ANNALES
POLONICI MATHEMATICI
XII (1962)

On the $C^{\alpha}|C^{\beta}$ convergence

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1. Given two complex sequences $x = (x_0, x_1, \ldots)$ and $y = (y_0, y_1, \ldots)$ we denote by $x \ast y$ their convolution, that is, the sequence defined by the n-th term of the form

$$(x \ast y)_n = \sum_{\nu=0}^{n} x_{n-\nu} y_\nu .$$

The convolution has the following properties:

$$\text{(i)} \qquad x \ast y = y \ast x ,$$

$$\text{(ii)} \qquad (x \ast y) \ast z = x \ast (y \ast z) ,$$

$$\text{(iii)} \qquad (x+y) \ast z = x \ast y + y \ast z ,$$

for arbitrary sequences x, y, z.

By A^{α} we denote Cesàro's sequence of order α, that is, the sequence with terms

$$A_0^{\alpha} = 1 , \quad A_n^{\alpha} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!} \quad \text{for} \quad n \geqslant 1 .$$

It is well known that for any α and β

$$\text{(iv)} \qquad A^{\alpha} \ast A^{\beta} = A^{\alpha+\beta+1} .$$

The operator S^{α}, which transforms the sequence x into the sequence $A^{\alpha-1} \ast x$, is called *summation operator* of order α and the sequence $S^{\alpha}(x) = A^{\alpha-1} \ast x$ is called the α-th *sum* of the sequence x. From (iv) it follows immediately that for any α and β

$$\text{(v)} \qquad S^{\alpha}(A^{\beta}) = A^{\alpha+\beta} .$$

For any sequence x and for any α and β we have $S^{\alpha}\big(S^{\beta}(x)\big) = S^{\alpha+\beta}(x)$, which may be written

$$\text{(vi)} \qquad S^{\alpha} S^{\beta} = S^{\alpha+\beta} .$$

Since $S^0(x) = x$ for any sequence x, we have $S^0 = E$, where E denotes the unit operator. From (vi) follows $S^\alpha S^{-\alpha} = S^{-\alpha} S^\alpha = E$, so that the operator $S^{-\alpha}$ is inverse with respect to S^α. In particular

$$S^{-1}(x) = (x_0, x_1 - x_0, x_2 - x_1, \ldots),$$

so that the operator S^{-1} may be called the *first difference*. It is to be noted that the operator S^{-1} differs from the operator Δ of the same name, for

$$\Delta(x) = (x_0 - x_1, x_1 - x_2, \ldots).$$

2. Let x be an arbitrary sequence. If the sequence

$$C^\alpha(x) = \frac{S^\alpha(x)}{A^\alpha}$$

defined for $\alpha \neq -1, -2, \ldots$, converges to the limit s, we say that the sequence x belongs to the field of convergence of the operator C^α with the limit s and we write

$$x \in \operatorname{Conv} C^\alpha | s .$$

If the sequence $C^\alpha(x)$ is bounded and K is the upper bound of the numbers $|C_n^\alpha(x)|$, we say that the sequence x belongs to the field of boundness of the operator C^α with the upper bound K and we write

$$x \in \operatorname{Bound} C^\alpha | K .$$

When the limit or the upper bound are not essential, we write simply $x \in \operatorname{Conv} C^\alpha$ or $x \in \operatorname{Bound} C^\alpha$. Since $C^0 = E$, $x \in \operatorname{Conv} E$ or $x \in \operatorname{Bound} E$ denotes that the sequence x is convergent or bounded in the common sense.

The operator C^α is regular for any $\alpha \geqslant 0$, that is, for any $x \in \operatorname{Conv} E | s$ we have $x \in \operatorname{Conv} C^\alpha | s$. For any $\beta > \alpha > -1$ the operator C^β is an extension of the operator C^α, that is, for any $x \in \operatorname{Conv} C^\alpha | s$ we have $x \in \operatorname{Conv} C^\beta | s$. For any $\alpha > -1$, $\beta > -1$ and $\alpha + \beta > -1$ the operators $C^\alpha C^\beta$ and $C^{\alpha+\beta}$ are equivalent, that is, for any $x \in \operatorname{Conv} C^\alpha C^\beta | s$ we have $x \in \operatorname{Conv} C^{\alpha+\beta} | s$ and reciprocally. Moreover, if $x \in \operatorname{Bound} C^\alpha C^\beta$, then $x \in \operatorname{Bound} C^{\alpha+\beta}$ and reciprocally (see Hardy [2]).

3. If there exists such a number s that the sequence $C^\alpha | C^\beta(x) - s|$ tends to 0, we say that the sequence x belongs to the field of convergence $C^\alpha | C^\beta$ with the limit s and we write

$$x \in \operatorname{Conv} C^\alpha | C^\beta | s .$$

If the sequence $C^\alpha | C^\beta(x)|$ is bounded and its upper bound is K, we write

$$x \in \operatorname{Bound} C^\alpha | C^\beta | K .$$

We recognize in the case $\alpha = 1$, $\beta = 0$ the strong convergence put forward in 1916 by M. Fekete and in the case $\alpha = 1$, $\beta > -1$ the strong summability of the order $\beta+1$ first introduced and studied in 1933 by C. E. Winn [1].

In this note we consider the $C^\alpha|C^\beta$ convergence for $\alpha > -1$ and $\beta > -1$ (A. Zygmund pointed out to me that only the case $-1 < \alpha < 1$ is to be considered, for, in virtue of Hardy's Theorem, from the convergence of $C^\alpha|C^\beta(x)-s|$ to 0 for $\alpha > 1$ it follows that $C^1|C^\beta(x)-s|$ converges to 0).

THEOREM I. *If $\alpha > -1$, $\beta > -1$ and $\alpha+\beta > -1$, then*

$$\operatorname{Conv} C^\alpha|C^\beta|s \subset \operatorname{Conv} C^{\alpha+\beta}|s\,,$$
$$\operatorname{Bound} C^\alpha|C^\beta \subset \operatorname{Bound} C^{\alpha+\beta}\,.$$

Proof. We have, for $\alpha > -1$,

$$\left|C^\alpha\big(C^\beta(x)\big)-s\right| = \left|C^\alpha\big(C^\beta(x)-s\big)\right| \leqslant C^\alpha|C^\beta(x)-s|$$

and, therefore, if $x \in \operatorname{Conv} C^\alpha|C^\beta|s$, then $x \in \operatorname{Conv} C^\alpha C^\beta|s$. Since the operators $C^\alpha C^\beta$ and $C^{\alpha+\beta}$ are equivalent, we obtain $x \in \operatorname{Conv} C^{\alpha+\beta}|s$. If $x \in \operatorname{Bound} C^\alpha|C^\beta$, then from $|C^\alpha C^\beta(x)| \leqslant C^\alpha|C^\beta(x)|$ it follows that $x \in \operatorname{Bound} C^\alpha C^\beta$ and, therefore, $x \in \operatorname{Bound} C^{\alpha+\beta}$.

THEOREM II. *If $\alpha' > \alpha > -1$, $\beta > -1$, then*

$$\operatorname{Conv} C^\alpha|C^\beta|s \subset \operatorname{Conv} C^{\alpha'}|C^\beta|s\,,$$
$$\operatorname{Bound} C^\alpha|C^\beta \subset \operatorname{Bound} C^{\alpha'}|C^\beta\,.$$

Proof. The theorem follows from the fact that the operator $C^{\alpha'}$ is an extension of the operator C^α.

LEMMA 1. *If, for $\alpha > 0$, $C_n^\alpha(x)$ is $o(1)$ or $O(1)$ for $n\to\infty$, then, for $\beta > -1$, $C_n^\alpha(A^\beta x)$ is $o(A_n^\beta)$ or $O(A_n^\beta)$ respectively.*

Proof. From Zygmund's remark it follows that for our purposes it is enough to consider the case $0 < \alpha < 1$, the Lemma being true for any $\alpha > 0$.

We have

$$\frac{C^\alpha(A^\beta x)}{A^\beta} = \frac{S^\alpha(A^\beta x)}{A^\alpha A^\beta} = \frac{A^{\alpha-1} * A^\beta x}{A^\alpha A^\beta}\,,$$

whence

$$\frac{C_n^\alpha(A^\beta x)}{A_n^\beta} = \frac{1}{A_n^\alpha A_n^\beta}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}A_\nu^\beta x_\nu = \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}x_\nu +$$
$$+\frac{1}{A_n^\alpha A_n^\beta}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}(A_\nu^\beta - A_n^\beta)x_\nu = C_n^\alpha(x) + \frac{1}{A_n^\alpha A_n^\beta}\sum_{\nu=0}^{n}\varepsilon_{n\nu}x_\nu\,,$$

where

$$\varepsilon_{n\nu} = A^{\alpha-1}_{n-\nu}(A^{\beta}_{\nu} - A^{\beta}_{n}) .$$

The sequence $C^{\alpha}_{n}(x)$ being $o(1)$ or $O(1)$ we show that $\sum_{\nu=0}^{n}\varepsilon_{n\nu}x_{\nu}$ is $o(A^{\alpha+\beta}_{n})$ or $O(A^{\alpha+\beta}_{n})$ respectively.

Applying Abel's transformation to the sum $\sum_{\nu=0}^{n}\varepsilon_{n\nu}x_{\nu}$ we obtain

$$\sum_{\nu=0}^{n}\varepsilon_{n\nu}x_{\nu} = -\sum_{\nu=0}^{n} S^{-1}_{\nu+1}(\varepsilon)\, S^{1}_{\nu}(x) + \varepsilon_{n,n+1} S^{1}_{n}(x) .$$

Since $\varepsilon_{n,n+1} = 0$ and for $0 \leqslant \nu \leqslant n$

$$\begin{aligned} S^{-1}_{\nu+1}(\varepsilon) &= A^{\alpha-1}_{n-(\nu+1)}(A^{\beta}_{\nu+1} - A^{\beta}_{n}) - A^{\alpha-1}_{n-\nu}(A^{\beta}_{\nu} - A^{\beta}_{n}) \\ &= A^{\alpha-1}_{n-(\nu+1)} A^{\beta-1}_{\nu+1} - A^{\alpha-2}_{n-\nu}(A^{\beta}_{\nu} - A^{\beta}_{n}) , \end{aligned}$$

we have

$$\sum_{\nu=0}^{n}\varepsilon_{n\nu}x_{\nu} = -\sum_{\nu=0}^{n} A^{\alpha+1}_{n-(\nu-1)} A^{\beta-1}_{\nu+1} S^{1}_{\nu}(x) + $$

$$+ \sum_{\nu=0}^{n} A^{\alpha-2}_{n-\nu}(A^{\beta}_{\nu} - A^{\beta}_{n}) S^{1}_{\nu}(x) = \mathrm{I} + \mathrm{II} .$$

By hypothesis $C^{\alpha}_{n}(x) = o(1)$, $0 < \alpha < 1$, whence $C^{1}_{n}(x) = o(1)$ and, therefore, $S^{1}_{n}(x) = o(n)$. In virtue of the well-known fact that for any $p > -1$ and $q > -1$

$$\sum_{\nu=0}^{n} A^{p}_{n-\nu}\, o(A^{q}_{\nu}) = o(A^{p+q+1}_{n})$$

we obtain

$$|\mathrm{I}| \leqslant \sum_{\nu=0}^{n} A^{\alpha-1}_{n-(\nu+1)} A^{\beta-1}_{\nu+1}\, o(\nu) = \sum_{\nu=0}^{n} A^{\alpha-1}_{n-1-\nu}\, o(A^{\beta}_{\nu}) = o(A^{\alpha+\beta}_{n}) .$$

Since $A^{\beta}_{n} - A^{\beta}_{\nu} = \sum_{\mu=\nu+1}^{n} A^{\beta-1}_{\mu}$, we have, for $\beta > 1$, $|A^{\beta}_{n} - A^{\beta}_{\nu}| \leqslant (n-\nu) A^{\beta-1}_{n}$ and, for $\beta < 1$, $|A^{\beta}_{n} - A^{\beta}_{\nu}| \leqslant (n-\nu) A^{\beta-1}_{\nu}$, so that, for $\beta > 1$,

$$\begin{aligned} |\mathrm{II}| &\leqslant A^{\beta-1}_{n} \sum_{\nu=0}^{n} |A^{\alpha-2}_{n-\nu}| (n-\nu)\, o(\nu) \\ &= A^{\beta-1}_{n} \sum_{\nu=0}^{n} A^{\alpha-1}_{n-\nu}\, o(A^{1}_{\nu}) = A^{\beta-1}_{n}\, o(A^{\alpha+1}_{n}) = o(A^{\alpha+\beta}_{n}) \end{aligned}$$

and, for $\beta < 1$,

$$|\mathrm{II}| \leqslant \sum_{\nu=0}^{n} |A_{n-\nu}^{\alpha-2}|(n-\nu)A_\nu^{\beta-1}o(\nu) = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}o(A_\nu^\beta) = o(A_n^{\alpha+\beta}) .$$

Thus the lemma is established for $0 < \alpha < 1$.

LEMMA 2. *If $\beta > \beta' > -1$, $\varepsilon > \varepsilon' > -\beta'-1$, then for any $\gamma > -\varepsilon'-1$ there is such a constant K that for any positive integers μ, ν*

$$\frac{A_\mu^\beta A_\nu^{\varepsilon+\gamma}}{A_{\nu+\mu}^{\beta+\varepsilon}} \leqslant K \frac{A_\mu^{\beta'} A_\nu^{\varepsilon'+\gamma}}{A_{\nu+\mu}^{\beta'+\varepsilon'}} .$$

Proof. Since for any $\beta > \beta'$ and $\varepsilon > \varepsilon'$

$$\frac{\mu^\beta \nu^\varepsilon}{(\mu+\nu)^{\beta+\varepsilon}} = \left(\frac{\mu}{\mu+\nu}\right)^\beta \left(\frac{\nu}{\mu+\nu}\right)^\varepsilon \leqslant \left(\frac{\mu}{\mu+\nu}\right)^{\beta'} \left(\frac{\nu}{\mu+\nu}\right)^{\varepsilon'},$$

we have, for μ and ν sufficiently great,

$$\frac{A_{\mu+\nu}^{\beta+\varepsilon}}{A_\mu^\beta A_\nu^{\alpha+\gamma}} \cong \frac{\Gamma(\beta+\varepsilon+1)}{\Gamma(\beta+1)\Gamma(\varepsilon+\gamma+1)} \cdot \frac{\mu^\beta \nu^{\alpha+\gamma}}{(\mu+\nu)^{\beta+\varepsilon}} \leqslant \frac{\Gamma(\beta+\varepsilon+1)}{\Gamma(\beta+1)\Gamma(\varepsilon+\gamma+1)} \cdot \frac{\mu^{\beta'}\nu^{\varepsilon'+\gamma}}{(\mu+\nu)^{\beta'+\varepsilon'}}$$

$$\cong \frac{\Gamma(\beta+\varepsilon+1)}{\Gamma(\beta+1)\Gamma(\varepsilon+\gamma+1)} \cdot \frac{\Gamma(\beta'+1)\Gamma(\varepsilon'+\gamma+1)}{\Gamma(\beta'+\varepsilon'+1)} \cdot \frac{A_\mu^{\beta'} A_\nu^{\varepsilon'+\gamma}}{A_{\mu+\nu}^{\beta'+\varepsilon'}},$$

whence it follows that we can find such a constant K that the inequality of the lemma is valid for any positive μ and ν.

THEOREM III. *If $\alpha > 0$, $\beta > -1$, then for any $\varepsilon > 0$*

$$\operatorname{Conv} C^\alpha|C^\beta|s \subset \operatorname{Conv} C^\alpha|C^{\beta+\varepsilon}|s ,$$

$$\operatorname{Bound} C^\alpha|C^\beta \subset \operatorname{Bound} C^\alpha|C^{\beta+\varepsilon} .$$

Proof. Since

$$C^\alpha|C^{\beta+\varepsilon}(x)-s| = C^\alpha\left|\frac{S^{\beta+\varepsilon}(x)}{A^{\beta+\varepsilon}} - s\right| = C^\alpha\left|\frac{S^\varepsilon\big(A^\beta(C^\beta(x)-s+s)\big)}{A^{\beta+\varepsilon}} - s\right|$$

$$= C^\alpha\left|\frac{S^\varepsilon\big(A^\beta(C^\beta(x)-s)\big)}{A^{\beta+\varepsilon}}\right| \leqslant C^\alpha \frac{S^\varepsilon\big(A^\beta|C^\beta(x)-s|\big)}{A^{\beta+\varepsilon}},$$

we have

$$C_n^\alpha|C^{\beta+\varepsilon}(x)-s| \leqslant \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}\frac{A_{n-\nu}^{\alpha-1}}{A_\nu^{\beta+\varepsilon}}\sum_{\mu=0}^{\nu} A_{\nu-\mu}^{\varepsilon-1}A_\mu^\beta|C_\mu^\beta(x)-s|$$

$$= \frac{1}{A_n^\alpha}\sum_{\mu=0}^{n} A_\mu^\beta|C_\mu^\beta(x)-s| \sum_{\nu=\mu}^{n}\frac{A_{n-\nu}^{\alpha-1}A_{\nu-\mu}^{\varepsilon-1}}{A_\nu^{\beta+\varepsilon}}$$

$$= \frac{1}{A_n^\alpha}\sum_{\mu=0}^{n} A_\mu^\beta|C_\mu^\beta(x)-s| \sum_{\nu=0}^{n-\mu}\frac{A_{n-\mu-\nu}^{\alpha-1}A_\nu^{\varepsilon-1}}{A_{\mu+\nu}^{\beta+\varepsilon}} .$$

If $\beta+\varepsilon \leqslant 0$, then

$$C_n^\alpha|C^{\beta+\varepsilon}(x)-s| \leqslant \frac{1}{A_n^\alpha A_n^{\beta+\varepsilon}} \sum_{\mu=0}^{n} A_\mu^\beta |C_\mu^\beta(x)-s| A_{n-\mu}^{\alpha+\varepsilon-1}$$
$$= \frac{A_n^{\alpha+\varepsilon}}{A_n^\alpha A_n^{\beta+\varepsilon}} C_n^{\alpha+\varepsilon}(A^\beta|C^\beta(x)-s|).$$

Now, if $x \in \operatorname{Conv} C^\alpha|C^\beta|s$, then, for $\varepsilon > 0$, $C_n^{\alpha+\varepsilon}|C^\beta(x)-s| = o(1)$ and, in virtue of lemma 1, $C_n^{\alpha+\varepsilon}(A^\beta|C^\beta(x)-s|) = o(A_n^\beta)$, whence $C_n^\alpha|C^{\beta+\varepsilon}(x)-s| = o(1)$. If $x \in \operatorname{Bound} C^\alpha|C^\beta$, then, for $\varepsilon > 0$, $C_n^{\alpha+\varepsilon}|C^\beta(x)| = O(1)$ and, in virtue of lemma 1, $C_n^{\alpha+\varepsilon}(A^\beta|C^\beta(x)|) = O(A_n^\beta)$, whence $C_n^\alpha|C^{\beta+\varepsilon}(x)| = O(1)$.

If $\beta+\varepsilon > 0$, we choose β' and ε' in such a way that $\beta' \leqslant \beta$, $\varepsilon' \leqslant \varepsilon$, $\beta' > -1$, $\varepsilon' > 0$ and $\beta'+\varepsilon' < 0$. In virtue of lemma 2 there is such a constant K that, for any μ, ν,

$$\frac{A_\mu^\beta A_\nu^{\varepsilon-1}}{A_{\nu+\mu}^{\beta+\varepsilon}} \leqslant K \frac{A_\mu^{\beta'} A_\nu^{\varepsilon'-1}}{A_{\nu+\mu}^{\beta'+\varepsilon'}},$$

whence

$$C_n^\alpha|C^{\beta+\varepsilon}(x)-s| \leqslant \frac{K}{A_n^\alpha A_n^{\beta'+\varepsilon'}} \sum_{\mu=0}^{n} A_\mu^{\beta'} |C_\mu^\beta(x)-s| A_{n-\mu}^{\alpha+\varepsilon'-1}$$
$$= K \frac{A_n^{\alpha+\varepsilon'}}{A_n^\alpha A_n^{\beta'+\varepsilon'}} C_n^{\alpha+\varepsilon'}(A^{\beta'}|C^\beta(x)-s|),$$

and, therefore, this case is reduced to the previous one. Thus the theorem is established.

THEOREM IV. *If* $\alpha > 0$, $\beta > -1$ *and* $0 < \varepsilon < 1$, *then*

$$\operatorname{Conv} C^{\alpha+\varepsilon}|C^\beta|s \subset \operatorname{Conv} C^\alpha|C^{\beta+\varepsilon}|s,$$
$$\operatorname{Bound} C^{\alpha+\varepsilon}|C^\beta \subset \operatorname{Bound} C^\alpha|C^{\beta+\varepsilon}.$$

Proof. If $\beta+\varepsilon < 0$ (in this case from $\beta > -1$ it follows $\varepsilon < 1$), then referring to the proof of theorem III we have

$$C_n^\alpha|C^{\beta+\varepsilon}(x)-s| \leqslant \frac{A_n^{\alpha+\varepsilon}}{A_n^\alpha A_n^{\beta+\varepsilon}} C_n^{\alpha+\varepsilon}(A^\beta|C^\beta(x)-s|),$$

whence, in virtue of the lemma 1, $C_n^\alpha|C^{\beta+\varepsilon}(x)-s| = o(1)$. If $\beta+\varepsilon > 0$, we can choose such β' that $\beta' > -1$, $\beta' < \beta$ and $\beta'+\varepsilon < 0$ (for $\varepsilon \geqslant 1$ we have $\beta'+\varepsilon > 0$). Applying lemma 2 we obtain (as in the proof of theorem III)

$$C_n^\alpha|C^{\beta+\varepsilon}(x)-s| \leqslant K \frac{A_n^{\alpha+\varepsilon}}{A_n^\alpha A_n^{\beta'+\varepsilon}} C_n^{\alpha+\varepsilon}(A^{\beta'}|C^\beta(x)-s|),$$

whence, in virtue of lemma 1, $C_n^\alpha|C^{\beta+\varepsilon}(x)-s| = o(1)$. Supposing $x \in \mathrm{Bound}\, C^{\alpha+\varepsilon}|C^\beta$ by the analogous argument we obtain $x \in \mathrm{Bound}\, C^\alpha|C^{\beta+\varepsilon}$.

THEOREM V. *If $\alpha > 0$, $\beta > -1$, then for any $\varepsilon > 0$*

$$\mathrm{Bound}\, C^\alpha|C^\beta \cap \mathrm{Conv}\, C^\alpha|C^{\beta+1}|s \subset \mathrm{Conv}\, C^\alpha|C^{\beta+\varepsilon}|s\,.$$

Proof. It is enough to prove the theorem for $0 < \varepsilon < 1$, since for $\varepsilon \geqslant 1$ the theorem follows from theorem III. We have

$$\begin{aligned} C_n^\alpha|C^{\beta+\varepsilon}(x)-s| &= C_n^\alpha\left|\frac{S^\varepsilon\big(S^\beta(x)\big)-sA^{\beta+\varepsilon}}{A^{\beta+\varepsilon}}\right| \\ &= C_n^\alpha\left|\frac{S^\varepsilon\big(S^\beta(x)-sA^\beta\big)}{A^{\beta+\varepsilon}}\right| \\ &= \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}\frac{A_{n-\nu}^{\alpha-1}}{A_\nu^{\beta+\varepsilon}}\left|\sum_{\mu=0}^{\nu}A_{\nu-\mu}^{\varepsilon-1}\big(S_\mu^\beta(x)-sA_\mu^\beta\big)\right| \\ &= \frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}\frac{A_{n-\nu}^{\alpha-1}}{A_\nu^{\beta+\varepsilon}}\left|\sum_{0\leqslant\mu\leqslant\nu\omega}+\sum_{\nu\omega<\mu\leqslant\nu}\right|, \end{aligned}$$

where ω is some real number of the interval $(0,1)$ and which will be chosen in a suitable manner.

If $\beta+\varepsilon \leqslant 0$, we have

$$C_n^\alpha|C^{\beta+\varepsilon}(x)-s| \leqslant \frac{1}{A_n^\alpha A_n^{\beta+\varepsilon}}\left(\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}\left|\sum_{0\leqslant\mu\leqslant\nu\omega}\right|+\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}\left|\sum_{\nu\omega<\mu\leqslant\nu}\right|\right) = \mathrm{I}+\mathrm{II}\,,$$

where

$$\begin{aligned} \mathrm{II} &\leqslant \frac{1}{A_n^\alpha A_n^{\beta+\varepsilon}}\sum_{\nu=0}^{n}A_{n-\nu}^{\alpha-1}\sum_{\nu\omega<\mu\leqslant\nu}A_{\nu-\mu}^{\varepsilon-1}A_\mu^\beta|C_\mu^\beta(x)-s| \\ &= \frac{1}{A_n^\alpha A_n^{\beta+\varepsilon}}\Big[\sum_{0\leqslant\mu\leqslant n\omega}A_\mu^\beta|C_\mu^\beta(x)-s|\sum_{\mu\leqslant\nu<\mu/\omega}A_{n-\nu}^{\alpha-1}A_{\nu-\mu}^{\varepsilon-1}+ \\ &\qquad+\sum_{n\omega<\mu\leqslant n}A_\mu^\beta|C_\mu^\beta(x)-s|\sum_{\mu\leqslant\nu\leqslant n}A_{n-\nu}^{\alpha-1}A_{\nu-\mu}^{\varepsilon-1}\Big] = \mathrm{II}'+\mathrm{II}''\,, \end{aligned}$$

$$\begin{aligned} \mathrm{II}'' &= \frac{1}{A_n^\alpha A_n^{\beta+\varepsilon}}\sum_{n\omega<\mu\leqslant n}A_\mu^\beta|C_\mu^\beta(x)-s|A_{n-\mu}^{\alpha+\varepsilon-1} \\ &= \frac{1}{A_n^\alpha A_n^{\beta+\varepsilon}}\sum_{n\omega<\mu\leqslant n}A_{n-\mu}^{\alpha-1}A_\mu^\beta|C_\mu^\beta(x)-s|\cdot\frac{A_{n-\mu}^{\alpha+\varepsilon-1}}{A_{n-\mu}^{\alpha-1}}\,. \end{aligned}$$

Since, for $n\omega < \mu \leqslant n$,

$$\frac{A_{n-\mu}^{\alpha+\varepsilon-1}}{A_{n-\mu}^{\alpha-1}} = O(A_{n-\mu}^{\varepsilon}) = O(A_{[n(1-\omega)]}^{\varepsilon}),$$

we have

$$\mathrm{II}'' = O\left(\frac{A_{[n(1-\omega)]}^{\varepsilon}}{A_n^{\alpha}A_n^{\beta+\varepsilon}}\sum_{\mu=0}^{n} A_{n-\mu}^{\alpha-1}A_{\mu}^{\beta}|C_{\mu}^{\beta}(x)-s|\right)$$

$$= O\left(\frac{A_{[n(1-\omega)]}^{\varepsilon}}{A_n^{\beta+\varepsilon}}\, C_n^{\alpha}\big(A^{\beta}|C^{\beta}(x)-s|\big)\right),$$

and therefore, in virtue of lemma 1,

$$\mathrm{II}'' = O\left(\frac{A_{[n(1-\omega)]}^{\varepsilon}A_n^{\beta}}{A_n^{\beta+\varepsilon}}\right) = O\big((1-\omega)^{\varepsilon}\big).$$

Now, for $0 < \alpha < 1$,

$$\mathrm{II}' \leqslant \frac{1}{A_n^{\alpha}A_n^{\beta+\varepsilon}}\sum_{0\leqslant\mu\leqslant n\omega} A_{\mu}^{\beta}|C_{\mu}^{\beta}(x)-s|A_{n-[\mu/\omega]}^{\alpha-1}A_{[\mu(1-\omega)/\omega]}^{\varepsilon}$$

$$\leqslant \frac{A_{[n(1-\omega)]}^{\varepsilon}}{A_n^{\alpha}A_n^{\beta+\varepsilon}}\sum_{0\leqslant\mu\leqslant n\omega} A_{\mu}^{\beta}|C_{\mu}^{\beta}(x)-s|A_{[n\omega-\mu]}^{\alpha-1}\cdot\frac{A_{[(n\omega-\mu)/\omega]}^{\alpha-1}}{A_{[n\omega-\mu]}^{\alpha-1}}$$

$$= O\left(\frac{A_{[n(1-\omega)]}^{\varepsilon}}{A_n^{\alpha}A_n^{\beta+\varepsilon}}\,\frac{A_{[n\omega]}^{\alpha}}{\omega^{\alpha-1}}\,C_{[n\omega]}^{\alpha}\big(A^{\beta}|C^{\beta}(x)-s|\big)\right)$$

$$= O\left(\frac{A_{[n(1-\omega)]}^{\varepsilon}A_{[n\omega]}^{\alpha}A_{[n\omega]}^{\beta}}{A_n^{\alpha}A_n^{\beta+\varepsilon}\omega^{\alpha-1}}\right) = O\big((1-\omega)^{\varepsilon}\omega^{\beta+1}\big).$$

It is easy to prove that for $\alpha \geqslant 1$ $\mathrm{II}'' = O\big((1-\omega)^{\varepsilon}\big)$, but, according to Zygmund's remark, this case may be ommitted.

Thus $\mathrm{II} = \mathrm{II}'+\mathrm{II}'' = O\big((1-\omega)^{\varepsilon}\big)$, whence it follows that, if $|1-\omega|$ is small enough, the sum II is arbitrarily small.

Let ω be fixed. Applying Abel's transformation to the sum I we obtain

$$\mathrm{I} = \frac{1}{A_n^{\alpha}A_n^{\beta+\varepsilon}}\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\Big|\sum_{0\leqslant\mu<\nu\omega} A_{\nu-\mu}^{\varepsilon-2}\big(S_{\mu}^{\beta+1}(x)-sA_{\mu}^{\beta+1}\big) + $$
$$+A_{[\nu(-\omega)]}^{\varepsilon}\big(S_{[\nu\omega]}^{\beta+1}(x)-sA_{[\nu\omega]}^{\beta+1}\big)\Big|$$

$$\leqslant \frac{1}{A_n^{\alpha}A_n^{\beta+\varepsilon}}\Big[\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}\sum_{0\leqslant\mu<\nu\omega}|A_{\nu-\mu}^{\varepsilon-2}|A_{\mu}^{\beta+1}|C_{\mu}^{\beta+1}(x)-s| + $$
$$+\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1}A_{[\nu(1-\omega)]}^{\varepsilon-1}A_{[\nu\omega]}^{\beta+1}|C_{[\nu\omega]}^{\beta+1}(x)-s|\Big] = \mathrm{I}'+\mathrm{I}''.$$

Since, for any ν and for fixed ω,

$$A^{\varepsilon-1}_{[\nu(1-\omega)]}A^{\beta+1}_{[\nu\omega]} \leqslant K A^{\beta+\varepsilon}_{[\nu\omega]},$$

where K is some constant, we have

$$\mathrm{I}'' \leqslant \frac{K}{A^\alpha_n A^{\beta+\varepsilon}_n}\sum_{\nu=0}^{n} A^{\alpha-1}_{n-\nu}A^{\beta+\varepsilon}_{[\nu\omega]}|C^{\beta+1}_{[\nu\omega]}(x)-s|\,.$$

Replacing the index $[\nu\omega]$ by λ we obtain

$$\begin{aligned}\mathrm{I}'' &\leqslant \frac{K}{A^\alpha_n A^{\beta+\varepsilon}_n}\cdot\frac{1}{\omega}\sum_{\lambda=0}^{[n\omega]} A^{\alpha-1}_{[(n\omega-\lambda)/\omega]}A^{\beta+\varepsilon}_{\lambda}|C^{\beta+1}_{\lambda}(x)-s| \\ &\leqslant \frac{K'}{A^\alpha_n A^{\beta+\varepsilon}_n}\cdot A^\alpha_{[n\omega]}C^\alpha_{[n\omega]}\big(A^{\beta+\varepsilon}|C^{\beta+1}(x)-s|\big),\end{aligned}$$

whence, in virtue of lemma 1,

$$\mathrm{I}'' = o\left(\frac{A^\alpha_{[n\omega]}A^{\beta+\varepsilon}_{[n\omega]}}{A^\alpha_n A^{\beta+\varepsilon}_n}\right) = o(1)\,.$$

Next, we have

$$\mathrm{I}' = \frac{1}{A^\alpha_n A^{\beta+\varepsilon}_n}\sum_{0\leqslant\mu\leqslant n\omega} A^{\beta+1}_\mu|C^{\beta+1}_\mu(x)-s|\sum_{\mu/\omega\leqslant\nu\leqslant n} A^{\alpha-1}_{n-\nu}|A^{\varepsilon-2}_{\nu-\mu}|\,.$$

If $0<\alpha<1$, then, for an arbitrary $\theta\in(0,1)$,

$$\begin{aligned}\mathrm{I}' = \frac{1}{A^\alpha_n A^{\beta+\varepsilon}_n}\Big[&\sum_{0\leqslant\mu\leqslant n\omega\theta} A^{\beta+1}_\mu|C^{\beta+1}_\mu(x)-s|\Big(\sum_{\mu/\omega\leqslant\nu\leqslant n\theta}+\sum_{n\theta<\nu\leqslant n}\Big)+ \\ &+\sum_{n\omega\theta<\mu\leqslant n\omega} A^{\beta+1}_\mu|C^{\beta+1}_\mu(x)-s|\sum_{\mu/\omega<\nu\leqslant n} A^{\alpha-1}_{n-\nu}|A^{\varepsilon-2}_{\nu-\mu}|\Big] = \mathrm{I}_1'+\mathrm{I}_2'+\mathrm{I}_3',\end{aligned}$$

where

$$\begin{aligned}\mathrm{I}_1' &\leqslant \frac{1}{A^\alpha_n A^{\beta+\varepsilon}_n}\sum_{0\leqslant\mu\leqslant n\omega\theta} A^{\beta+1}_\mu|C^{\beta+1}_\mu(x)-s|A^{\alpha-1}_{[n(1-\theta)]}\sum_{\nu\geqslant\mu/\omega}|A^{\varepsilon-2}_{\nu-\mu}| \\ &\leqslant K\frac{A^{\alpha-1}_n}{A^\alpha_n A^{\beta+\varepsilon}_n}\sum_{0\leqslant\mu\leqslant n\omega\theta} A^{\beta+1}_\mu|C^{\beta+1}_\mu(x)-s|A^{\varepsilon-1}_{[(1-\omega)\mu/\omega]} \\ &\leqslant K'\frac{A^{\alpha-1}_n}{A^\alpha_n A^{\beta+\varepsilon}_n}\sum_{0\leqslant\mu\leqslant n\omega\theta} A^{\beta+1}_\mu|C^{\beta+1}_\mu(x)-s| \\ &\leqslant K'\frac{A^{\alpha-1}_n A^1_n}{A^\alpha_n A^{\beta+\varepsilon}_n}C^1_n\big(A^{\beta+1}|C^{\beta+1}(x)-s|\big)\,.\end{aligned}$$

Since, for $0 < \alpha < 1$, $C_n^\alpha(A^{\beta+1}|C^{\beta+1}(x)-s|) = o(A_n^{\beta+\varepsilon})$, this approximation holds also for $\alpha = 1$, whence $\mathrm{I}_1' = o(1)$. Next we have

$$\mathrm{I}_2' \leqslant \frac{1}{A_n^\alpha A_n^{\beta+\varepsilon}} \sum_{0\leqslant\mu\leqslant n\omega\theta} A_\mu^{\beta+1}|C_\mu^{\beta+1}(x)-s|\,|A_{[n(1-\theta)]}^{\varepsilon-2}| \sum_{n\theta\leqslant\nu\leqslant n} A_{n-\nu}^{\alpha-1}$$
$$\leqslant K\frac{|A_n^{\varepsilon-2}|}{A_n^\alpha A_n^{\beta+\varepsilon}} \sum_{0\leqslant\mu\leqslant n\omega\theta} A_\mu^{\beta+1}|C_\mu^{\beta+1}(x)-s|A_{[n(1-\theta)]}^\alpha$$
$$\leqslant K'\frac{|A_n^{\alpha+\varepsilon-2}|A_n^1}{A_n^\alpha A_n^{\beta+\varepsilon}} C_n^1(A^{\beta+1}|C^{\beta+1}(x)-s|) = o(1)$$

and

$$\mathrm{I}_3' = \frac{1}{A_n^\alpha A_n^{\beta+\varepsilon}} \sum_{n\omega\theta<\mu<n\omega} A_\mu^{\beta+1}|C_\mu^{\beta+1}(x)-s| \sum_{\mu/\omega<\nu\leqslant n} A_{n-\nu}^{\alpha-1}|A_{\nu-\mu}^{\varepsilon-2}|$$
$$\leqslant \frac{1}{A_n^\alpha A_n^{\beta+\varepsilon}} \sum_{n\omega\theta<\mu<n\omega} A_\mu^{\beta+1}|C_\mu^{\beta+1}(x)-s|\,|A_{[(1-\omega)\mu/\omega]}^{\varepsilon-2}|A_{n-[\mu/\omega]}^\alpha$$
$$\leqslant K\frac{|A_{[n(1-\omega)\theta]}^{\varepsilon-2}|}{A_n^\alpha A_n^{\beta+\varepsilon}} \sum_{0\leqslant\mu\leqslant n\omega} A_\mu^{\beta+1}|C_\mu^{\beta+1}(x)-s|A_{[n\omega]-\mu}^\alpha$$
$$\leqslant K'\frac{|A_n^{\varepsilon-2}|A_{[n\omega]}^{\alpha+1}}{A_n^\alpha A_n^{\beta+\varepsilon}} C_{[n\omega]}^{\alpha+1}(A^{\beta+1}|C^{\beta+1}(x)-s|)$$
$$= o\left(\frac{A_n^{\varepsilon-2}A_{[n\omega]}^{\alpha+1}A_{[n\omega]}^{\beta+1}}{A_n^\alpha A_n^{\beta+\varepsilon}}\right) = o(1),$$

so that $\mathrm{I}_1'+\mathrm{I}_2'+\mathrm{I}_3' = o(1)$ for $0<\alpha<1$. Thus the sum $\mathrm{I}+\mathrm{II}$ is arbitrarily small for $\beta+\varepsilon\leqslant 0$ and $0<\alpha<1$. If $\beta+\varepsilon>0$, we choose such β', ε' that $\beta>\beta'>-1$, $0<\varepsilon'<1$, $\beta'+\varepsilon'<0$ and we apply lemma 2.

THEOREM VI. *If $\alpha>0$, $\beta>-1$, $\gamma>0$, then for any $\varepsilon>0$*

$$\mathrm{Bound}\,C^\alpha|C^\beta \cap \mathrm{Conv}\,C^\alpha|C^{\beta+\gamma}|s \subset \mathrm{Conv}\,C^\alpha|C^{\beta+\varepsilon}|s.$$

Proof. If $\gamma\leqslant 1$, then, in virtue of theorem III, $x\in\mathrm{Conv}\,C^\alpha|C^{\beta+\varepsilon}|s$, and theorem VI follows from the theorem V. If $1<\gamma<2$, then, in virtue of theorem III, $x\in\mathrm{Bound}\,C^\alpha|C^{\beta+\gamma-1}$ and, in virtue of theorem V, $x\in C^\alpha|C^{\beta+\gamma-1+\varepsilon}|s$ for any $\varepsilon>0$. Now, making $\varepsilon=2-\gamma$, we obtain again $x\in\mathrm{Conv}\,C^\alpha|C^{\beta+1}|s$. If $\gamma=2$, then, taking into account that $x\in\mathrm{Bound}\,C^\alpha|C^{\beta+1}$, we obtain, in virtue of theorem V, $x\in\mathrm{Conv}\,C^\alpha|C^{\beta+1+\varepsilon}|s$ for any $\varepsilon>0$. Let $\varepsilon\in(0,1)$ and $\gamma'=1+\varepsilon$; then $1<\gamma'<2$ and $x\in\mathrm{Conv}\,C^\alpha|C^{\beta+\gamma'}|s$, whence $x\in\mathrm{Conv}\,C^\alpha|C^\beta|s$. This reasoning may be continued for $\gamma>2$ in the obvious way.

THEOREM VII. *If $\alpha>0$, $\beta>-1$ and $\gamma>-1$, then for any $\varepsilon>0$*

$$\mathrm{Bound}\,C^\alpha|C^\beta \cap \mathrm{Conv}\,C^\gamma|s \subset \mathrm{Conv}\,C^\alpha|C^{\beta+\varepsilon}|s.$$

Proof. Since $C_n^\gamma(x)-s\to 0$, we have, for any $\alpha>0$ and $\varepsilon'>0$, $C_n^\alpha|C^{\gamma+\varepsilon'}(x)-s|\to 0$. Now, let us choose such $\varepsilon'>0$ that $\gamma+\varepsilon'>\beta$. Applying theorem VI we obtain theorem VII.

4. Cauchy's product of the series $\sum_{n=0}^{\infty}a_n$ and $\sum_{n=0}^{\infty}b_n$ is defined as the series $\sum_{n=0}^{\infty}(a\divideontimes b)_n$, where $a=(a_0, a_1, \dots)$ and $b=(b_0, b_1, \dots)$. This classical definition may be formulated in terms of sequences. If we denote $x=S^1(a)$ and $y=S^1(b)$, then the sequence $S^1(a\divideontimes b)$ is Cauchy's product of the series $S^1(a)$ and $S^1(b)$ and may be called *Cauchy's product* of the sequences x and y. Denoting this product by $x\circ y$ we have

$$x\circ y = S^1(a\divideontimes b) = S^1\big(S^{-1}(x)\divideontimes S^{-1}(y)\big) = S^{-1}(x\divideontimes y)\,.$$

THEOREM VIII. *If $x\in \operatorname{Conv} C^\alpha|C^\beta|s$ for $\alpha>0$, $\beta>-1$ and $y\in \operatorname{Conv} C^\gamma|t$ for $\gamma\geqslant 0$, then $x\circ y\in \operatorname{Conv} C^{\alpha+\beta+\gamma}|st$.*

Proof. It is to be shown that the sequence

$$C_n^{\alpha+\beta+\gamma}(x\circ y) = \frac{S_n^{\alpha+\beta+\gamma-1}(x\divideontimes y)}{A_n^{\alpha+\beta+\gamma}} = \frac{\big(S^{\alpha+\beta-1}(x)\divideontimes S^\gamma(y)\big)_n}{A_n^{\alpha+\beta+\gamma}}$$

$$= \frac{1}{A_n^{\alpha+\beta+\gamma}}\sum_{\nu=0}^{n} S_{n-\nu}^{\alpha+\beta-1}(x)\,A_\nu^\gamma\, C_\nu^\gamma(y)$$

converges to st. Let us suppose that $t=0$. Since $C_\nu^\gamma(y)\to 0$, it is sufficient to prove that the matrix

$$a_{n\nu} = \frac{S_{n-\nu}^{\alpha+\beta-1}(x)\,A_\nu^\gamma}{A_n^{\alpha-\beta+\gamma}}$$

is a Toeplitz matrix, that is, it fulfills the following Toeplitz conditions:

(i) $$\lim_{n\to\infty} a_{n\nu}=0 \quad \text{for} \quad \nu=0,1,2,\dots;$$

(ii) $$\sum_{\nu=0}^{n}|a_{n\nu}|\leqslant H \quad \text{for} \quad n=0,1,2,\dots$$

Since

$$S^{\alpha+\beta-1}(x) = S^{\alpha-1}\big[A^\beta\big(C^\beta(x)-s\big)\big]+sA^{\alpha+\beta-1}\,,$$

we have

$$|a_{n\nu}|\leqslant \frac{S_{n-\nu}^{\alpha-1}\big(A^\beta|C^\beta(x)-s|\big)A_\nu^\gamma}{A_n^{\alpha+\beta+\gamma}} + |s|\frac{A_{n-\nu}^{\alpha+\beta-1}A_\nu^\gamma}{A_n^{\alpha+\beta+\gamma}}\,.$$

Now, in virtue of lemma 1, $C_n^\alpha(A^\beta|C^\beta(x)-s|)=o(A_n^\beta)$, whence

$$C_n^{\alpha-1}(A^\beta|C^\beta(x)-s|)=o(A_n^{\beta+1}), \qquad S_n^{\alpha-1}(A^\beta|C^\beta(x)-s|)=o(A_n^{\alpha+\beta}),$$

and therefore $a_{n\nu}=o(1)$ for $n\to\infty$. Next, in virtue of lemma 1,

$$\sum_{\nu=0}^{n}|a_{n\nu}|\leqslant\frac{S^{\alpha+\gamma}(A^\beta|C^\beta(x)-s|)}{A_n^{\alpha+\beta+\gamma}}+|s|$$
$$=\frac{A_n^{\alpha+\gamma}}{A_n^{\alpha+\beta+\gamma}}C_n^{\alpha+\gamma}(A^\beta|C^\beta(x)-s|)+|s|$$
$$=\frac{A_n^{\alpha+\gamma}}{A_n^{\alpha+\beta+\gamma}}o(A_n^\beta)+|s|=O(1).$$

Thus the Toeplitz conditions are fulfilled and the theorem is established in the case $t=0$. If $t\neq 0$, we introduce the sequence $y'=(y_0-t, y_1-t, y_2-t, \ldots)$. Since $C_n^{\alpha+\beta+\gamma}(x)\to s$ and

$$C_n^{\alpha+\beta+\gamma}(x\circ y)=C_n^{\alpha+\beta+\gamma}(x\circ y')+tC_n^{\alpha+\beta+\gamma}(x),$$

we have, in virtue of the result just obtained, $C_n^{\alpha+\beta+\gamma}(x\circ y)\to t\cdot s$.

THEOREM IX. *If* $x\in C^\alpha|C^\beta|s$ *for* $\alpha>0$, $\beta>-1$ *and* $y\in C^\alpha|C^{\beta'}|t$ *for* $\beta'>-1$, *then* $x\circ y\in C^\alpha|C^{\beta+\beta'+1}|st$.

Proof. We have

$$C^{\beta+\beta'+1}(x\circ y)-st=\frac{1}{A^{\beta+\beta'+1}}[S^{\beta+\beta'}(x * y)-stA^{\beta+\beta'+1}]$$
$$=\frac{1}{A^{\beta+\beta'+1}}[S^\beta(x)*S^{\beta'}(y)-stA^\beta*A^{\beta'}]$$
$$=\frac{1}{A^{\beta+\beta'+1}}\{[S^\beta(x)-sA^\beta]*[S^{\beta'}(y)-tA^{\beta'}]+$$
$$+[S^\beta(x)-sA^\beta]*tA^{\beta'}+sA^\beta*[S^{\beta'}(y)-tA^{\beta'}]\}$$
$$=\frac{1}{A^{\beta+\beta'+1}}\{A^\beta[C^\beta(x)-s]*A^{\beta'}[C^{\beta'}(y)-t]+$$
$$+A^\beta[C^\beta(x)-s]*tA^{\beta'}+sA^\beta*A^{\beta'}[C^{\beta'}(y)-t]\},$$

whence

$$C_n^\alpha|C^{\beta+\beta'+1}(x\circ y)-st|$$
$$\leqslant\frac{1}{A_n^\alpha}\sum_{\nu=0}^{n}\frac{A_{n-\nu}^{\alpha-1}}{A_\nu^{\beta+\beta'+1}}\Big\{\sum_{\mu=0}^{\nu}A_\mu^\beta|C_\mu^\beta(x)-s|A_{\nu-\mu}^{\beta'}|C_{\nu-\mu}^{\beta'}(y)-t|+$$
$$+|t|\sum_{\mu=0}^{\nu}A_\mu^\beta|C^\beta(x)-s|A_{\nu-\mu}^{\beta'}+|s|\sum_{\mu=0}^{\nu}A_{\nu-\mu}^\beta A_\mu^{\beta'}|C_\mu^{\beta'}(y)-t|\Big\}$$

$$= \frac{1}{A_n^\alpha}\Big\{\sum_{\mu=0}^{n} A_\mu^\beta |C_\mu^\beta(x)-s| \sum_{\nu=0}^{n-\mu} \frac{A_{n-\nu-\mu}^{\alpha-1} A_\nu^{\beta'}}{A_{\nu+\mu}^{\beta+\beta'+1}} |C_\nu^{\beta'}(y)-t| +$$

$$+ |t| \sum_{\mu=0}^{n} A_\mu^\beta |C_\mu^\beta(x)-s| \sum_{\nu=0}^{n-\mu} \frac{A_{n-\nu-\mu}^{\alpha-1} A_\nu^{\beta'}}{A_{\nu+\mu}^{\beta+\beta'+1}} +$$

$$+ |s| \sum_{\mu=0}^{n} A_\mu^{\beta'} |C_\mu^{\beta'}(y)-t| \sum_{\nu=0}^{n-\mu} \frac{A_{n-\nu-\mu}^{\alpha-1} A_\nu^{\beta}}{A_{\nu+\mu}^{\beta+\beta'+1}} \Big\}.$$

If $\beta+\beta'+1<0$, then, in virtue of lemma 1,

$$C_n^\alpha|C^{\beta+\beta'+1}(x\circ y)-st|$$

$$\leqslant \frac{1}{A_n^\alpha A_n^{\beta+\beta'+1}}\Big\{\sum_{\mu=0}^{n} A_\mu^\beta |C_\mu^\beta(x)-s| A_{n-\mu}^\alpha C_{n-\mu}^\alpha\big(A^{\beta'}|C^{\beta'}(y)-t|\big) +$$

$$+ |t| A_n^{\alpha+\beta'+1} C_n^{\alpha+\beta'+1}\big(A^\beta|C^\beta(x)-s|\big) + |s| A_n^{\alpha+\beta+1} C_n^{\alpha+\beta+1}\big(A^{\beta'}|C^{\beta'}(y)-t|\big)\Big\}$$

$$= \frac{1}{A_n^\alpha A_n^{\beta+\beta'+1}}\Big\{\sum_{\mu=0}^{n} A_\mu^\beta |C_\mu^\beta(x)-s| o(A_{n-\mu}^{\alpha+\beta'}) +$$

$$+ |t| o(A_n^{\alpha+\beta+\beta'+1}) + |s| o(A_n^{\alpha+\beta+\beta'+1})\Big\}$$

$$= \frac{1}{A_n^\alpha A_n^{\beta+\beta'+1}}\Big\{o\Big(S_n^{\alpha+\beta'+1}\big(A^\beta|C^\beta(x)-s|\big)\Big) + o(A_n^{\alpha+\beta+\beta'+1})\Big\}$$

$$= \frac{o(A_n^{\alpha+\beta+\beta'+1})}{A_n^\alpha A_n^{\beta+\beta'+1}} = o(1)\,.$$

If $\beta+\beta'+1>0$, we choose such $\tilde\beta, \tilde\beta'$ that $\beta>\tilde\beta>-1$, $\alpha'>\tilde\beta'>-1$ and $\tilde\beta+\tilde\beta'+1<0$. Then, in virtue of lemma 2, there is such a constant K that for any μ,ν

$$\frac{A_\mu^\beta A_\mu^{\beta'}}{A_{\nu+\mu}^{\beta+\beta'+1}} \leqslant K\,\frac{A_\mu^{\tilde\beta} A_\nu^{\tilde\beta'}}{A_{\nu+\mu}^{\tilde\beta+\tilde\beta'+1}}$$

and therefore we get the case just considered.

References

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Reçu par la Rédaction le 2. 11. 1960