)}{\dfrac{zG'(z)}{\lambda G(z)}+c}+P(z) = e^{i\mu/\lambda}h^{1/\lambda}(z) \quad \text{in } E,$$

(1) $$\left|\arg\left(\frac{zP'(z)}{(zG'(z)/\lambda G(z))+c}+P(z)\right)\right| = \frac{1}{\lambda}|\arg e^{i\mu}h(z)| \leqslant \frac{\pi\alpha}{2\lambda} \quad \text{in } E.$$

Since $P(0) = 1$ and $P(z)\in H(E)$, we can write

$$P(z) = \left(\frac{1-\omega(z)}{1+\omega(z)}\right)^{\alpha/\lambda},$$

where $\omega(z)\in H(E)$, $\omega(0) = 0$ and $1+\omega(z) \neq 0$ in E and a suitable branch on the right-hand side is chosen. It is enough to show that $|\omega(z)| < 1$ in E. If not, there exists a point ζ_0 in E such that $|\omega(\zeta_0)| = 1$ and $\zeta_0\omega'(\zeta_0) = k\,\omega(\zeta_0)$, $k \geqslant 1$ by Jack's lemma [3]. Now,

$$\frac{\zeta_0 P'(\zeta_0)}{\dfrac{\zeta_0 G'(\zeta_0)}{\lambda G(\zeta_0)}+c}+P(\zeta_0) = T_1+T_2,$$

say. Let $\omega(\zeta_0) = e^{i\theta}$, so

$$\arg P(\zeta_0) = \frac{\alpha}{\lambda}\arg\left(\frac{1-e^{i\theta}}{1+e^{i\theta}}\right) = \begin{cases} -\pi\alpha/2\lambda & \text{if } 0<\theta<\pi, \\ \pi\alpha/2\lambda & \text{if } -\pi<\theta<0; \end{cases}$$

$$\arg\left(\frac{\zeta_0 P'(\zeta_0)}{P(\zeta_0)}\right) = \begin{cases} -\frac{1}{2}\pi & \text{if } 0<\theta<\pi, \\ \frac{1}{2}\pi & \text{if } -\pi<\theta<0. \end{cases}$$

When $\beta \geqslant \alpha \geqslant 0$, $\lambda > 0$ and $\operatorname{Re} c \geqslant (\beta-\alpha)/2\lambda$, $G(z) \in \pi_{\alpha-\beta}$, and hence $\operatorname{Re}(zG'(z)/\lambda G(z)+c) > 0$ which means $|\arg(zG'(z)/\lambda G(z)+c)| < \frac{1}{2}\pi$. Hence if $-\pi < \theta < 0$,

$$\frac{\pi\alpha}{2\lambda}\min(\alpha, \beta) < \arg T_1 < \pi+\frac{\pi\alpha}{2\lambda}.$$

If $-\pi < \theta < 0$, T_1 is a complex number in the half plane determined by

$$\frac{\pi\alpha}{2\lambda} < \arg W < \pi+\frac{\pi\alpha}{2\lambda}. \tag{2}$$

$T_2 = P(\zeta_0)$ lies on the ray $\arg W = \pi\alpha/2\lambda$. Hence $T_1 + T_2$ is a complex number lying in the half plane given by (2). If $0 < \theta < \pi$, $-\pi\alpha/2\lambda-\pi < \arg T_1 < -\pi\alpha/2\lambda$; or T_1 is a complex number in the half plane determined by

$$\frac{-\pi\alpha}{2\lambda}-\pi < \arg W < \frac{-\pi\alpha}{2\lambda}. \tag{3}$$

$T_2 = P(\zeta_0)$ lies on the ray $\arg W = -\pi\alpha/2\lambda$. Hence T_1+T_2 is a complex number lying in the half plane given by (3). Thus in both the cases $0 < \theta < \pi$ and $-\pi < \theta < 0$, there is a contradiction to (1). Hence $|\omega(z)| < 1$ in E. Now,

$$\frac{\alpha}{\lambda}\left|\arg\left(\frac{1-\omega(z)}{1+\omega(z)}\right)\right| < \frac{\pi\alpha}{2\lambda}$$

implying that $|\arg P(z)| < \pi\alpha/2\lambda$, where $e^{i\mu/\lambda}H^{1/\lambda}(z) = P(z)$. Since $F(z) = G(z)H(z)$, where $G \in \pi_{\alpha-\beta}$ and $|\arg e^{i\mu}H(z)| < \pi\alpha/2$, we have $F \in K(\alpha, \beta)$ for $z \in E$ when $\beta \geqslant \alpha > 0$, $\lambda \geqslant \alpha$, $\operatorname{Re} c \geqslant (\beta-\alpha)/2\lambda$ and $\lambda = \mu/2\pi n$, n being a nonzero integer of the same sign as μ.

Now, Theorem 2 leads to the following lemmas which are vital for the subsequent results in this paper.

LEMMA 1. *Let* $p \in H(E)$, $p(0) = 1$ *and let further* $p(z)$ *satisfy*

$$-\alpha\pi+\frac{(\alpha-\beta)}{2}(\theta_2-\theta_1) \leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{p(re^{i\theta})-1+\frac{re^{i\theta}p'(re^{i\theta})}{(p(re^{i\theta})-1)/\lambda+c}\right\}d\theta \tag{4}$$

$$\leqslant \frac{(\alpha-\beta)}{2}(\theta_2-\theta_1)+\beta\pi,$$

$\forall r \in (0, 1)$, $0 < \theta_1 < \theta_2 < \theta_1+2\pi$, $\beta \geqslant \alpha > 0$ *and* $\lambda \geqslant \alpha$.

Then F *defined by*

$$F(z) = \exp\left\{\int_0^z \frac{p(t)-1}{t}\,dt\right\}$$

is in $K(\alpha, \beta)$; *that is,* $F(z) = G(z)H(z)$, *where* $G(z) \in \pi_{\alpha-\beta}$ *and* $|\arg e^{i\mu} H(z)| \leqslant \pi\alpha/2$ *for a suitable real* μ, *provided* $\operatorname{Re} c > (\beta-\alpha)/2\lambda$, $\lambda = \mu/2\pi n$, n *being a nonzero integer of the same sign as* μ.

Proof. The function

$$F(z) = \exp \int_0^z \frac{p(t)-1}{t} dt = 1 + a_1 z + \ldots$$

is regular in E and we have $zF'(z)/F(z) = p(z)-1$, $F(z) \neq 0$ in E. Consider the function $f(z)$ defined for $z \in E$ by

$$f(z) = \left\{F^{1/\lambda}(z) + \frac{z}{c}(F^{1/\lambda}(z))'\right\}^{\lambda} = \frac{F(z)}{c^{\lambda}}\left(c + \frac{p(z)-1}{\lambda}\right)^{\lambda}. \tag{5}$$

Formula (5) implies

$$F^{1/\lambda}(z) = \frac{c}{z^c} \int_0^z t^{c-1} f^{1/\lambda}(t)\, dt. \tag{6}$$

On logarithmic differentiation (5) yields:

$$\frac{zf'(z)}{f(z)} = p(z) - 1 + \frac{zp'(z)}{\dfrac{p(z)-1}{\lambda} + c}. \tag{7}$$

From (4) and (7) it follows that $f \in K(\alpha, \beta)$. Hence f can be written as $f(z) = g(z)h(z)$, where $g(z) \in \pi_{\alpha-\beta}$ and $|\arg e^{i\mu} h(z)| \leqslant \pi\alpha/2$ for a suitable real μ. By Theorem 2, $F(z)$ related to $f(z)$ by (5) is also in $K(\alpha, \beta)$ provided $\beta \geqslant \alpha > 0$, $\operatorname{Re} c \geqslant (\beta-\alpha)/2\lambda$, $\lambda \geqslant \alpha$ and $\lambda = \mu/2\pi n$, where n is a nonzero integer of the same sign as μ.

LEMMA 2. *Let* $p \in H(E)$, $p(0) = 1$ *and let* p *satisfy* (4), *for every* $r \in (0, 1)$, $\beta \geqslant \alpha > 0$, $\lambda \geqslant \alpha$, $0 < \theta_1 < \theta_2 < \theta_1 + 2\pi$. *Then*

$$-\alpha\pi + \frac{(\alpha-\beta)}{2}(\theta_2 - \theta_1) \leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}(p(re^{i\theta}) - 1)\, d\theta \leqslant \frac{(\alpha-\beta)}{2}(\theta_2 - \theta_1) + \beta\pi,$$

whenever $\operatorname{Re} c \geqslant (\beta-\alpha)/2\lambda$ *and* $\lambda = \mu/2\pi n$, *where* μ *is determined as in Lemma* 1 *and* n *is a nonzero integer of the same sign as* μ.

Proof. By the previous lemma, we get

$$F(z) = \left(\exp \int_0^z \frac{p(t)-1}{t} dt\right) \in K(\alpha, \beta)$$

provided $\beta \geqslant \alpha > 0$, $\operatorname{Re} c \geqslant (\alpha-\beta)/2\lambda$, $\lambda = \mu/2\pi n$ (where μ is a real number determined by $F \in K(\alpha, \beta)$), n being a nonzero integer of the same sign as μ. Hence

$$-\alpha\pi + \frac{(\alpha-\beta)}{2}(\theta_2-\theta_1) \leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}\left(\frac{re^{i\theta}F'(re^{i\theta})}{F(re^{i\theta})}\right)d\theta \leqslant \frac{(\alpha-\beta)}{2}(\theta_2-\theta_1)+\beta\pi$$

under the stated conditions on α, β, c and λ; that is

$$-\alpha\pi + \frac{(\alpha-\beta)}{2}(\theta_2-\theta_1) \leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}(p(re^{i\theta})-1)\,d\theta \leqslant \frac{(\alpha-\beta)}{2}(\theta_2-\theta_1)+\beta\pi$$

under the stated conditions on α, β, c and λ.

Now, we introduce a new class $K_\lambda(\alpha, \beta)$ defined as follows:

DEFINITION 1. Let $f \in H(E)$ and let $f(z) = z + a_2 z^2 + \ldots$, with $f(z)/z \neq 0$ in E. f is said to be in the *class* $K_\lambda(\alpha, \beta)$ if and only if

$$-\alpha\pi \leqslant \int_{\theta_1}^{\theta_2}\left(\operatorname{Re}\left\{(1-\lambda)\frac{re^{i\theta}f'(re^{i\theta})}{f(re^{i\theta})}+\lambda\left(1+\frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right)\right\}-\frac{\alpha-\beta+2}{2}\right)d\theta \leqslant \beta\pi$$

with $\alpha \geqslant 0$, $\beta \geqslant 0$, $0 < \theta_1 < \theta_2 < \theta_1 + 2\pi$, for every $r \in (0, 1)$ and λ being a non-negative real number.

Remark 1. When $\lambda = 0$ we note that $f/z \in K(\alpha, \beta)$. $g(z) = z + a_1 z^2 + \ldots$ is a close-to-convex function of order α if and only if $g' \in K_1(\alpha, \alpha+2)$. $K_\lambda(1, 3) \equiv P(\lambda)$ of λ-close-to-convex functions of Bharati [1].

Remark 2. $f \in K_\lambda(\alpha, \beta)$ if and only if there exist a $g \in K(\alpha, \beta)$ such that $(f(z)/z)^{1-\lambda} f'(z) = g(z)$. Since $g \in K(\alpha, \beta)$, we can write $g(z) = g_1(z)\ h_1(z)$, where $g_1 \in \pi_{\alpha-\beta}$ and $|\arg e^{i\mu} h_1(z)| \leqslant \frac{1}{2}\pi \min(\alpha, \beta)$ for a suitable real μ. Thus to every $f \in K_\lambda(\alpha, \beta)$ there corresponds a real number μ.

THEOREM 3. *Let $f \in K_\lambda(\alpha, \beta)$ for $\beta \geqslant \alpha > 0$. Then for every c with $\operatorname{Re} c \geqslant (\beta-\alpha-2)/2\lambda$, $\lambda \geqslant \alpha$ and $\lambda = \mu/2\pi n$ with the corresponding μ of $f(z)$, n being a nonzero integer of the same sign as μ, F defined by*

$$F(z) = \left\{\frac{c+1/\lambda}{z^c}\int_0^z t^{c-1} f^{1/\lambda}(t)\,dt\right\}^\lambda,$$

$z \in E$ is also in $K_\lambda(\alpha, \beta)$.

Proof.

$$F(z) = \left\{\frac{c+1/\lambda}{z^c}\int_0^z t^{c-1} f^{1/\lambda}(t)\,dt\right\}^\lambda$$

is holomorphic in a neighbourhood of $z = 0$ and $F(0) = 0 = F'(0) - 1$. Thus there exists a $R > 0$ such that $F(z) \neq 0$ in $0 < |z| < R$. Hence

$$\left(\frac{F(z)}{z}\right)^{1/\lambda} = \frac{c+1/\lambda}{z^{c+1/\lambda}} \int_0^z t^{c-1} f^{1/\lambda}(t)\, dt;$$

differentiating with respect to z and simplifying we get

$$\frac{zF'(z) F^{1/\lambda - 1}(z)}{\lambda} + cF^{1/\lambda}(z) = \left(c + \frac{1}{\lambda}\right) f^{1/\lambda}(z);$$

again differentiating with respect to z, we get

$$\frac{F'(z) F^{1/\lambda-1}(z)}{\lambda} \left\{ \lambda \left(1 + \frac{zF''(z)}{F'(z)}\right) + (1-\lambda)\frac{zF'(z)}{F(z)} + c\lambda \right\}$$
$$= \left(c + \frac{1}{\lambda}\right) f^{1/\lambda - 1}(z) f'(z).$$

Now putting

$$p(z) = (1-\lambda)\frac{zF'(z)}{F(z)} + \lambda\left(1 + \frac{zF''(z)}{F'(z)}\right),$$

$p(0) = 1$ and

$$\frac{F^{1/\lambda-1}(z) F'(z)}{\lambda}(p(z) + c\lambda) = \left(c + \frac{1}{\lambda}\right) f^{1/\lambda - 1}(z) f'(z).$$

By logarithmic differentiation of the above equality we obtain

$$p(z) - 1 + \frac{zp'(z)}{\dfrac{p(z)-1}{\lambda} + c + \dfrac{1}{\lambda}} = (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1.$$

Since $f \in K_\lambda(\alpha, \beta)$, this gives, for $|z| < R$,

$$-\alpha\pi + \frac{(\alpha-\beta)}{2}(\theta_2 - \theta_1) \leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{(1-\lambda)\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} + \lambda\left(1 + \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} - 1\right)\right\} d\theta$$
$$= \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{p(re^{i\theta}) - 1 + \frac{re^{i\theta} p'(re^{i\theta})}{\left((p(re^{i\theta}) - 1)/\lambda + c + 1/\lambda\right)}\right\} d\theta$$
$$\leqslant \frac{(\alpha-\beta)}{2}(\theta_2 - \theta_1) + \beta\pi.$$

An application of Lemma 2 yields for $|z| < R$

$$-\alpha\pi + \left(\frac{\alpha-\beta}{2}\right)(\theta_2-\theta_1) \leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}\left(p(re^{i\theta}) - 1\right) d\theta \leqslant \left(\frac{\alpha-\beta}{2}\right)(\theta_2 - \theta_1) + \beta\pi$$

provided $\beta \geqslant \alpha > 0$, $\operatorname{Re} c \geqslant (\beta-\alpha-2)/2\lambda$, $\lambda \geqslant \alpha$, $\lambda = \mu/2\pi n$ with the μ corresponding to f, n being a nonzero integer of the same sign as μ. This implies that in $|z| < R$

$$-\alpha\pi + \frac{(\alpha-\beta)}{2}(\theta_2-\theta_1)$$

$$\leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{(1-\lambda)\frac{re^{i\theta}F'(re^{i\theta})}{F(re^{i\theta})} + \lambda\left(1+\frac{re^{i\theta}F''(re^{i\theta})}{F(re^{i\theta})}\right)-1\right\}d\theta$$

$$\leqslant \frac{(\alpha-\beta)}{2}(\theta_2-\theta_1)+\beta\pi$$

under the stated conditions on α, β, λ and c in $|z| < R$. Hence by Remark 2, $(F(z)/z)^{1-\lambda}F'(z) = G(z)H(z)$, where $G \in \pi_{\alpha-\beta}$ and $|\arg e^{i\mu}H(z)| \leqslant \pi\alpha/2$ in $|z| < R$. We know that $G(z) \neq 0$ in E. $H(z)$ can also be proved to be nonzero in E as in Theorem 2. Hence $F(z)/z \neq 0$ in E. Proceeding as above, in E we have $F \in K_\lambda(\alpha, \beta)$ in E under the stated conditions on α, β, c and λ.

Remark 3. For $\beta = 3$ and $\alpha = 1$ the above result implies that the class $P(\lambda)$ of λ-close-to-convex functions is closed under the transform by the Ruscheweyh operator when $\lambda \geqslant 1$ and $\operatorname{Re} c \geqslant 0$ [4].

THEOREM 4. *For $2 \geqslant \beta-\alpha$ and $\beta \geqslant \alpha > 0$ and $\lambda \geqslant \alpha$, $\lambda = \mu/2\pi n$, where μ and n are as in the previous theorem, $f \in K_\lambda(\alpha, \beta)$ implies $f \in K_0(\alpha, \beta)$.*

Proof. Let $f \in K_\lambda(\alpha, \beta)$. Then there is a real number μ associated with f. Taking $p(z) = zf'(z)/f(z)$ and differentiating, we get

$$p(z)+\frac{zp'(z)}{p(z)} = 1+\frac{zf''(z)}{f(z)}.$$

Since $f \in K_\lambda(\alpha, \beta)$, we have

$$-\alpha\pi + \frac{(\alpha-\beta)}{2}(\theta_2-\theta_1)$$

$$\leqslant \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{(1-\lambda)re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})} + \lambda\left(1+re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right)-1\right\}d\theta$$

$$= \int_{\theta_1}^{\theta_2} \operatorname{Re}\left\{p(re^{i\theta})-1+\frac{re^{i\theta}p'(re^{i\theta})}{(p(re^{i\theta})-1)/\lambda+1/\lambda}\right\}d\theta$$

$$\leqslant \frac{(\alpha-\beta)}{2}(\theta_2-\theta_1)+\beta\pi.$$

An application of Lemma 2 yields

$$-\alpha\pi+\frac{(\alpha-\beta)}{2}(\theta_2-\theta_1)\leqslant\int_{\theta_1}^{\theta_2}\operatorname{Re}(p(re^{i\theta})-1)\,d\theta\leqslant\frac{(\alpha-\beta)}{2}(\theta_2-\theta_1)+\beta\pi,$$

provided $\beta\geqslant\alpha>0$, $2\geqslant\beta-\alpha$, $\lambda\geqslant\alpha$ and $\lambda=\mu/2\pi n$, where μ and n are as stated in the theorem. Hence

$$-\alpha\pi\leqslant\int_{\theta_1}^{\theta_2}\operatorname{Re}\left\{re^{i\theta}\frac{f'(re^{i\theta})}{f(re^{i\theta})}-\frac{\alpha-\beta+2}{2}\right\}d\theta\leqslant\beta\pi$$

provided $\beta\geqslant\alpha>0$, $2\geqslant\beta-\alpha$ and $\lambda\geqslant\alpha$. This means $f\in K_0(\alpha,\beta)$ under the above conditions on α, β and λ given in the theorem.

Remark 4. For $\beta=3$ and $\alpha=1$ we get the inclusion relation $K_\lambda(1,3)=P(\lambda)\subset K_0(1,3)=CS^*$ of close-to-star functions for $\lambda\geqslant 1$, a result obtained in an earlier paper [5].

References

[1] R. Bharati, *On α-close-to-convex functions*, Proc. Indian Acad. Sci. 88 A (1979), 93–103.

[2] P. Eenigenburg, S. S. Millar, P. T. Mocanu and M. O. Reade, *On a Briot-Bouquet differential subordination*, Rev. Roumaine Math. Pures Appl. 29 (1984), 567–573.

[3] I. S. Jack, *Functions starlike and convex of order α''*, J. London Math. Soc. (2) 3 (1971), 469–474.

[4] K. S. Padmanabhan and R. Bharati, *On α close-to-convex functions II*, Glasinik Mathematici 16 (1981), 235–244.

[5] R. Parvatham and S. Radha, *On certain classes of analytic functions*, this fasc., 31–34.

[6] T. Sheil-Small, *The Hadamard product and linear transformations of classes of analytic functions*, J. Analyse Math. 34 (1978), 204–234.

[7] —, *Some remarks on Basilevic functions*, ibidem 43 (1983/84), 1–11.

RAMANUJAN INSTITUTE
UNIVERSITY OF MADRAS
MADRAS, INDIA
J.B.A.S. COLLEGE FOR WOMEN, MADRAS

Reçu par la Rédaction le 30.05.1986