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Distributions invariant under compact Lie groups

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Abstract. A characterization of invariant smooth functions and distributions in terms of smooth functions and distributions on the orbit space is given. To this end the concept of an elliptic space is introduced.

Introduction. In this paper I give a characterization of functions and distributions invariant under a smooth action of a compact Lie group. The problem itself has rather a long history. One of the main steps was the theorem on invariant polynomials which expresses such polynomials in terms of basic invariants (see Weyl [10]). In some special cases, when there are no functional relations among the basic invariants, also smooth functions are expressable in terms of those invariants, the same being true for distributions; cf. Schwartz [9] and Oksak [6]. In these cases the basic invariants define a sort of differentiable structure on the orbit space. Still, when functional bounds on invariants do appear, the orbit space is a manifold with singularities on its boundary, and trying to define a differentiable structure on it proves to be a rather difficult task.

Given an invariant differentiable function f, one of the problems is to express in the language of the orbit space what it means to say that f is differentiable. I suggest a solution of this problem in terms of invariant elliptic operators. Namely, given a continuous function f and an elliptic operator P, we can express the fact that f is smooth in the form: $P^i f$ is continuous for all iterates of P; this follows immediately from the regularity theorem for elliptic operators. Now, assuming all objects invariant, the fact that $P^i f$ is continuous can easily be expreseed in terms of the continuity of the respective functions on the orbit space and of the projection L of the operator P onto the orbit space. This shows that invariant smooth functions are characterized in terms of an operator defined on the "interior" of the orbit space and it is only the topology at the "boundary" that matters. This leads to the idea of introducing spaces with a distinguished elliptic operator. Such spaces will be called *elliptic spaces*. Many concepts of differential geometry can be defined in such spaces. In this paper, however, we confine ourselves only to those which are necessary for a characterization of invariant functions and distributions. Such a characterization is provided by

Theorems 1 and 3 which state that there is a natural isomorphism between smooth functions (distributions) on the orbit space and invariant smooth functions (invariant distributions).

1. Notation and definitions. In the sequel G will always denote a compact Lie group. M is a paracompact connected n-dimensional C^∞-manifold. We shall denote by $R\colon G \times M \to M$ a fixed smooth action of G on M. We also write $R_g\colon M \to M$, $R_g(x) = R(g, x)$. The symbol M/G or M^* will denote the orbit space of M under the action of G together with the natural orbit space topology.

Greek letters α, β denote densities on M^*, while ω (possibly with subscripts) stands for densities on M.

By an *elliptic operator of order m* we shall understand a C^∞ real linear differential elliptic operator acting on C^∞-functions on M. In a local chart such an operator has the following form:

$$L = \sum_{k=0}^{m} \sum_{|\gamma|=k} A_\gamma(x) \frac{\partial^k}{\partial x^\gamma}$$

with real C^∞-coefficients A_γ and the characteristic form

$$\sum_{|\gamma|=m} A_\gamma(x)\xi^\gamma \neq 0 \quad \text{for real } \xi \neq 0,\ x \in M.$$

Actually, the ellipticity of L will be applied only to ensure the validity of the following theorem:

THEOREM. *Let f be an integrable function on an open set $\Omega \subset R^n$ and P an elliptic operator acting on f in the sense of distributions. Then f is equal a.e. to a smooth function iff $P^i f$ is integrable for all iterates P^i of P.*

Moreover, for a continuous function f, if $P^i f$ is continuous, then $f \in C^{im-[n/2]}(\Omega)$, where m is the order of P.

The proof of this theorem can be found in John [3]. In fact, in the theorem given by John all functions are assumed continuous. However, this assumption may be weakened assuming integrability instead. The proof in [4] is then transformed, by obvious modifications, so as to work in this case; also see the Remark in [4].

2. Elliptic spaces. Let N be a topological Hausdorff space in which there exists an open dense subset $A \subset N$ such that A is a p-dimensional manifold. We shall denote any such space by (A, N). Let L be an elliptic operator on A. The triple (A, N, L) will be called an *elliptic space*.

Let $C_N^0(N)$ be the space of continuous functions on N, which are C^∞ on A. Write $L^i = \underbrace{L \circ \ldots \circ L}_{i \text{ times}}$ for $i > 0$ and $L^0 = \mathrm{id}$. Write $L^{-i}(Z)$ for the counter-image of a $Z \subset C^0(N)$ under L^i. Define:

$$C_{AN}^{\infty,k}(N) = \bigcap_{i=0}^{k} L^{-i}(C^0(N)), \qquad C^\infty(N) = \bigcap_{i=0}^{\infty} L^{-i}(C^0(N)).$$

Remark 1. It follows from the ellipticity of L that $C^{\infty}(N) = \bigcap_{i=0}^{\infty} L^{-i}(C(N))$, where $C(N)$ is the set of continuous functions on N and L acts on $C(N)$ in the distributional sense.

DEFINITION 1. The set $C^{\infty}(N) = \bigcap_{i=0}^{\infty} L^{-i}(C^{0}(N))$ will be called the *set of smooth functions on* N. Thus a function $f \in C^{0}(N)$ is smooth on N provided $L^{i} f \in C^{0}(N)$ for all positive integers i.

For a compact set $K \subset N$ define $C^{\infty k}_{AN}(K) = \{f \in C^{\infty k}_{AN}(N)\colon \operatorname{supp} f \subset K\}$. Let $C_0^{\infty}(N)$ denote the set of compactly supported functions in $C^{\infty}(N)$. In $C^{\infty k}_{AN}(K)$ we introduce the topology given by the norm $\|f\|^{k} = \sum_{i=0}^{k} \sup_{x} |L^{i} f|$ which turns $C^{\infty k}_{AN}(K)$ into a Banach space. In $C_0^{\infty}(N)$ we define a topology in the natural way, i.e., we write $C_0^{\infty}(N) \ni f_j \underset{j\to\infty}{\to} 0$ if there is a compact set K such that $\operatorname{supp} f_j \subset K$ for $j = 1, 2, \ldots$ and $\|f_j\|^{k} \underset{j\to\infty}{\to} 0$ for every k. It is not difficult to prove that $C_0^{\infty}(N)$ is complete.

Now we pass to the definition of smooth densities on N.

Denote by $\Omega^{\infty 0}_{AN}(N)$ the set of smooth, integrable p-densities on A. Denoting by L^{*} the adjoint of L (L^{*} acts on smooth densities on A) we define, as in the case of functions,

$$\Omega^{\infty}(N) = \bigcap_{i=0}^{\infty} (L^{*})^{-i}\big(\Omega^{\infty 0}_{AN}(N)\big).$$

Remark 2. As in the case of functions, $\Omega^{\infty}(N) = \bigcap_{i=0}^{\infty} (L^{*})^{-i}\big(\Omega(N)\big)$ where $\Omega(N)$ is the set of integrable densities on A where densities equal a.e. are identified. L^{*} acts on such densities in the distributional way.

Let $\Omega_0^{\infty}(N)$ be the set of compactly supported densities in $\Omega^{\infty}(N)$. We shall define a topology in $\Omega_0^{\infty}(N)$, but we begin with some general remarks.

Remark 3. Let α be a p-density (p-form) on a p-dimensional manifold A; A is not assumed to be orientable. Then one can define the integral $\int|\alpha|$ of the absolute value of α, in the following way. Choose open disjoint subsets U_i such that $\bigcup_{i=1}^{\infty} \bar{U}_i = A$, $\bigcup_{i=1}^{\infty} \operatorname{bd} U_i$ has measure zero and each U_i is in the domain of some chart. Let $\alpha_i = \alpha|_{U_i}$ and let $a_i(x)$ be the coefficient of α_i in the chart (Φ, U_i). Define

$$\int|\alpha_i| = \int_{\Phi(U_i)} |a_i(x)|$$

if a_i is integrable. Given another chart Ψ on U_i, if $b_i(x)$ is the coordinate of α_i in Ψ, we get from the definition of a p-density (p-form) that $|a_i| = |b_i| \cdot |J_H|$, where J_H is the Jacobian of $H = \Psi \circ \Phi^{-1}$, which shows that the above integral is independent of the choice of a particular chart. Finally, we set

$$\int_A |\alpha| = \sum_{i=1}^{\infty} \int |\alpha_i|,$$

provided the right-hand side make sense and is independent of the choice of the sets U_i.

For $\alpha \in \Omega_0^\infty(N)$ set

$$\|\alpha\|^k = \sum_{i=1}^{k} \int_A |(L^*)^i \alpha|.$$

Define convergence in $\Omega_0^\infty(N)$ in the following sequential way. Let $\alpha_j \in \Omega_0^\infty(N)$. We say that α_j tends to zero if $\|\alpha_j\|^k$ tends to zero for every k and there exists a compact set $K \subset N$ such that $\operatorname{supp} \alpha_j \subset K$ for all j.

Proposition 1. *The space $\Omega_0^\infty(N)$ with the above-defined convergence is complete.*

Proof. Fix a compact set $K \subset N$. Set $\Omega^\infty(K) = \{f \in \Omega^\infty(N)$: $\operatorname{supp} f \subset K\}$. Also define $Z_k(K) = \bigcap_{i=0}^{k} (L^*)^{-i}(\Omega(K))$ with $\Omega(K)$ being the set of p-densities integrable on N with support in K. It is enough to show that $\Omega^\infty(K)$ is complete. But in view of Remark 2, $\Omega^\infty(K) = \bigcap_{i=1}^{\infty} Z_i(K)$ and, since obviously every $Z_i(K)$ is complete in the norm $\| \ \|^i$, we are done.

Assuming A orientable, all the above constructions can be repeated for p-forms on N. In particular, the notion of a smooth p-form is then defined. Also one can introduce the concept of orientability on an elliptic space N.

Definition 2. Consider (A, N) with A orientable. We say that N is *orientable* if there exists a smooth p-form α_0 on N such that every other smooth p-form α on N is representable in the form $\alpha = h \cdot \alpha_0$ with a continuous function h.

In fact, all such functions h have properties analogous to those of functions in $C^\infty(N)$. Since in applications these two spaces coincide, we shall not treat them separately.

Definition 3. A linear functional on $\Omega_0^\infty(N)$ continuous in the topology of this space is called a *distribution on* N. The space of all distributions on N will be denoted by $D'(N)$.

The general theory of topological vector spaces applied to $\Omega_0^\infty(N)$ yields the following result.

Proposition 2. *$u \in D'(N)$ iff for every compact set $K \subset N$ there is a positive integer k and a constant C such that*

$$|u[\alpha]| \leqslant C\|\alpha\|_k \quad \text{for} \quad \alpha \in \Omega^\infty(K)$$

($u[\alpha]$ *denotes the value of u on a density α*).

We end this section by noting the general fact that when A is oriented, p-denstities are identified in a canonical way with p-forms. This is done with the help of the 0-density of orientation denoted in the sequel by ε_p (for details see de Rham [8]).

3. The orbit space of compact Lie groups. Let M and G be as in the preceding section. Since G is compact, M^* is a Hausdorff space (Bredon [2]). Denote by $\tilde{M}$ the union of orbits of principal type (see Bredon [2], p. 179). Then it follows from Theorem 3.1 in Bredon [2] that $\tilde{M}$ is open and dense in M; hence $A = \pi(\tilde{M})$ is open and dense in $M^* = \pi(M)$, where $\pi\colon M \to M^*$ is the natural projection onto the orbit space. Now the slice theorem implies that A is a C^∞-manifold. Denote by s the dimension of any orbit in A. Then the differential system of rank $p = n-s$ on $\tilde{M}$ defined by the subspaces tangent to the orbits in $\tilde{M}$ is involutive. Moreover, it is regular, which is also immediate in view of the slice theorem. Leaves of the foliation given by this system agree with the orbits of the group G^0, the identity component of G. Denote the space of leaves by $\tilde{M}/G^0$. It follows from the results of Palais [7] that $\tilde{M}/G^0$ is a C^∞-manifold (it is Hausdorff since the leaves are compact). Moreover, it was proved in Ziemian [11] that the projection $\tilde{\pi}\colon \tilde{M} - \tilde{M}/G^0$ induces a linear operation $\tilde{K}$ from the space of compactly supported densities on $\tilde{M}$ onto that on $\tilde{M}/G^0$. In fact, $\tilde{K}$ is determined by the identity $\int\limits_{\tilde{M}} (\tilde{h}\circ\tilde{\pi})\cdot\omega = \int\limits_{\tilde{M}/G^0} \tilde{h}\cdot\tilde{K}\omega$ for every continuous function $\tilde{h}$ on $\tilde{M}/G^0$ and a density ω on $\tilde{M}$. We define an operation $K\colon \Omega_0^\infty(\tilde{M}) \to \Omega_0^\infty(A)$ by summing up the values of K over all components in $\tilde{M}/G^0$ of any orbit of G. Obviously, the operation thus defined satisfies $\int\limits_{\tilde{M}} h\circ\pi\cdot\omega = \int\limits_{A} h\cdot K\omega$ for a continuous h on A and $\omega \in \Omega_0^\infty(\tilde{M})$.

Note that K extends in a natural way to the space $\Omega'(\tilde{M})$.

Now let us devote a bit of attention to elliptic operators. Let P be an invariant elliptic operator on M. G being compact, this ensures that such an operator always exists. P restricted to $\tilde{M}$ induces an operator on A – simply by operating with P on invariant functions. Denote this operator by L. It can be proved (see e.g. Atiyah [1]) that L is also an elliptic operator on A. Summing up the above, we see that (A, M^*, L) is an elliptic space. It will be clear from the theorems which follow that the functional structure on (A, M^*, L), i.e. the spaces of smooth functions and smooth densities on M^*, do not depend on the choice of a particular operator P. In the sequel, P will denote an arbitrary fixed elliptic operator.

With the above notation we have the following theorem:

THEOREM 1. *The set $C^\infty(M^*)$ is topologically isomorphic to the set $C^\infty_{\text{inv}}(M)$ of C^∞-invariant functions on M. This isomorphism is given by*

$$C^\infty(M^*) \ni h \overset{J}{\mapsto} h\circ\pi \in C^\infty_{\text{inv}}(M).$$

Proof. If $h \in C^{\infty}(M^*)$ then h is continuous on M^* and so is $L^i h$ for all i. Thus $f = h \circ \pi$ is continuous on M and, for every i, $P^i f = (L^i h) \circ \pi$ is continuous. (P acts on f in the distributional sense.) Since P is elliptic, it follows from the regularity theorem that f is C^{∞}. Now $f = h \circ \pi$; hence $f \in C^{\infty}_{\mathrm{inv}}(M)$. To prove the continuity of J and J^{-1}, we note that this is equivalent to proving that the topology of local uniform convergence with all derivatives in $C^{\infty}(M)$ is equivalent to the topology of local uniform convergence of the iterates $P^i f$ for $f \in C^{\infty}(M)$, and this follows immediately from Friedrichs' inequality and Sobolev's lemma (see [5]).

Now we shall establish the connection between smooth densities on M and C^{∞}-densities on the elliptic space (A, M^*, L).

LEMMA 1. *There exists an orientable invariant neighbourhood of every orbit.*

Proof. It is enough to prove the lemma in the case where G is connected since every component of an orbit can be treated separately. Let $x \in M$ and let U be the domain of a chart at x. Then the neighbourhood $\operatorname{Inv} U = \bigcup_{g \in G} R_g(U)$ contains the orbit of x and is orientable since it is obtained by translating U by diffeomorphisms belonging to a connected Lie group and thus having positive "transition" functions.

LEMMA 2. (a) *Let α be a p-density on $A \subset M^*$. If α is integrable on A, then there exists a p-density ω on M integrable on M and such that $K\omega = \alpha$ and*

$$\int_A \alpha = \int_M \omega. \tag{1}$$

(b) *Let ω be an n-density on M. If ω is integrable, then $\alpha = K\omega$ is integrable and satisfies* (1).

Proof. Suppose that α is an integrable density on A. It follows from properties of the operation K (see [11]) that one can construct a density ω on M which is integrable on compact subsets contained in $\pi^{-1}(A) \subset M$ and such that $K\omega = \alpha$ and $\int K|\omega| = \int |\alpha|$. Denote by A_δ for $\delta > 0$ a family of compact sets contained in A such that $\bigcup_\delta A_\delta = A$. We have $\int_{A_\delta} \alpha = \int_{M_\delta} \omega$ where $M_\delta = \pi^{-1}(A_\delta)$, and since the integral $\int_A \alpha$ exists, we get that $\int_M \omega$ exists and satisfies (1). The proof of (b) is analogous.

LEMMA 3. *Let ω_1, ω_2 be two integrable n-densities on M such that $K\omega_1 = K\omega_2$ on A. Then*

$$\operatorname{Inv} \omega_1 = \operatorname{Inv} \omega_2 \tag{2}$$

where $\operatorname{Inv} \omega = \int_G R_g^* \omega \, d\mu$ is the invariant averaging of a form ω over the group G with its normalized Haar measure μ.

Proof. Let ω_1, ω_2 be two integrable n-densities. We shall prove (2) locally. For this end let x be a point in M. Let ε_x be a local 0-density of orientation in some neighbourhood of the orbit of x given by Lemma 1. Denote by ω_0 an invariant n-form non-vanishing on this neighbourhood. Then

$$\text{(3)} \qquad \operatorname{Inv} \omega_i = f_i \cdot \omega_0 \cdot \varepsilon_x, \quad i = 1, 2,$$

where f_i are invariant integrable functions. Thus there are h_i on A such that $f_i = h_i \circ \pi$. Since K is invariant, we have $K\omega_i = K \operatorname{Inv} \omega_i$, $i = 1, 2$. Now (3) implies that $K \operatorname{Inv} \omega_i = h_i \cdot K(\omega_0 \cdot \varepsilon_x)$. Since by assumption $K \operatorname{Inv} \omega_1 = K \operatorname{Inv} \omega_2$, it follows that $h_1 = h_2$ a.e. for $K(\omega_0 \cdot \varepsilon_x) \neq 0$ at every point. Thus $f_1 = f_2$, which proves the lemma.

Now we can prove the following analogue of Theorem 1.

THEOREM 2. *The set $\Omega^\infty(M^*)$ ($\Omega_0^\infty(M^*)$) is isomorphic to $\Omega_{\text{inv}}^\infty(M)$ ($\Omega_{0,\text{inv}}^\infty(M)$). The isomorphism is given by $\Omega_{\text{inv}}^\infty(M) \ni \omega \overset{K}{\mapsto} K\omega \in \Omega^\infty(M^*)$.*

Proof. We shall prove that K^{-1} is well defined and continuous. To prove this fix $\alpha \in \Omega^\infty(M^*)$ and denote by ω_i (for $i = 0, 1, 2, \ldots$) the unique invariant form, existing by Lemmas 2 and 3, such that $(L^*)^i \alpha = K\omega_i$. Since $\omega_0 \in \Omega(M)$, we have $K(P^*)^i \omega_0 = (L^*)^i K\omega_0$. It follows in view of Lemma 3 (both $(P^*)^i \omega_0$ and ω_i being invariant) that $\omega_i = (P^*)^i \omega_0$. Thus $(P^*)^i \omega_0$ is integrable for all i. Applying locally, on account of Lemma 1, the regularity theorem for elliptic operators, we get that $\omega_0 \in \Omega^\infty(M)$. The proof of the continuity of K^{-1} is analogous to the case of functions. To prove that K is continuous, note that for all $i = 0, 1, 2, \ldots$ and $\omega \in \Omega^\infty(M)$

$$\int |(L^*)^i K\omega| = \int |K(P^*)^i \omega| \leqslant \int K|(P^*)^i \omega| = \int |(P^*)^i \omega|.$$

PROPOSITION 2. *The elliptic space (M^*, L) is orientable if only and if M is orientable.*

Proof. Suppose that M is orientable and let ω_0 be a C^∞ non-zero invariant n-form([1]) on M. Then it follows from Theorem 2 that $\alpha_0 = K\omega_0$ is C^∞ on M^*. Given $\alpha \in \Omega^\infty(M^*)$, there is $\omega \in \Omega_{\text{inv}}^\infty(M)$ such that $K\omega = \alpha$. Since $\omega = f \cdot \omega_0$, we see that f is both $C^\infty(M)$ and invariant; thus it is of the form $f = h \circ \pi$ for $h \in C^\infty(M^*)$. We get $\alpha = h \cdot \alpha_0$, which shows that M^* is orientable.

Conversely, let M^* be orientable and let α_0 be the orientation form. We shall prove that $\omega_0 = K^{-1} \alpha_0 \in \Omega_{\text{inv}}^\infty(M)$ is non-vanishing on M. Let $x \in M$. It follows from Lemma 1 that there exists an orientation form ω_1 on some neighbourhood of the orbit of x. Also ω_1 can be assumed C^∞ and invariant.

([1]) Such a form always exists, which can be shown by using e.g. an invariant Riemannian metric on M.

By assumption there exists a continuous h on M^* such that $K\omega_1 = h\cdot K\omega_0$, ω_1, ω_0 being invariant; this implies that $\omega_1 = (h\circ\pi)\cdot\omega_0$. Since $\omega_1(x)\neq 0$, we have $h\circ\pi(x)\neq 0$ and $\omega_0(x)\neq 0$, which was to be proved.

On account of Theorem 2 we can establish the following characterization of invariant distributions.

THEOREM 3. *The space $D'(M^*)$ is isomorphic to the space $D'_{\mathrm{inv}}(M)$ of invariant distributions on M in the following way:*

$$D'(M^*)\ni T\overset{K'}{\mapsto} u\in D'_{\mathrm{inv}}(M),$$

where

(4) $$u[\omega] = T[K\omega] \quad \textit{for} \quad \omega\in\Omega_0^\infty(M).$$

Proof. First we remark that there is a natural isomorphism between the space of invariant distributions on M and the space of continuous linear functionals on $\Omega^\infty_{0,\mathrm{inv}}(M)$ of compactly supported invariant C^∞-densities on M, defined by the operation of averaging. Given $T\in D'(M^*)$ and $\omega\in\Omega_0^\infty(M)$, define u by (4). It follows from Theorem 2 that u is well-defined and continuous.

Conversely, given $u\in D'_{\mathrm{inv}}(M)$ and $\alpha\in\Omega_0^\infty(M^*)$, we take the unique invariant ω (existing by Theorem 3) such that $K\omega=\alpha$ and set $T[\alpha]=u[\omega]$. Theorem 2 shows again that u is well-defined and continuous. The continuity of K and K^{-1} is obvious.

In case of M being orientable, we get the following version of Theorem 3.

THEOREM 4. *Let M be orientable. Then the space $(C^\infty_{0,\mathrm{inv}}(M))'$ of invariant linear continuous functionals on $C_{0,\mathrm{inv}}(M)$ is isomorphic to the space $(C_0^\infty(M^*))'$ of continuous linear functionals on $C_0^\infty(M^*)$. The isomorphism is given by*

$$(C_0^\infty(M^*))'\ni T\overset{\pi'}{\mapsto} u\in(C^\infty_{0,\mathrm{inv}}(M))'$$

where $u[h\circ\pi] = T[h]$, $h\in C_0^\infty(M^*)$.

Proof. This follows from Theorem 3, in view of Proposition 2, since the space of functions h on M^* of the form $\alpha = h\cdot\alpha_0$ for $\alpha\in\Omega_0^\infty(M^*)$, where α_0 is an orientation form on M^*, is isomorphic to $C_0^\infty(M^*)$.

EXAMPLE. Let $G=\mathrm{SO}(n)$, $M=R^n$ and suppose that G acts on R^n in the natural way. Set $P=\Delta$, the Laplace operator. We shall express the orbit space $M^*=R^n/\mathrm{SO}(n)$ with the help of the "chart" $\bar{R}_+\ni s\overset{S^{-1}}{\mapsto}\{x\colon |x|^2 = s\}\subset R^n$, where R_+ is the set of positive real numbers. It is easily checked that $S(M^*)=\bar{R}_+$, $S(\Delta)=R_+$. S can be regarded as a "diffeomorphism" between the elliptic spaces (M^*, L) and $(\bar{R}_+, L_S)$, where $L_S = 4s\,d^2/ds^2 + 2n\,d/ds$. It follows from Theorem 1 that the set of C^∞ rotation invariant functions is isomorphic to the set of continuous functions h on $\bar{R}_+$ such that $(L_S)^i h$ is continuous for all $i=1,2,\ldots$ We shall find this set.

First note that for every differentiable function h on R_+, if $L_S h = f$ is continuous on $\bar{R}_+$, then

$$h(s) = \frac{1}{4}\int_0^s u^{-n/2}\Big(\int_0^u f(t)\cdot t^{n/2-1}\,dt + C_1\Big)du + C_2.$$

If $n \geqslant 2$ and h is continuous, then $C_1 = 0$ and $C_2 = h(0)$. We see that

$$h'(s) = \frac{1}{4}s^{-n/2}\int_0^s f(t)\cdot t^{n/2-1}\,dt.$$

Writing $f(t) = f(0)+g(t)$, $g(0) = 0$, it is immediate to see that $h'(s)$ is continuous at 0.

Thus the set of C^r-functions on the elliptic space $(\bar{R}_+, L_S)$ coincides with the set of functions continuous with all derivatives on $\bar{R}_+$. In this case Theorem 3 is just the theorem on rotation invariant distributions given by Schwartz in [9].

4. Concluding remarks. The paper shows that it is possible to define smooth functions and distributions on the orbit space of compact Lie groups in inner terms, i.e. without passing to the higher level of M, and the objects just defined have all the desired natural properties. The concept of an elliptic space allows one to translate singularities of the orbit space into a loss of ellipticity on the boundary for an elliptic operator.

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