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Extremal plurisubharmonic functions in C^N

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Abstract. Useful basic properties of some extremal plurisubharmonic functions are obtained. The extremal functions are used to obtain results giving: (1) necessary and sufficient conditions for a subset of C^N to be a C^N-polar set, (2) characterization of holomorphic functions in a neighbourhood of a compact C^N-regular subset of C^N, (3) analytic continuation of separately holomorphic functions on cross-like subsets of C^N, (4) a condition for single-valuedness of analytic functions of N complex variables.

Contents

Introduction. Let PSH (D) denote the set of all plurisubharmonic (plsh) functions in an open subset D of the space C^N of N complex variables. Put

$$L := \{u \in \mathrm{PSH}\,(C^N)\colon\ u(x) \leqslant \beta + \log(1+|x|) \text{ in } C^N\},$$

$$L^+ := \{u \in \mathrm{PSH}\,(C^N)\colon\ \alpha + \log(1+|x|) \leqslant u(x) \leqslant \beta + \log(1+|x|) \text{ in } C^N\},$$

where α and β are real constants that may depend on u, and $|x| := \max\limits_{1 \leqslant j \leqslant N} |x_j|$ for $x = (x_1, \ldots, x_N) \in C^N$.

Given a subset E of C^N and a real function $b\colon C^N \to [-\infty, +\infty)$, define for all $x \in C^N$

$$V_{E,b}(x) := \sup\{u(x)\colon u \in L, u \leqslant b \text{ on } E\},$$
$$V_{E,b}^{+}(x) := \sup\{u(x)\colon u \in L^{+}, u \leqslant b \text{ on } E\}.$$

The function $V_{E,b}$ is called *L-extremal function* corresponding to E and b. If $b = 0$ on E, we write V_E instead of $V_{E,0}$ and call the *L-extremal function of E*.

If E is bounded, we define

$$\Phi_{E,b}(x) := \sup_{n \geqslant 1} [\sup\{|f(x)|^{1/n}\colon f \in \mathscr{F}_n(E, b)\}], \quad x \in C^N,$$

where $\mathscr{F}_n(E, b)$ denotes the family of polynomials f of degree $\leqslant n$ such that $|f_n(x)| \leqslant \exp[nb(x)]$ on E. If $b = 0$ on E, we write Φ_E instead of $\Phi_{E,0}$.

The extremal function $\Phi_{E,b}$ was first defined and applied to solve some questions of the complex analysis of several complex variables in [24].

If $N = 1$, the extremal function $\Phi_{E,b}$ was first defined (in a different way, based on Fekete–Leja extremal points) and investigated by F. Leja and next by various authors, mostly Leja's students (see [23] for references). The method of extremal points and extremal functions on the complex plane was applied to various questions of the complex analysis of one variable (e.g. conformal mappings, Dirichlet problem, theory of interpolation; see [23] and [24] for references).

In particular, if E is a compact subset of C with the positive logarithmic capacity, then $\log \Phi_E$ is the Green function of the unbounded component of $C \setminus E$ with pole at infinity.

If $N \geqslant 2$, the extremal function $\log \Phi_E$ is a natural counterpart of the Green function, not only by its definition but also due to its applications. In particular, analogously as on the complex plane, also for $N \geqslant 2$ the function Φ_E is very useful in the theory of interpolation and approximation by polynomials. E.g. Φ_E permits to extend the Bernstein–Walsh theorem on characterization of analytic functions on compact subsets of C to the case of compact subsets of C^N, $N \geqslant 2$.

The purpose of this paper is to start a systematic study of the classes L, L^{+} and of the extremal functions $V_{E,b}$ and $\Phi_{E,b}$.

In our study an important role is played by the simple Proposition 1.2, and by some results (see Propositions 1.3 and 1.4) that may be derived as special cases from some theorems contained in Ferrier's book [8]. We have proved that if $E \subset C^N$ is compact and b is continuous then

(i) $V_{E,b} = \log \Phi_{E,b}$ in C^N;

(ii) If $V_{E,b}$ is continuous at every point of E, then it is continuous at every point of C^N.

If $b = 0$ on E these two results were earlier obtained by a different method by Zaharjuta [33].

We say that a subset E of C^N is:

1° *locally C^N-polar*, if for every point $a \in E$, there exists a function W plsh in a neighbourhood U_a of a such that $W = -\infty$ on $E \cap U_a$;

2° *globally C^N-polar*, if there exists $W \in \mathrm{PSH}(C^N)$ such that $W = -\infty$ on E;

3° *L-polar*, if there exists $W \in L$ such that $W = -\infty$ on E.

B. Josefson [14] has recently proved that conditions 1° and 2° are equivalent. In this paper we prove that 2° $\Leftrightarrow$ 3°, so that all the three notions are equivalent.

As applications of the developed method of the L-extremal functions the paper contains a new proof of an extended Bernstein–Walsh theorem [24], a modified version of a theorem on separate analyticity [26], and finally a sufficient condition for single-valuedness of analytic functions of several complex variables.

The main contents of this paper were a subject of a few lectures given by the author during his stay (February–March, 1976) at the Uppsala University as an invited professor. The author is greatly indebted for the invitation.

1. Some families of plurisubharmonic functions in C^N

Given any open subset G of C^N (the space of N complex variables), we denote by PSH (G) the set of all plurisubharmonic (plsh) functions in G.

1.1. We shall be interested in the following families of plsh functions in C^N.

$$L := \{u \in \mathrm{PSH}(C^N)\colon\ u(x) \leqslant \beta + \log(1+|x|) \text{ in } C^N\},$$

$$L^+ := \{u \in \mathrm{PSH}(C^N)\colon\ \alpha + \log(1+|x|) \leqslant u(x) \leqslant \beta + \log(1+|x|) \text{ in } C^N\},$$

α and β being real constants that may depend on u, and $|x| := \max\limits_{1 \leqslant j \leqslant N} |x_j|$ for any $x = (x_1, \ldots, x_N) \in C^N$.

It is obvious that $L^+ \subset L$ and both families are convex subsets of PSH (C^N). The elements of L are sometimes called plsh functions with minimal growth at infinity of type 1 (see [18], [19], where L is denoted by S_1).

Observe that if f is a non-zero polynomial of N complex variables of degree $\leqslant n$, then $(1/n)\log|f| \in L$. Indeed, let $M := \sup\{|f(x)|\colon |x| \leqslant 1\}$; then by the Cauchy inequalities

$$|f(x)| \leqslant M(1+|x|+\ldots+|x|^n) \leqslant M_1(1+|x|^n), \qquad M_1 = \text{const} > 0,$$

whence the result follows.

Put $\omega(x) = C_N \exp(-1/(1-|x|^2))$ for $|x| \leqslant 1$ and $\omega(x) = 0$ for $|x| \geqslant 1$, where the positive constant C_N is chosen so that $\int \omega(x)\,dx = 1$, the integration being taken with respect to $2N$-dimensional Lebesgue measure in C^N. Given any $\lambda > 0$ put $\omega_\lambda(x) = \lambda^{-2N}\omega(\lambda^{-1}x)$. Then $\int \omega_\lambda(x)\,dx = 1$ and $\omega_\lambda(x) = 0$ for $|x| \geqslant \lambda$.

1.2. PROPOSITION. *If $u \in L$ (resp. $u \in L^+$), then $u_\lambda := u * \omega_\lambda$ given by*

$$(u * \omega_\lambda)(x) := \int u(x+y)\omega_\lambda(y)\,dy, \qquad x \in C^N,$$

is a $\mathscr{C}^\infty$-function in C^N belonging to L (resp. to L^+). Moreover,

$$u_\lambda \downarrow u \quad as\ \lambda \downarrow 0.$$

Proof. It is well known [13] that u_λ is $\mathscr{C}^\infty$, $u_\lambda \in \mathrm{PSH}(C^N)$ and $u_\lambda \downarrow u$ as $\lambda \downarrow 0$ in C^N. It follows directly from the definition of u_λ that $u_\lambda \in L$ (resp. $u_\lambda \in L^+$).

1.3. PROPOSITION. *Given a function $u \in L^+$, put $\delta := e^{-u}$ and $\delta_\lambda(x) := \inf_{y \in C^N} [\delta(y) + (1/\lambda)|y-x|]$, $x \in C^N$, $\lambda > 0$. Then*

(i) $|\delta_\lambda(x) - \delta_\lambda(y)| \leqslant (1/\lambda)|x-y|$, $x, y \in C^N$;

(ii) $u_\lambda := -\log \delta_\lambda \in L^+$, *if* $0 < \lambda < e^\beta$;

(iii) $u_\lambda \downarrow u$ *in* C^N *as* $\lambda \downarrow 0$.

Proof. (i) may be shown by an elementary calculation.

(ii) The function u_λ is plsh in C^N by Lemma 2, p. 48 of [8]. One may easily check that

$$u(x) \leqslant u_\lambda(x) \leqslant \beta + \lg(1+|x|), \qquad 0 < \lambda < e^\beta.$$

(iii) It is enough to show that $\delta_\lambda \uparrow \delta$ as $\lambda \downarrow 0$. It is obvious that $\delta_{\lambda'} \leqslant \delta_{\lambda''} \leqslant \delta$ for $0 < \lambda' \leqslant \lambda''$. Fix $x \in C^N$ and $\varepsilon > 0$. Take λ_0 so small that

$$\delta_\lambda(x) = \inf_{y \in C^N} [\delta(y) + (1/\lambda)|x-y|], \qquad 0 < \lambda < \lambda_0.$$

By the lower-semicontinuity of δ there exists a neighbourhood U of x such that

$$\delta(y) \geqslant \delta(x) - \varepsilon, \qquad y \in U.$$

We may choose λ_0 so small that the ball $B = B(x, \lambda M)$, where $M := \sup_{x \in C^N} \delta(x)$, is contained in U for $0 < \lambda < \lambda_0$. Then

$$\delta_\lambda(x) = \inf_{y \in B} [\delta(y) + (1/\lambda)|x-y|] \geqslant \delta(x) - \varepsilon, \qquad 0 < \lambda < \lambda_0. \qquad \text{Q.E.D.}$$

The following Proposition follows as a special case from Theorem 2, p. 82 of [8].

1.4. PROPOSITION. *If $u \in L^+$ and $\delta := e^{-u}$ is Lipschitz (with Lipschitz constant $= 1$), then there exist a sequence of positive numbers $\{c_n\}$ and a positive integer k such that for every n there exists a family F_n of holomorphic functions in C^N such that*

$$(e^u)^n \leqslant \sup_{f \in F_n} |f| \leqslant c_n (e^u)^{n+k} \quad in\ C^N$$

and $\lim \sqrt[n]{c_n} = 1$.

1.5. Remark. It follows from the Liouville theorem that each function $f \in F_n$ is a polynomial of degree $\leqslant n+k$.

2. L-extremal functions

2.1. Let E be any subset of C^N and $b\colon C^N \to [-\infty, +\infty)$ any real function defined in C^N. The function b may take value $-\infty$, but we do not allow it to attain value $+\infty$.

Put

$$L(E, b) := \{u \in L\colon u \leqslant b \text{ on } E\},$$
$$L^+(E, b) := \{u \in L^+\colon u \leqslant b \text{ on } E\}.$$

$L(E, -\infty)$ will stand for $L(E, b)$ with $b \equiv -\infty$ on E. Observe that $L(E, -\infty)$ may be empty, if E is too large, e.g. if $\operatorname{int} E \neq \emptyset$ or more generally if E is not a polar set (see Section 3).

Let us define for every $x \in C^N$

$$V(x) \equiv V(x, E, b) \equiv V_{E,b}(x) := \sup\{u(x)\colon u \in L(E, b)\},$$
$$V^+(x) \equiv V^+(x, E, b) \equiv V_{E,b}^+(x) := \sup\{u(x)\colon u \in L^+(E, b)\}.$$

We shall write V_E or V_E^+, if $b = 0$ on E.

2.2. Definition. The function $V_{E,b}$ (resp. $V_{E,b}^+$) will be called *L-extremal* (resp. *L^+-extremal*) *function* associated with E and b.

The following three properties of the extremal function V are direct consequence of its definition

2.3. *Monotonicity with respect to b*: $V_{E,b_1} \leqslant V_{E,b_2}$ in C^N, if $b_1 \leqslant b_2$ on E.

2.4. *Monotonicity with respect to E*: $V_{F,b} \leqslant V_{E,b}$ in C^N, if $E \subset F$.

2.5. $V_{E,b+c} = c + V_{E,b}$ in C^N for every real constant c.

2.6. If $E = B(a, r) := \{x \in C^N\colon \|x-a\| \leqslant r\}$ is a ball with center a and radius r, where $\|\ \|$ is any norm in C^N, then

$$V_E(x) = \log^+ \|x-a\|/r.$$

Indeed, it is obvious that $\log^+ \|x-a\|/r \leqslant V_E(x)$ in C^N. In order to obtain the opposite inequality take a fixed $x \in C^N$ with $\|x-a\| > r$, and observe that for every $u \in L(E, 0)$ the function

$$w(\lambda) := u\big(a+\lambda(x-a)\big) - \log^+ |\lambda|\, \|x-a\|/r,$$

is bounded and subharmonic for $|\lambda| > r/\|x-a\|$, and $w(\lambda) \leqslant 0$ as $|\lambda| = r/\|x-a\|$. By putting $w(\infty) := \lim_{\lambda\to\infty} w(\lambda)$, the function w becomes subharmonic at ∞.

Therefore by the maximum principle we obtain the inequality $w(\lambda) \leqslant 0$

for all $|\lambda| \geqslant r/\|x-a\|$. In particular we may take $\lambda = 1$ and so obtain the required inequality.

2.7. If the set E is bounded and the function b is lower-bounded on E, then

$$V_{E,b} = V_{E,b}^{+} \quad \text{in } \mathbf{C}^N.$$

In particular, if E is bounded, then $V_E = V_E^{+}$.

Indeed, put $m := \inf\{b(x)\colon x \in E\}$ and let $E \subset B(0, r)$. Then the function $\max\{u(x), m+\log^{+}\|x\|/r\}$ belongs to $L^{+}(E, b)$ for every $u \in L(E, b)$. Hence $V_{E,b} \leqslant V_{E,b}^{+}$. The opposite inequality is obvious.

The following property is obvious.

2.8. If $b = \frac{1}{\alpha}[\alpha_1 b_1 + \alpha_2 b_2]$, where α_1, α_2 are non-negative real numbers such that $\alpha := \alpha_1+\alpha_2 > 0$, then

$$\alpha_1 V_{E,b_1} + \alpha_2 V_{E,b_2} \leqslant \alpha V_{E,b}.$$

2.9. If $a\colon \mathbf{C}^N \to [-\infty, +\infty)$ is a real function such that $V_{E,a}$ is finite at every point of $\mathbf{C}^N$, then

$$\frac{1}{\lambda}[V_{E,a+\lambda b} - V_{E,a}] \leqslant \frac{1}{\lambda'}[V_{E,a+\lambda' b} - V_{E,a}] \quad \text{in } \mathbf{C}^N \text{ as } 0 < \lambda' < \lambda.$$

This inequality follows from 2.8 by putting $b_1 = a+\lambda b$, $b_2 = a$, $\alpha_1 = 1/\lambda$, $\alpha_2 = 1/\lambda' - 1/\lambda$.

2.10. If $-\infty < m \leqslant b(x) \leqslant M < +\infty$ on E, then

$$m + V_E \leqslant V_{E,b} \leqslant M + V_E \quad \text{in } \mathbf{C}^N.$$

In particular, if b is bounded, then $V_{E,b}(x)$ is finite if and only if $V_E(x)$ is finite.

2.11. *Bernstein–Walsh inequality.* If f is a polynomial of N complex variables of degree $\leqslant n$ such that $|f(x)| \leqslant M\exp[nb(x)]$ on E, then

$$|f(x)| \leqslant M\exp[nV_{E,b}(x)], \quad x \in \mathbf{C}^N.$$

Indeed if $f \neq 0$, then $(1/n)(\log|f| - \log M) \in L(E, b)$, so the result is an immediate consequence of the definition of $V_{E,b}$.

2.12. Proposition. *If E is compact and $b|E$ is lower-semicontinuous, then $V_{E,b}$ is lower-semicontinuous in $\mathbf{C}^N$.*

Proof. Fix $u \in L(E, b)$ and $\varepsilon > 0$. By the compactness of E and lower-semicontinuity of b one can find $\lambda = \lambda(\varepsilon) > 0$ so small that

$$u_\lambda := u * \omega_\lambda \leqslant b+\varepsilon \quad \text{on } E.$$

Hence $u_\lambda - \varepsilon \in L(E, b)$ and so $u_\lambda - \varepsilon \leqslant V_{E,b}$ in C^N. It follows that $V_{E,b}$ is an upper envelope of continuous functions $u * \omega_\lambda - \varepsilon$, where $u \in L(E, b)$, $\varepsilon > 0$, $\lambda = \lambda(\varepsilon, u) > 0$. Therefore $V_{E,b}$ is lower-semicontinuous.

2.13. PROPOSITION. *If E is compact and the extremal function $V = V_{E,b}$ is continuous at every point of E, then it is continuous in C^N. In particular $V_{E,b} \in L(E, b)$.*

Proof. Since $V^*(x) := \limsup_{y \to x} V(y) = V(x)$ for $x \in E$, we can find a ball $B = B(a, r)$ with $a \in E$ such that $V \leqslant V^* \leqslant M = \text{const}$ on B. Therefore, by 2.6,

$$V \leqslant M + \log^+ \frac{|x-a|}{r} \quad \text{in } C^N.$$

Hence $V^* \in L$. Now, since $V_\lambda := V^* * \omega_\lambda \in \mathscr{C}^\infty \cap L$ and $V_\lambda \downarrow V^*$ in C^N, in particular $V_\lambda \downarrow V$ on E, as $\lambda \downarrow 0$, the Dini theorem implies

$$V_\lambda \leqslant V + \varepsilon \quad \text{on } E \text{ as } 0 < \lambda < \lambda_0 = \lambda_0(\varepsilon).$$

Thus $V_\lambda - \varepsilon \leqslant V \leqslant b$ on E and finally

$$V_\lambda - \varepsilon \leqslant V \leqslant V^* \leqslant V_\lambda \quad \text{in } C^N,\ 0 < \lambda < \lambda_0.$$

Therefore V is a uniform limit of $\mathscr{C}^\infty$ functions $V_\lambda (\lambda \downarrow 0)$. Q.E.D.

2.14. PROPOSITION. *If E is compact and b is continuous, then*

$$V_{E^r,b} \uparrow V_{E,b} \quad \textit{in } C^N \textit{ as } r \downarrow 0,$$

where $E^r := \bigcup_{a \in E} B(a, r) = \{x \in C^N: \operatorname{dist}(x, E) \leqslant r\}$, $B(a, r) := \{x \in C^N: |x-a| \leqslant r\}$.

Proof. Take any $u \in L(E, b)$. Given $\varepsilon > 0$, we may find $\lambda > 0$ such that $u_\lambda := u * \omega_\lambda \leqslant b + \varepsilon$ on E. Since u_λ and b are continuous we may find $r_0 > 0$ such that

$$u_\lambda < b + 2\varepsilon \quad \text{on } E^r,\ 0 < r < r_0 = r_0(u).$$

Hence $u \leqslant u_\lambda \leqslant 2\varepsilon + V_{E^r,b}$ in C^N, $0 < r < r_0$, and finally $V_{E,b} \leqslant 2\varepsilon + \lim_{r \to 0} V_{E^r,b}$. Since $V_{E^r,b} \leqslant V_{E,b}$, we get the result.

2.15. DEFINITION. We say that a subset E of C^N is 1° *locally L-regular at a point* $a \in \bar{E}$, if for every $r > 0$ the extremal function $V_{E \cap B(a,r)}$ is continuous at a; 2° *locally L-regular*, if it is locally L-regular at every point $a \in \bar{E}$.

2.16. PROPOSITION. *If E is a compact set locally L-regular, then for every real continuous function b the extremal function $V = V_{E,b}$ is continuous in C^N.*

Proof. First observe that $V^* \leqslant b$ on E. Indeed, given $a \in E$ and $\varepsilon > 0$, we have

$$V(x) \leqslant V_{E \cap B(a,r),\, b(a)+\varepsilon} = b(a) + \varepsilon + V_{E \cap B(a,r)} \quad \text{in } C^N,$$

where $r > 0$ is so small that $b(x) \leqslant b(a)+\varepsilon$ in $B(a, r)$. Hence $V^*(a) \leqslant b(a)+\varepsilon$, and by the arbitrariness of $\varepsilon > 0$ we get $V^*(a) \leqslant b(a)$.

Now

$$V^* \leqslant V_\lambda := V^* * \omega_\lambda \leqslant b+\varepsilon \quad \text{on } E \text{ as } 0 < \lambda < \lambda_\varepsilon,$$

whence $V_\lambda - \varepsilon \in L(E, b)$ and

$$V_\lambda - \varepsilon \leqslant V \leqslant V^* \leqslant V_\lambda \quad \text{in } C^N,\ 0 < \lambda < \lambda_\varepsilon.$$

Therefore V is a uniform limit of the $\mathscr{C}^\infty$ functions V_λ as $\lambda \downarrow 0$.

2.17. Corollary from the proof of 2.16. *If E is compact and b is a real continuous function such that $V_{E,b}^* \leqslant b$ on E, then $V_{E,b}$ is continuous in C^N.*

2.18. If f is a non-zero polynomial of degree $\leqslant k$ and $b := \frac{1}{k} \log |f|$, then, for every subset E of C^N, $V_{E,b} = b$ on E.

In particular if $E = \partial D$, D being a bounded domain such that $f(x) \neq 0$ for $x \in \bar{D}$, then

$$V_{E,b}(x) = \frac{1}{k} \log |f(x)|, \quad x \in \bar{D}.$$

Proof. By 1.1, $(1/k) \log |f| \in L(E, b)$. Therefore $b(x) = (1/k) \log |f(x)| \leqslant V_{E,b}(x)$ in C^N. On the other hand $V_{E,b} \leqslant b = (1/k) \log |f|$ on E. By the maximum principle the last inequality holds true in $\bar{D}$. Q.E.D.

3. L-polar sets

3.1. Definition. We say that a subset E of C^N is 1° *locally C^N-polar*, if for every point $a \in E$ there exists a plsh function W in an open neighbourhood U_a of a such that $W = -\infty$ on $E \cap U_a$; 2° *globally C^N-polar*, if there exists a function W plsh in C^N such that $W = -\infty$ on E; 3° *L-polar*, if there exists a function $W \in L$ such that $W = -\infty$ on E (i.e. if $L(E, -\infty) \neq 0$).

3.2. Definition. Given a subset E of C^N and an open set $G \subset C^N$ we put for every $x \in G$

$$h(x, E, G) \equiv h_{EG}(x) := \sup\{u(x): u \in \mathrm{PSH}(G),\ u \leqslant 0 \text{ on } E \cap G,\ u \leqslant 1 \text{ on } G\}.$$

The function h_{EG}^* is plsh in G. If $h_{EG}^*(a) < 1$ for a point $a \in G$, then $h_{EG}^*(x) < 1$ for all x in the connected component of G containing the point a.

3.3. Proposition. *$E \subset C^N$ is locally C^N-polar if and only if for every $a \in E$ there exists a domain $D \ni a$ such that*

$$h_{ED}^*(x) := \limsup_{y \to x} h_{ED}(y) = 1 \quad \text{for all } x \in D.$$

Proof. 1° If E is locally C^N-polar, then for every $a \in E$ one can find a neighbourhood U_a of a and a plsh function W in U_a such that $W = -\infty$ on $E \cap U_a$. Let D be a relatively compact subdomain of U_a containing a. We may assume $W \leqslant 0$ on D. Then

$$\frac{1}{k} W + 1 \leqslant h_{ED} \quad \text{in } D \text{ for all } k \geqslant 1.$$

Hence $h_{ED} = 1$ in a dense subset of D, i.e. $h_{ED}^* \equiv 1$.

2° Assume now that D is a domain such that $h_{ED}^* = 1$ in D. Since the set $\{x \in D\colon h_{ED}(x) < h_{ED}^*(x)\}$ is of Lebesgue measure zero, there exists a point $\xi \in D$ such that for every $k \in N$ one can find $u_k \in \mathrm{PSH}(D)$ with $u_k = 0$ on $E \cap D$, $u_k \leqslant 1$ in D and $u_k(\xi) \geqslant 1 - 2^{-k}$. We claim that the function

$$W(x) := \sum_{k \geqslant 1} [u_k(x) - 1], \qquad x \in D,$$

is plsh in D and $W = -\infty$ on $E \cap D$. Indeed, the sequence of the partial sums of the series is decreasing and $W(\xi) \geqslant -1$. Hence $W \in \mathrm{PSH}(D)$. It is obvious that $W = -\infty$ on $E \cap D$.

3.4. LEMMA (comp. with Théorème 3 of [19]). *Let $(u_i)_{i \in I}$ be a family of functions belonging to L. Put*

$$u := \sup\{u_i\colon i \in I\} \quad \text{in } C^N.$$

Then the following conditions are equivalent:

(1) *There exist real numbers $R > 0$ and $M > 0$ such that $u \leqslant M$ in the ball $B = B(0, R)$;*

(2) *There exist real numbers $R > 0$ and $M > 0$ such that*

$$u(x) \leqslant M + \log^+ |x|/R \quad \text{in } C^N;$$

(3) *There exist an open non-empty subset D of C^N and a real constant $M > 0$ such that $u \leqslant M$ on D;*

(4) *u is bounded from above on any compact subset of C^N;*

(5) *$u^* \in L$.*

If, moreover, u_i is continuous for every $i \in I$, then each of conditions (1)–(5) *is equivalent to the condition*

(6) *$u(x) < +\infty$ for every $x \in D$, D being a non-empty open subset of C^N.*

Proof. Implications (1) ⇒ (2) and (4) ⇒ (5) follow from 2.6. Implications (2) ⇒ (3) ⇒ (4), (5) ⇒ (1) and (2) ⇒ (6) are obvious.

If u_i, $i \in I$, are continuous and (6) is satisfied, then u being lower semicontinuous there exist a ball $B = B(a, R) \subset D$ and a positive constant M such that $u \leqslant M$ on B. Thus (3) is satisfied. Q.E.D.

3.5. THEOREM. (Compare with Théorème 4 of [19].) *Given any family*

$(u_i)_{i\in I} \subset L$, put $u := \sup_i u_i$ and $A_u := \{x\in C^N\colon u(x) < +\infty\}$. Then $u^*\in L$ if and only if A_u is not L-polar.

Proof. 1° If $u^*\in L$, then $A_u = C^N$ and A_u is not L-polar. 2° Assume now $u^*\notin L$. Then by Lemma 3.4

$$\sup\{u(x)\colon x\in B_1 = B(0,1)\} = +\infty.$$

Hence for every n there exists $i_n\in I$ such that $\sup_{B_1} u_{i_n} \geq n$. Put $v_n := u_{i_n}$ and $M_n := \sup_{B_1} v_n$. Then $\lim_{n\to\infty} M_n = +\infty$ and $v_n - M_n \leq \log^+|x|$ in C^N.

We claim that there exist $\varepsilon > 0$ and $\xi\in C^N$ such that

$$(*)\qquad \limsup_{n\to\infty} \exp[v_n(\xi)-M_n] \geq \varepsilon.$$

Otherwise $\limsup_{n\to\infty}\exp[v_n(x)-M_n] \leq 0$ for all $x\in C^N$. Hence by the Hartogs lemma

$$\exp[v_n(x)-M_n] \leq \varepsilon,\qquad x\in B_1,\ \varepsilon>0,\ n\geq n_\varepsilon.$$

If $0<\varepsilon<1$, this gives a contradiction with the definition of M_n.

Let us now fix $\varepsilon > 0$ and $\xi\in C^N$ satisfying $(*)$ and take a sequence of integers $n_k < n_{k+1}$, $k\geq 1$, such that

$$\lim_{k\to\infty}\exp[v_{n_k}(\xi)-M_{n_k}] \geq \varepsilon \quad\text{ and }\quad M_{n_k}\geq 2^k \quad (k\geq 1).$$

We claim that the function W defined by

$$W(x) := \sum_{k\geq 1} 2^{-k}[v_{n_k}(x)-M_{n_k}],\qquad x\in C^N,$$

belongs to $L(A_v, -\infty)$, where $v = \sup_{n\geq 1} v_n$ and $A_v := \{x\in C^N\colon v(x) < +\infty\}$.

Indeed, given any $R>1$ we have $2^{-k}[v_{n_k}(x)-M_{n_k}] - 2^{-k}\log^+ R \leq 0$, $k\geq 1$, on $B(0,R)$. Therefore W is upper-semicontinuous in $B(0,R)$. Hence it is uppersemicontinuous in C^N.

If $x\in A_v$, then $2^{-k}[v_{n_k}(x)-M_{n_k}] \leq 2^{-k}v_{n_k}(x) - 1 \leq -\frac{1}{2}$, $k\geq k_0$. Hence $W = -\infty$ in A_v. If $x=\xi$, then $W(\xi) > -\infty$. And finally $W(x)\leq \log^+|x|$ in C^N. Therefore $W\in L(A_v, -\infty)$, and in particular $W\in L(A_u,-\infty)$ because $A_u\subset A_v$.

3.6. THEOREM. *If $E = \bigcup_{n\geq 1} E_n$ and $L(E_n,-\infty)\neq\emptyset$ $(n\geq 1)$, then $L(E,-\infty)\neq\emptyset$. In other words: A countable union of L-polar sets is an L-polar set.*

Proof. Taking $E_1\cup\ldots\cup E_n$, we may assume that $E_n\subset E_{n+1}$. Now for each n take $u_n\in L(E_n,-\infty)$, put $M_n := \sup_{B_1} u_n$ and observe that there exist

$\varepsilon > 0$ and $\xi \in C^N$ such that $\limsup_{n\to\infty} \exp[u_n(\xi) - M_n] \geqslant \varepsilon$. Next define $W(x) := \sum_{k\geqslant 1} 2^{-k}[u_{n_k}(x) - M_{n_k}]$. Then $W \in L(E, -\infty)$. Q.E.D.

3.7. COROLLARY. *If $E \subset C^N$ is not L-polar, then there exists a point $a \in C^N$ such that $E \cap B(a, r)$ is not L-polar for any $r > 0$.*

3.8. DEFINITION. Given any subset E of C^N the number

$$c(E) := \liminf_{|x|\to\infty} |x| \exp(-V_E(x))$$

will be called the *L-capacity* of E.

If E is a compact subset of the complex plane C, then $c(E)$ is the logarithmic capacity (transfinite diameter) of E (see [23], [15]).

By applying Lemma 3.4 and Theorem 3.5 to the family $\{u \in L\colon u \leqslant 0$ on $E\}$ we get the following

3.9. COROLLARY. *If E is any subset of C^N, the following conditions are equivalent:*

(i) $c(E) = 0$;

(ii) $V_E^* \notin L$;

(iii) $V_E^* \equiv +\infty$;

(iv) *E is L-polar.*

If $c(E) > 0$, then there exists $a \in C^N$ such that $c(E \cap B(a, r)) > 0$ for all $r > 0$.

3.10. THEOREM. *For every subset E of C^N the following conditions are equivalent:*

(a) *E is locally C^N-polar;*

(b) *E is globally C^N- polar;*

(c) *E is L-polar;*

(d) *For every bounded domain $D \subset C^N$, $h_{ED}^* = 1$ in D.*

Proof. The most difficult part of the theorem is the implication (a)$\Rightarrow$(b) that was recently proved by Josefson [14].

(b)$\Rightarrow$(c). By Theorem 3.6 we may assume that E is bounded. Let W be any function plsh in C^N such that $W = -\infty$ on E.

Suppose E is not L-polar. Then, by Corollary 3.9, $V_E^* \in L$. Moreover, by 2.7, $V_E^* \in L^+$. So given any $\varrho > M$, we can find $R > 0$ so large that $E \subset B = B(0, R)$ and

$$V_E^*(x) \geqslant M + \varrho \quad \text{on } \partial B, \quad \text{where} \quad M := \sup_{x\in E} V_E^*(x).$$

We may assume $W \leqslant 0$ on B. Given any positive integer k, put

$$v_k(x) := V_E^*(x)/(M+\varrho) \quad \text{for } |x| \geqslant R$$

and

$$v_k(x) := \max\{(1/k)W(x)+1,\ V_E^*(x)/(M+\varrho)\} \quad \text{for } |x| \leqslant R.$$

Then $(M+\varrho)v_k \leqslant M$ on E and $(M+\varrho)v_k \in L$. Therefore $(M+\varrho)v_k \leqslant M+V_E$ in C^N. In particular, $(M+\varrho)\left(\frac{1}{k}W+1\right) \leqslant M+V_E$ in B for $k \geqslant 1$. Hence $M+\varrho \leqslant M+V_E$ in B, in particular we get $\varrho \leqslant M$ for $x \in E$. This contradiction implies that E is L-polar.

Implications (c) $\Rightarrow$ (d) $\Rightarrow$ (a) follow from Proposition 3.3 (or from its proof).

3.11. PROPOSITION. *If F is L-polar, then for every bounded set E and for every function b: $C^N \to [-\infty, +\infty)$*

$$V_{E\cup F,b}^* = V_{E,b}^* \quad \textit{in } C^N.$$

Proof. It is sufficient to show that $V_{E,b}^* \leqslant V_{E\cup F,b}^*$. Take any $u \in L(E,b)$ and any $W \in L(F, -\infty)$. We may assume that $W \leqslant 0$ on E. Therefore $\frac{1}{k}W+u \in L(E\cup F, b)$ and

$$\frac{1}{k}W+u \leqslant V_{E\cup F,b} \quad \text{in } C^N \text{ for } k \geqslant 1.$$

Hence

$$u \leqslant V_{E\cup F,b} \quad \text{in } C^N \setminus W^{-1}(\{-\infty\})$$

and finally

$$V_{E,b}^* \leqslant V_{E\cup F,b}^*.$$

4. L-extremal functions associated with compact subsets of C^N

4.1. Let $\mathscr{P}_n$ denote the vector space of all complex-valued polynomials of N complex variables of degree $\leqslant n$. Let

$$\varkappa\colon \{1,2,\ldots\} \ni j \to \varkappa(j) = (\varkappa_1(j), \ldots, \varkappa_N(j)) \in Z_+^N$$

denote a fixed one-to-one mapping such that

$$|\varkappa(j)| \leqslant |\varkappa(j+1)|, \quad j \geqslant 1,$$

where $|\alpha| = \alpha_1 + \ldots + \alpha_N$ for $\alpha = (\alpha_1, \ldots, \alpha_N) \in Z_+^N$.

It is obvious that for $N = 1$ we have $\varkappa(j) = j+1$.

Let h_n denote the number of the elements of the set $\{j\colon |\varkappa(j)| = n\}$. We may also say that h_n is the number of monomials $x^\alpha = x_1^{\alpha_1}\ldots x_N^{\alpha_N}$ of degree n, or in other words h_n is the number of coefficients of a homogeneous polynomial of degree n. One may check that

$$h_n = \binom{N+n-1}{n-1}.$$

Let m_n denote the number of monomials x^{α} of degree $\leqslant n$. Then $m_n = \binom{N+n}{n}$ = the number of coefficients of any $f \in \mathscr{P}_n$.

The sequence of monomials

$$e_j(x) := x^{\varkappa(j)} = x_1^{\varkappa_1(j)} \dots x_N^{\varkappa_N(j)}, \quad j = 1, 2, \dots$$

is a basis for the vector space of all polynomials of N complex variables.

The set of monomials $e_1, \dots, e_{m_n}$ is a basis for the vector space $\mathscr{P}_n$, the set $\{e_{m_{n-1}+1}, \dots, e_{m_n}\}$ is a basis for the vector space $\mathscr{H}_n$ of all homogeneous polynomials of degree n.

If $f \in \mathscr{P}_n$, $g \in \mathscr{H}_n$, then

$$f = \sum_{j=1}^{m_n} a_j e_j, \quad g = \sum_{j=m_{n-1}+1}^{m_n} b_j e_j,$$

where the complex coefficients a_j and b_j are uniquely determined by f and g, respectively.

Let $x^{(n)} = \{x_1, \dots, x_n\}$ denote a system of n points $x_1, \dots, x_n$ of C^N. The determinant

$$V(x^{(n)}) \equiv V(x_1, \dots, x_n) := \det [e_j(x_k)] \equiv \begin{vmatrix} 1 & , \dots, 1 \\ e_2(x_1), & \dots, e_2(x_n) \\ \vdots & \vdots \\ e_n(x_1), & \dots, e_n(x_n) \end{vmatrix},$$

will be called the *Vandermondian* of the points system $x^{(n)}$.

If $V(x^{(n)}) \neq 0$, we define the Lagrange interpolating polynomials by

$$L^{(j)}(x, x^{(n)}) := \frac{V(x_1, \dots, x_{n-1}, x, x_{j+1}, \dots, x_n)}{V(x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_n)}, \quad j = 1, \dots, n.$$

It is clear that $L^{(j)}(x_k, x^{(n)}) = \delta_{jk}$, where $\delta_{jk} = 0$ for $j \neq k$ and $\delta_{jj} = 1$.

Hence we obtain the following *Lagrange interpolation formula*:

$$f(x) = \sum_{j=1}^{m_n} f(x_j) L^{(j)}(x, x^{(m_n)}), \quad x \in C^N, \ f \in \mathscr{P}_n,$$

where $x^{(m_n)}$ is any system of m_n points $x_1, \dots, x_{m_n}$ of C^N such that $V(x^{(m_n)}) \neq 0$.

4.2. Definition. We say that a subset E of C^N is

1° *unisolvent* (or *determining* for polynomials) *of order* n, if the following implication is true:

$$f \in \mathscr{P}_n, \quad f = 0 \quad \text{on } E \Rightarrow f = 0 \quad \text{in } C^N;$$

2° *unisolvent*, if it is unisolvent of any order $n \geqslant 1$.

4.3. Proposition. *For every* n *the following conditions are equivalent*:

(a) E *is unisolvent of order* n;

(b) $V_k(E) := \sup\{|V(x_1, \ldots, x_k)|: \{x_1, \ldots, x_k\} \subset E\} \neq 0$, $k = 1, \ldots, m_n$;

(c) $V_{m_n}(E) \neq 0$.

Proof. (a) $\Rightarrow$ (b). It is obvious that $V_1(E) = 1$. Suppose $V_k(E) \neq 0$, $k < m_n$ and let $\{\xi_1, \ldots, \xi_k\} \subset E$ be a system of k points of E such that $V(\xi_1, \ldots, \xi_k) \neq 0$. Then

$$V(\xi_1, \ldots, \xi_k, x) = V(\xi_1, \ldots, \xi_k) e_{k+1}(x) + \sum_{j=1}^{k} c_j e_j(x) \not\equiv 0.$$

Therefore $V_{k+1}(E) \geqslant \sup_{x \in E} |V(\xi_n, \ldots, \xi_k, x)| > 0$.

(b) $\Rightarrow$ (c) is clear, and (c) $\Rightarrow$ (a) follows from the Lagrange interpolation formula.

4.4. Example. If E_j $(j = 1, \ldots, N)$ is a subset of the complex plane C containing at least $n+1$ different points, then $E := E_1 \times \ldots \times E_N$ is unisolvent of order n. If a set E is unisolvent, then every set $F \supset E$ is also unisolvent.

4.5. Definition. Let E be a compact subset of C^N and b: $C^N \to R$ – a real continuous function defined in C^N. Any system $\xi^{(n)}$ of n points $\xi_1, \ldots, \xi_n$ of E will be called a *system of extremal points of order* n corresponding to E and b, if

$$W(\xi^{(n)}, b) \geqslant W(x^{(n)}, b) \quad \text{for all } x^{(n)} = \{x_1, \ldots, x_n\} \subset E,$$

where

$$W(x^{(n)}, b) := |V(x^{(n)})| \exp\{-|\varkappa(n)| [b(x_1) + \ldots + b(x_n)]\}.$$

If $b = 0$ on E, $\xi^{(n)}$ is called a system of extremal points of E of order n.

The following theorem gives an answer to an old problem due to F. Leja.

4.6. Theorem (Zaharjuta [32]). *If* $V_n(E) := |V(\xi^{(n)})|$, $n \geqslant 1$, *where* $\xi^{(n)}$ *is an arbitrary system of extremal points of* E *of order* n, *then the sequence*

$$d_n(E) := V_n(E)^{1/l_n}, \quad n \geqslant 1, \tag{1}$$

where $l_n := |\varkappa(1)| + \ldots + |\varkappa(n)|$, *is convergent.*

The limit $d(E)$ of sequence (1) is called C^N-*transfinite diameter of* E.

If $N = 1$, $d(E)$ is identical with the Fekete transfinite diameter of E (or with the logarithmic capacity, see [23]).

One may check that if A: $C^N \to C^N$ is an affine mapping, then

$$d(A(E)) = |\det A| \, d(E).$$

It is also known ([22]) that

$$d(E_1 \times \ldots \times E_N) = [d(E_n) \ldots d(E_N)]^{1/N},$$

E_j $(j = 1, \ldots, N)$ denoting compact subsets of the complex plane.

4.7. Given a compact set $E \subset C^N$ and a real function $b\colon C^N \to R$ bounded on E, put for every $n \geqslant 1$

$$F_n(E, b) := \{f \in \mathscr{P}_n\colon |f(x)| \leqslant e^{nb(x)} \text{ on } E\},$$

$$\Phi_n(x) \equiv \Phi_n(x, E, b) := \sup\{|f(x)|\colon f \in F_n(E, b)\}, \quad x \in C^N.$$

It is clear that

$$\Phi_k \Phi_l \leqslant \Phi_{k+l} \quad \text{and} \quad \Phi_n^k \leqslant \Phi_{k \cdot n} \quad \text{in } C^N \text{ for } k, l, n \geqslant 1.$$

Hence we obtain the following

4.8. PROPOSITION ([23], [24]). *For every $x \in C^N$ there exists $\Phi(x)$ with $1 \leqslant \Phi(x) \leqslant +\infty$ such that*

$$\Phi(x) = \lim_{n \to \infty} \sqrt[n]{\Phi_n(x)} = \sup_{n \geqslant 1} \sqrt[n]{\Phi_n(x)}.$$

4.9. DEFINITION. $\Phi(x) \equiv \Phi(x, E, b) \equiv \Phi_{E,b}(x) := \sup_{n \geqslant 1} \sqrt[n]{\Phi_n(x)}$, $x \in C^N$, is called an *extremal polynomial function of* E with respect to b. If $b \equiv 0$ on E, we write $\Phi_E(x) := \Phi(x, E, 0)$.

4.10. Let E be a unisolvent compact subset of C^N. Given any system of extremal points $\xi^{(m_n)} = \{\xi_1, \ldots, \xi_{m_n}\}$ of order m_n corresponding to E and b, we define for all $x \in C^N$

$$\Phi_n^{(1)}(x) := \max_{1 \leqslant j \leqslant m_n} |L^{(j)}(x, \xi^{(m_n)}) e^{nb(\xi_j)}|,$$

$$\Phi_n^{(2)}(x) := \sum_{j=1}^{m_n} |L^{(j)}(x, \xi^{(m_n)}) e^{nb(\xi_j)}|.$$

Next we define for all $x \in C^N$

$$\Phi_n^{(3)}(x) := \inf\{\max_{1 \leqslant j \leqslant m_n} |L^{(j)}(x, x^{(m_n)}) e^{nb(x_j)}|\},$$

$$\Phi_n^{(4)}(x) := \inf \sum_{j=1}^{m_n} |L^{(j)}(x, x^{(m_n)}) e^{nb(x_j)}|,$$

the inf being taken over all $x^{(m_n)} = \{x_1, \ldots, x_{m_n}\} \subset E$ with $V(x^{(m_n)}) \neq 0$.

We claim that

$$\Phi_n \leqslant \Phi_n^{(4)} \leqslant m_n \Phi_n^{(3)} \leqslant m_n \Phi_n^{(1)} \leqslant m_n \Phi_n^{(2)} \leqslant m_n^2 \Phi_n. \tag{2}$$

Indeed, the first inequality follows from the Lagrange interpolation formula. The last inequality follows from the interpolation formula and from the following inequalities

$$|L^{(j)}(x, \xi^{(m_n)}) e^{nb(\xi_j)}| \leqslant e^{nb(x)}, \quad x \in E,\ j = 1, \ldots, m_n,$$

that are direct consequence of Definition 4.5. The remaining inequalities of (2) are obvious.

We have just proved the following

4.11. PROPOSITION [24]. *If E is a unisolvent compact subset of C^N, then for every real continuous function $b\colon C^N \to R$ the sequences $\{\sqrt[l]{\Phi_n(x)}\}$ and $\{\sqrt[l]{\Phi_n^{(j)}(x)}\}_{n\geq 1}$ $(j = 1, 2, 3, 4)$ tend to the same limit $\Phi(x, E, b)$ as $n \to \infty$ for all $x \in C^N$.*

The main result of this section is given by the following

4.12. THEOREM. *If E is a compact subset of C^N and $b\colon C^N \to R$ is continuous, then*

$$V_{E,b} = \log \Phi_{E,b} \quad \text{in } C^N. \tag{3}$$

Proof. It is clear that $\log \Phi_{E,b} \leqslant V_{E,b} = V_{E,b}^+$ (see 1.1 and 2.7). Let now u be a fixed function in $L^+(E, b)$. Then by Proposition 1.3, given $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that

$$u_\lambda(x) := -\log \delta_\lambda(x) < b(x)+\varepsilon \quad \text{on } E, \text{ as } 0 < \lambda < \lambda_0. \tag{4}$$

Hence, by Proposition 1.4 applied to the function $v_\lambda := \log \frac{1}{\lambda} + u_\lambda$, we get

$$\left(\frac{1}{\lambda} e^{u_\lambda(x)}\right)^n \leqslant \sup_{f\in F_n} |f(x)| \leqslant c_n \left(\frac{1}{\lambda} e^{u_\lambda(x)}\right)^n, \quad x \in C^N,$$

where $f \in F_n$ is a polynomial of degree $\leqslant k+n$, $k = \text{const}$, and $\sqrt[n]{c_n} \to 1$ $(n \to \infty)$. So by (4)

$$\lambda^n |f(x)| \leqslant \lambda^{-k} c_n e^{(k+n)[b(x)+\varepsilon]}, \quad x \in E,\ f \in F_n.$$

Hence

$$e^{u(x)} \leqslant e^{u_\lambda(x)} \leqslant \lambda \sup_{f\in F_n} |f(x)|^{1/n} \leqslant \lambda^{-k/n} \sqrt[n]{c_n}\, e^{(k/n+1)\varepsilon} \Phi_{E,b}^{1+k/n}(x),$$
$$x \in C^N,\ n \geqslant 1,$$

so that $u \leqslant \log \Phi_{E,b}$ in C^N for all $u \in L^+(E, b)$. Therefore $V_{E,b} \leqslant \log \Phi_{E,b}$. Q.E.D.

4.13. Remark. Under the assumption that $b = 0$ on E and V_E is continuous equation (3) was first proved by a different method by Zaharjuta [33].

4.14. COROLLARY. *If E is a compact subset of C^N, then*

(i) *$V_E = V_{\hat{E}}$, where $\hat{E} := \{x \in C^N\colon |f(x)| \leqslant \|f\|_E$ for all polynomials $f\}$ is the polynomial envelope of E;*

(ii) *$V_E > 0$ in $C^N \setminus \hat{E}$.*

5. Extremal function Φ_E for special subsets of C^N

5.1. DEFINITION. We say that

1° E is *N-circular*, if for every point $a \in E$ the set $\{x \in C^N\colon |x_j| = |a_j|\ (j = 1, \ldots, N)\}$ is contained in E;

2° E is *circular*, if for every point $x \in E$ the set $\{\lambda x\colon \lambda \in C,\ |\lambda| = 1\}$ is contained in E.

5.2. Given any bounded subset E of C^N we put

$$M(x) \equiv M_E(x) := \sup\{|c_\alpha x^\alpha|^{1/|\alpha|}\}, \qquad x \in C^N,$$

the sup being taken over all monomials $c_\alpha x^\alpha$ such that $|\alpha| \geqslant 1$ and $|c_\alpha x^\alpha| \leqslant 1$ on E. It is obvious that $M(\lambda_1 x_1, \ldots, \lambda_N x_N) = rM(x_1, \ldots, x_N)$ if $x = (x_1, \ldots, x_N) \in C^N$ and $\lambda_j \in C$, $|\lambda_j| = r$ $(j = 1, \ldots, N)$.

5.3. Given any bounded subset E of C^N we put

$$H(x) \equiv H_E(x) := \sup_{n \geqslant 1}\{\sup_f |f(x)|^{1/n}\}, \qquad x \in C^N,$$

where $\sup_f$ denotes supremum taken over all homogeneous polynomials f of degree n. It is obvious that

$$H(\lambda x) = |\lambda|\, H(x), \qquad \text{if } x \in C^N,\ \lambda \in C.$$

5.4. PROPOSITION [24]. *If E is an N-circular compact subset of C^N, then*

$$\Phi_E = \max\{1, M_E\} \qquad \text{in } C^N.$$

Proof. It is obvious that $\max\{1, M_E\} \leqslant \Phi_E$. Let $f(x) = \sum_{|\alpha| \leqslant n} c_\alpha x^\alpha$ be a polynomial of degree $\leqslant n$ such that $|f(x)| \leqslant 1$ on E. Then by the Cauchy inequalities $|c_\alpha x^\alpha| \leqslant 1$ on E. Hence

$$|c_\alpha x^\alpha| \leqslant M(x)^{|\alpha|} \qquad \text{in } C^N,$$

$$|f(x)| \leqslant \sum_{|\alpha| \leqslant n} M(x)^{|\alpha|} \leqslant m_n M(x)^n, \qquad \text{if } M(x) \geqslant 1,$$

and $|f(x)| \leqslant m_n$ as $M(x) \leqslant 1$. Hence $|f(x)|^{1/n} \leqslant \max\{\sqrt[n]{m_n}, \sqrt[n]{m_n}\, M(x)\}$ in C^N. Now, if we apply the last inequality to the polynomial f^k $(k \geqslant 1)$, we get

$$|f(x)|^{1/n} \leqslant \max\{1, M(x)\} \qquad \text{in } C^N,$$

so that $\Phi_E \leqslant \max\{1, M_E\}$ in C^N. Q.E.D.

5.5. EXAMPLE. If $E = \{x \in C^N\colon q(x) \leqslant 1\}$, where

$$q(x) = \|x\| := (|x_1|^2 + \ldots + |x_N|^2)^{1/2}$$

or $q(x) = |x| := \max_{1 \leqslant j \leqslant N} |x_j|$, then $\Phi_E(x) = \max\{1, q(x)\}$.

5.6. PROPOSITION [24]. *If E is a circular compact subset of C^N, then $\Phi_E = \max\{1, H_E\}$, H_E being given by* 5.3.

Proof. It is obvious that $\max\{1, H_E\} \leqslant \Phi_E$. Let $f(x) = \sum_{|\alpha|\leqslant n} c_\alpha x^\alpha = \sum_{k=0}^{n} \big(\sum_{|\alpha|=k} c_\alpha x^\alpha\big) = \sum_{k=0}^{n} f_k(x)$ be a polynomial of degree $\leqslant n$ such that $|f(x)| \leqslant 1$ on E. Then by the Cauchy inequalities $|f_k(x)| \leqslant 1$ on E for $k = 0, \ldots, n$. Hence

$$|f(x)| \leqslant n+1 \quad \text{as } H(x) \leqslant 1 \quad \text{and} \quad |f(x)| \leqslant (n+1)\, H(x)^n \quad \text{as } H(x) \geqslant 1.$$

Hence, by a standard reasoning, $\Phi_E \leqslant \max\{1, H_E\}$. Q.E.D.

5.7. EXAMPLE. If $E := \{\lambda x\colon x \in R^N,\ \|x\| = 1, \lambda \in C, |\lambda| = 1\}$, then

$$H_E(z) = [\langle z, z\rangle + |z \wedge \bar{z}|]^{1/2} = \big[\|x\|^2 + \|y\|^2 + 2\sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}\,\big]^{1/2},$$

where $z = x+iy$, $\langle z, z\rangle = \sum_{j=1}^{N} z_j \bar{z}_j$.

5.8. For any bounded open set G in C^N we define the scalar product

$$\langle f, g\rangle := \int_G f(x)\overline{g(x)}\, dx$$

for any two complex functions f and g with $\int_G |f|^2 dx < +\infty$, $\int_G |g|^2 dx < +\infty$, the integration being taken with respect to the $2N$-dimensional Lebesgue measure in C^N.

Given a fixed one-to-one mapping

$$\varkappa\colon \{1, 2, \ldots\} \ni j \to \varkappa(j) \in Z_+^N$$

such that $|\varkappa(j)| \leqslant |\varkappa(j+1)|$, put $e_j(x) = x^{\varkappa(j)}$ $(j = 1, 2, \ldots)$. By the standard Schmidt procedure of orthonormalization we define a multiple sequence of polynomials $\{p_\alpha\}$, $\alpha \in Z_+^N$, with the following properties:

1° $p_\alpha = \sum_{j=1}^{j(\alpha)} a_j e_j$, where $j(\alpha) = \varkappa^{-1}(\alpha)$, $\deg p_\alpha = |\alpha|$,

2° $e_j = \sum_{\varkappa^{-1}(\alpha)\leqslant j} c_\alpha p_\alpha$, $j \geqslant 1$;

3° $\langle p_\alpha, p_\beta\rangle = \delta_{\alpha\beta}$ for all $\alpha, \beta \in Z_+^N$.

Let f be a holomorphic function in G such that $\|f\|^2 := \int_G |f|^2 dx < +\infty$. Then, by the Cauchy integral formula for polydiscs, one obtains the following known inequality

$$(*) \qquad |f(a)|^2 \leqslant (\pi r^2)^{-N} \|f\|^2,$$

where a is any point of G such that $B(a, r) := \{x \in C^N\colon |x-a| < r\} \subset G$.

5.9. PROPOSITION. *If $E = E_1 \times \ldots \times E_s$, where E_j is a compact subset of C^{N_j}, then*

$$\Phi_E(x_1, \ldots, x_s) = \max\{\Phi_{E_1}(x_1), \ldots, \Phi_{E_s}(x_s)\}, \quad x = (x_1, \ldots, x_s) \in C^N,$$

where $N = N_1 + \ldots + N_s$.

Proof. Without loss of generality we may assume $s = 2$. Put $A = E_1$, $B = E_2$. Given any $r > 0$ define $A^r := \bigcup_{a \in A} B(a, r) = \{x \in C^{N_1}: \operatorname{dist}(x, A) \leqslant r\}$ and $B^r = \{y \in C^{N_2}: \operatorname{dist}(y, B) \leqslant r\}$. Let $\{p_\alpha\}$ $(\alpha \in Z_+^{N_1})$ and $\{q_\beta\}$ $(\beta \in Z_+^{N_2})$ be the orthonormal sequences of polynomials defined in 5.8, when $G = A^r$ and $G = B^r$, respectively.

It is obvious that

$$\{p_\alpha(x)\, q_\beta(y)\} \quad (\alpha \in Z_+^{N_1},\ \beta \in Z_+^{N_2})$$

is an orthonormal system of polynomials with respect to the scalar product

$$\langle f, g\rangle := \int_{A^r \times B^r} f(x, y)\,\overline{g(x, y)}\,dx\,dy.$$

If $f(x, y) = \sum_{|\alpha| + |\beta| \leqslant n} a_{\alpha\beta}\, x^\alpha y^\beta$ is a polynomial of degree $\leqslant n$, then

$$f(x, y) = \sum_{|\alpha| + |\beta| \leqslant n} c_{\alpha\beta}\, p_\alpha(x)\, q_\beta(y) \quad \text{in } C^{N_1} \times C^{N_2}$$

for suitably chosen $c_{\alpha\beta} \in C$. Moreover, $c_{\alpha\beta} = \langle f, p_\alpha q_\beta\rangle$. Hence

$$|c_{\alpha\beta}| \leqslant M_r \sqrt{v(A^r \times B^r)},$$

where $M_r = \sup\{|f(x, y)|: x \in A^r, y \in B^r\}$ and $v(A^r \times B^r)$ denotes the Lebesgue volume of $A^r \times B^r$. By $(*)$ of 5.8 we have

$$|p_\alpha(x)|^2 \leqslant (\pi r^2)^{-N_1} \quad \text{for } x \in A \quad \text{and} \quad |q_\beta(y)|^2 \leqslant (\pi r^2)^{-N_2} \quad \text{for } y \in B.$$

Therefore

$$|f(x, y)| \leqslant M_r \sqrt{v(A^r \times B^r)}\, m_n (\pi r^2)^{-N/2} \max\{\Phi_A(x), \Phi_B(y)\}^n,$$
$$x \in C^{N_1},\ y \in C^{N_2}.$$

Hence, by substituting f^k $(k \geqslant 1)$ in place of f, we obtain

$$|f(x, y)|^{1/n} \leqslant M_r \max\{\Phi_A(x), \Phi_B(y)\}.$$

If $r \downarrow 0$, then $M_r \equiv M_r(f) \downarrow M = M(f) := \sup\{|f(x, y)|: x \in A, y \in B\}$. Therefore for any polynomial $f(x, y)$ such that $|f| \leqslant 1$ on $A \times B$ we get

$$|f(x, y)|^{1/n} \leqslant \max\{\Phi_A(x), \Phi_B(y)\} \quad \text{in } C^{N_1} \times C^{N_2}.$$

Hence $\Phi_{A \times B}(x, y) \leqslant \max\{\Phi_A(x), \Phi_B(y)\}$. Since the opposite inequality is obvious, the proof is concluded.

5.10. Remark. If $N_j = 1$ ($j = 1, \ldots, s$), Proposition 5.9 was first proved by a different method in [24].

5.11. Proposition (M. Mazurek). *If E is a compact subset of C^N and $R > 1$, then*

$$V_{E_R} = \max\{0, V_E - \log R\} = V_{F_R},$$

where $E_R := \{x \in C^N\colon\ V_E(x) \leqslant \log R\}$, $D_R := \{x \in C^N\colon\ V_E(x) < \log R\}$ *and* $F_R := \bar{D}_R$.

Proof. 1° $V_{E_R} = \max\{0, V_E - \log R\}$. Indeed, it is obvious that $\max\{0, V_E - \log R\} \leqslant V_{E_R}$. In order to show the opposite inequality, given $r > 0$, put

$$E_R^r := \{x \in C^N\colon\ V_{E^r}(x) \leqslant \log R\}.$$

Next, given any $u \in L(E_R^r, 0)$, the function v defined by $v(x) := \max\{u(x) + \log R, V_{E^r}(x)\}$ as $x \in C^N \setminus E_R^r$ and $v(x) := V_{E^r}(x)$ as $x \in E_R^r$, belongs to $L(E^r, 0)$. Therefore

$$u(x) + \log R \leqslant V_{E^r}(x) \leqslant V_E(x), \quad x \in C^N \setminus E_R^r.$$

Hence

$$V_{E_R^r}(x) \leqslant V_E(x) - \log R \quad \text{in } C^N \setminus E_R^r.$$

By 2.14, $E_R^r \downarrow E_R$ and $V_{E_R^r} \uparrow V_{E_R}$ as $r \downarrow 0$. Thus $V_{E_R}(x) \leqslant V_E(x) - \log R$ in $C^N \setminus E_R$, and finally $V_{E_R} \leqslant \max\{0, V_E - \log R\}$ in C^N.

2° For every R' with $1 < R' < R$ we have $E_{R'} \subset D_R \subset F_R \subset E_R$. Hence, by 1°,

$$V_{E_R} \leqslant V_{F_R} \leqslant V_{E_{R'}} = \max\{0, V_E - \log R'\}, \quad 1 < R' < R.$$

If $R' \to R$, we get $V_{E_R} = V_{F_R} = \max\{0, V_E - \log R\}$. Q.E.D.

5.12. Corollary. *If V_E is continuous, then, for every $R > 1$, V_{E_R} is continuous.*

5.13. Question. Is $F_R^* := \bar{D}_R$ polynomially convex? (The positive answer implies that $E_R = \bar{D}_R$.)

5.14. Proposition. *Let q be a non-negative real continuous function in C^N, such that $q(\lambda x) = |\lambda| q(x)$ for $\lambda \in C$, $x \in C^N$ and $\log q \in L^+$. If $E := \{x \in C^N\colon\ q(x) \leqslant 1\}$, then*

$$V_E(x) = \log^+ q(x), \quad x \in C^N.$$

Proof. Let x be a fixed point in $C^N \setminus E$ and let u be a fixed function in L^+ such that $u \leqslant 0$ on E. Then the function $\lambda \to u(\lambda x) - \log q(\lambda x)$ is subharmonic in $\hat{C} \setminus \{0\}$ (where $\hat{C} = C \cup \{\infty\}$) and

$$u(\lambda x) - \log q(\lambda x) \leqslant 0 \quad \text{as } |\lambda| = r,$$

where r is a positive number such that $q(rx) = 1$. Now by the maximum property for subharmonic functions the last inequality holds true for all $\lambda \in C$ with $|\lambda| \geqslant r$. Hence $u(x) \leqslant \log q(x)$. Therefore $V_E \leqslant \log^+ q$. Since the opposite inequality is obvious, the proof is concluded.

5.15. COROLLARY. *If $q\colon C^N \to [0, +\infty)$ is a norm and $E := \{x \in C^N: q(x-a) \leqslant r\}$, then $V_E(x) = \log^+ q(x-a)/r$.*

6. L-regular sets in C^N

6.1. PROPOSITION. *Let E be a compact subset of C^N. Then the following conditions are equivalent*:

(a) *V_E is continuous in C^N*;

(b) *V_E is continuous at every point $a \in E$ (i.e. $V_E^* = V_E = 0$ on E)*;

(c) *$h_{E,D}^* = 0$ on E for every open bounded neighbourhood D of $\hat{E}$ (the polynomial envelope of E)*;

(Φ) *For every $\varepsilon > 0$ there exists an open neighbourhood U of E such that for every polynomial $f \in \mathscr{P}_n$, $n \geqslant 1$,*

$$\|f\|_U \leqslant \|f\|_E e^{n \cdot \varepsilon}.$$

Proof. (a)$\Leftrightarrow$(b) by Proposition 2.13.

(a)$\Rightarrow$(Φ). If $f \in \mathscr{P}_n$, then, by 2.11, $|f(x)| \leqslant \|f\| \exp(nV_E(x))$ in C^N. Hence (Φ) is satisfied with $U := \{x \in C^N: V_E(x) < \varepsilon\}$.

(Φ)$\Rightarrow$(a). If (Φ) is satisfied, then, by Theorem 4.12, $V_E(x) < \varepsilon$ for $x \in U$.

(a)$\Rightarrow$(c). Put $m := \inf\{V_E(x): x \in \partial D\}$. Then $m > 0$ and $V_E \geqslant m$ on ∂D. Let u be a plsh function in D such that $u \leqslant 0$ on E and $u \leqslant 1$ in D. Then the function v defined by

$$v(x) := \max\{mu(x), V_E(x)\} \quad \text{for } x \in D, \qquad v(x) := V_E(x) \quad \text{for } x \in C^N \setminus D$$

belongs to L and $v \leqslant 0$ on E. Hence

$$mh_{E,D}^*(x) \leqslant V_E(x) \quad \text{in } D.$$

In particular, $h_{E,D}^* = 0$ on E.

(c)$\Rightarrow$(b). If $h_{E,D}^* = 0$ on E, then by Theorem 3.10 and Corollary 3.9, $V_E^* \in L$. Put

$$M := \sup\{V_E^*(x): x \in \bar{D}\}.$$

Then for every $u \in L$ with $u \leqslant 0$ on E we have $u/M \leqslant h_{E,D}$ in D. Hence $V_E^* \leqslant Mh_{E,D}^*$ in D, so that $V_E^* = 0$ on E. Q.E.D.

6.2. PROPOSITION. *If E is a compact subset of C^N, then V_E is continuous at $a \in E$ if and only if for every bounded neighbourhood D of E the function $h_{E,D}$ is continuous at a (i.e. $V_E^*(a) = 0 \Leftrightarrow h_{E,D}^*(a) = 0$).*

Proof. 1° $V_E^*(a) = 0 \Rightarrow h_{E,D}^*(a) = 0$. Indeed, given any $\varepsilon > 0$ take $r > 0$

so small that $V_{E^r}(x) \geqslant m-\varepsilon$, where $m := \inf_{x\in\partial D} V_E(x)$. Then for every function u plsh in D such that $u \leqslant 0$ on E and $u \leqslant 1$ in D, we have

$$\max\{(m-\varepsilon)u, V_{E^r}\} \leqslant V_E \quad \text{in } D.$$

Hence $mh^*_{E,D} \leqslant V^*_E$ in D and $h^*_{E,D}(a) = 0$.

2° $h^*_{E,D}(a) = 0 \Rightarrow V^*_E(a) = 0$ because $V_E(x) \leqslant Mh_{ED}(x)$, where $M := \sup_{\bar{D}} V^*_E$.

6.3. Definition. Let E be a subset of C^N and let a be a point of its closure $\bar{E}$. We say that:

1° E is *locally L-regular at a*, if for every $r > 0$ the extremal function $V_{E\cap B(a,r)}$ (corresponding to the set $E\cap B(a,r)$) is continuous at a;

2° E satisfies the *polynomial condition* (L^1_0) at a, if for every $\varepsilon > 0$ there exists a neighbourhood U of a such that

$$\|f\|_U \leqslant \|f\|_E \exp(\varepsilon\cdot\deg f)$$

for every polynomial f;

3° E satisfies the *polynomial condition* (L^1) at a, if for every $r > 0$ the set $E\cap B(a,r)$ satisfies condition (L^1_0) at a;

4° E satisfies the *polynomial condition* (L_0) at a, if for every family $\mathscr{F}$ of polynomials f such that

$$\sup_{f\in\mathscr{F}} |f(x)| < +\infty, \quad x\in E,$$

and for every $\varepsilon > 0$ there exist two positive numbers M and δ such that

$$|f(x)| \leqslant M\exp(\varepsilon\cdot\deg f), \quad |x-a| < \delta,\ f\in\mathscr{F}.$$

5° E satisfies the *polynomial condition* (L) at a, if for every $r > 0$ the set $E\cap B(a,r)$ satisfies condition (L_0) at a;

6° E is *C^N-fat* at a (or E is not C^N-thin at a), if for every open connected neighbourhood U of a the following Proposition (Modified Hartogs Lemma) is true:

"Let $\{v_n\}$ be a sequence of functions defined in U by

$$v_n(x) := \sup\{\lambda_n \log|f_n(x,t)|:\ t\in T_n\}, \quad x\in U,\ n\geqslant 1,$$

where λ_n is a positive number, T_n is an arbitrary set of arbitrary elements and the function f_n is holomorphic with respect to $x\in U$ for each fixed $t\in T_n$.

If $v_n \leqslant M$ $(n\geqslant 1)$ in U and $\limsup_{n\to\infty} v_n \leqslant A$ on E, then for every $\varepsilon > 0$ there exists $\delta > 0$ and $n_0 = n_0(\varepsilon)$ such that

$$v_n(x) \leqslant A+\varepsilon, \quad n\geqslant n_0,\ |x-a| < \delta."$$

6.4. Proposition. *If $N = 1$ and E is compact, then conditions* 1°, 2°, 3°, 5° *and* 6° *are equivalent.*

If $N \geqslant 1$ and E is compact or not, then $1^\circ \Rightarrow 3^\circ \Rightarrow 2^\circ$, $5^\circ \Rightarrow 4^\circ \Rightarrow 2^\circ$ and $5^\circ \Rightarrow 6^\circ$. If $N \geqslant 1$ and E is compact, then $6^\circ \Rightarrow 1^\circ$.

Proof. For the case $N = 1$ see [23], [25], [26], [15]. If $N \geqslant 1$, the implications $1^\circ \Rightarrow 3^\circ \Rightarrow 2^\circ$ and $5^\circ \Rightarrow 4^\circ \Rightarrow 2^\circ$ are obvious. The proof of the implication $5^\circ \Rightarrow 6^\circ$ may be found in [26] (see the proof of Theorem 2.1 in [26]). Finally, if $E \subset C^N$ is compact, $6^\circ \Rightarrow 1^\circ$ follows from Theorem 4.12.

6.5. Remark. All conditions 1°–6° are invariant with respect to one-to-one mappings of C^N onto itself. Indeed, if $A: C^N \to C^N$ is any non-singular affine mapping and E is any subset of C^N, then

$$V_{A(E)} = V_A \circ A^{-1}.$$

If $F \supset E$ and E satisfies any of conditions 1°–6°, then F satisfies the same conditions.

6.6. Example 1 (F. Leja [16]). Let E be a subset of C and let a be a limit point of E. If there exist a positive number ϱ and a subset S of the interval $(0, \varrho)$ with the Lebesgue measure $m(S) = \varrho$ such that for every $r \in S$ the circle $\{z: |z-a| = r\}$ intersects E, then E satisfies (L) at a.

6.7. Example 2. If $F_1 \times \ldots \times F_N \subset E$, F_j being a compact subset of the complex plane such that F_j is locally L-regular at $a_j \in F_j$ $(j = 1, \ldots, N)$, then E satisfies all conditions 1°–6° at $a = (a_1, \ldots, a_N)$. For instance E satisfies 1°–6° at $a \in E$, if there exist continua F_j $(j = 1, \ldots, N)$, not reduced to the point a_j, such that $a \in F_1 \times \ldots \times F_N \subset E$.

Remark 6.5 and this example imply

6.8. Example 3. Let E be a subset of R^N (we identity R^N with the subset $R^N + i \cdot 0$ of C^N). If P is a parallelepiped with non-empty interior contained in E, then E satisfies all conditions 1°–6° at every point $a \in \bar{P}$.

6.9. Example 4 (Baouendi–Goulaouic [3]). *Homothety Criterion.* Let $I = [A, B]$ be any interval in C^N. For $h > 1$ denote by $I(h)$ the homothetic interval obtained from I by the homothety contered at $(A+B)/2$ and whose ratio of similitude is h.

If E is a subset of C^N denote by $E(h)$ the union of $I(h)$, where I is any interval in E, $E(h) := \bigcup_{I \subset E} I(h)$.

If $a \in R^N$ (resp. $a \in C^N$) and $E(h)$ is a neighbourhood of a in R^N (resp. in C^N) for every $h > 1$, then E satisfies condition (L_0^1) at a.

6.10. Example 5 (Baouendi–Goulaouic [3]). Let φ be a strictly increasing continuous function defined on $[0, 1]$ with $\varphi(0) = 0$. Then the compact set $E \subset R^2$ defined by

$$E = \{(x_1, x_2) \in R^2 \colon 0 \leqslant x_1 \leqslant r,\ 0 \leqslant x_2 \leqslant \varphi(x_1)\}, \quad 0 < r \leqslant 1,$$

satisfies the homothety criterion (and hence L^1) at $a = (0, 0)$.

6.11. EXAMPLE 6 (Dudley–Rundal [7]). If E is a compact convex subset of $\boldsymbol{R}^N$ with non-empty interior, then E satisfies 5° at every $a \in E$. The same remains true if E is a union of compact convex sets with non-empty interiors.

This follows easily from Example 1 and Remark 6.5.

7. Extremal functions of one complex variable

In this section we shall recall (without proof) some of the known properties of the extremal functions of one complex variable. More detailed informations and references may be found in [23].

7.1. Let b denote a real bounded function defined on a compact subset E of $\boldsymbol{C}$. If $c(E) > 0$ ($c(E)$ denotes the logarithmic capacity of E), then $V_{E,b}$ is harmonic in $\boldsymbol{C}\backslash E$.

Indeed, if $B(a, r)$ is a disc in $\boldsymbol{C}\backslash E$, one may replace every $u \in L(E, b)$ by $u' \in L(E, b)$ given by

$$u'(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2 - |z-a|^2}{|re^{it} - (z-a)|^2}\, u(a + re^{it})\, dt \qquad \text{in } B(a, r)$$

and $u' = u$ in $\boldsymbol{C}\backslash B(a, r)$. Hence $V_{E,b}$ is harmonic in $B(a, r)$.

7.2. If $c(E) > 0$, then V_E is the Green function of D_∞ with pole at ∞, where $D_\infty = D_\infty(E)$ denotes the unbounded component of $\hat{\boldsymbol{C}}\backslash E$, $\hat{\boldsymbol{C}} := \boldsymbol{C} \cup \{\infty\}$.

7.3. THEOREM [23]. *Let $E = \partial D_\infty(E)$ and let $D_\infty(E)$ be regular with respect to the Dirichlet problem. Let b denote a real continuous function defined on E.*

Then for every $\lambda > 0$

$$u_\lambda(z) := (1/\lambda)\,[V_{E,\lambda b}(z) - V_E(z)], \qquad z \in \hat{\boldsymbol{C}},$$

is continuous on $\hat{\boldsymbol{C}}$ ($u_\lambda(\infty) = \lim_{z\to\infty} u_\lambda(z)$), harmonic in $\hat{\boldsymbol{C}}\backslash E$ and $u_\lambda \to u := \lim_{\lambda \downarrow 0} u_\lambda$ uniformly on $\hat{\boldsymbol{C}}$ as $\lambda \downarrow 0$. The limit function u is equal to b on E, so that u is the solution of the Dirichlet problem for $\hat{\boldsymbol{C}}\backslash E$ with boundary values b.

Proof. By 7.1, u_λ is harmonic in $\hat{\boldsymbol{C}}\backslash E$. The continuity of u_λ on $\boldsymbol{C}$ follows from Proposition 2.16. The limit $u = \lim u_\lambda$ exists by 2.9.

By the Dini theorem and by the maximum principle for harmonic functions it is now enough to show that $u_\lambda \uparrow b$ on E as $\lambda \downarrow 0$. This is, however, a direct consequence of Proposition 2.17 and of the following known (e.g. [23])

7.4. PROPOSITION. *If $E = \partial D_\infty(E)$ and b: $E \to \boldsymbol{R}$ is continuous, then there exists a sequence of holomorphic polynomials $\{g_k\}$ such that*

$$\exp b(z) = \sup_{k \geq 1} |g_k(z)|, \qquad z \in E.$$

8. Analytic functions on compact subsets of C^N

8.1. Given a compact set $E \subset C^N$ and a complex continuous function f on E, let $T_n f \in \mathscr{P}_n$ denote a Čebyšev polynomial of best approximation to f on E of degree $\leq n$. The polynomial $T_n f$ is defined by the condition

$$\varrho_n(E, f) := \|f - T_n f\|_E \leq \|f - p\|_E := \sup_{x \in E} |f(x) - p(x)| \quad \text{for all } p \in \mathscr{P}_n.$$

Put

$$\varrho(E, f) := \limsup_{n \to \infty} \sqrt[n]{\varrho_n(E, f)}.$$

8.2. Remark. Put

$$(L_n f)(x) := \sum_{j=1}^{m_n} f(\xi_j) L^{(j)}(x, \xi^{(m_n)}),$$

where $\xi^{(m_n)}$ is any system of extremal points of E of order m_n. Then

$$\|f - L_n f\|_E \leq \|f - T_n f\|_E + \|L_n(f - T_n f)\|_E \leq (1 + m_n)\|f - T_n f\|_E, \quad n \geq 1,$$

whence

$$\varrho(E, f) = \limsup_{n \to \infty} \sqrt[n]{\|f - L_n f\|_E}.$$

We shall need the following known

8.3. Proposition [13]. *If f is holomorphic in a polynomial polyhedron*

$$D := \{x \in C^N \colon |x_j| < \delta \ (j = 1, \ldots, N), |p_i(x)| < \delta \ (i = 1, \ldots, k)\},$$

where p_i $(i = 1, \ldots, k)$ is a polynomial, δ is a positive number and k is a positive integer, then there exists a function F holomorphic in the polycylinder

$$\Delta := \{(x, y) \in C^N \times C^k \colon |x| < \delta, \ |y| < \delta\},$$

so that

$$f(x) = F\big(x, p(x)\big), \quad x \in E,$$

where $p(x) = \big(p_1(x), \ldots, p_k(x)\big)$.

8.4. Lemma. *Let p_i $(i = 1, \ldots, k)$ be polynomials of degree $\leq d$. Given $R > 1$ and $t > 0$, define*

$$D_t := \{x \in C^N \colon \varphi(x) < R^t\},$$

where $\varphi(x) := \max_{1 \leq i \leq k} |p_i(x)|^{1/d}$. Assume that D_t is bounded.

If $0 < s < t$ and f is holomorphic in D_t, then

$$\varrho(\bar{D}_s, f) \leq R^{s-t}.$$

Proof. Take $\delta > 0$ so large that D_t is contained in the polydisc $\{x \in C^N \colon |x| < \delta \cdot R^{s \cdot d}\}$. Then

$$D_t = \{x \in C^N \colon |x| < \delta \cdot R^{t \cdot d}, |\delta \cdot p_i(x)| < \delta \cdot R^{t \cdot d} \ (i = 1, \ldots, k)\}$$

so that D_t is a polynomial polyhedron. By 8.3 there exists a function

$$F(x, y) = \sum c_{\alpha\beta} x^\alpha y^\beta \quad (\alpha \in Z_+^N, \beta \in Z_+^k),$$

holomorphic in the polydisc

$$\Delta = \{(x, y) \in C^{N+k}: |x| < \delta \cdot R^{t \cdot d}, \quad |y| < \delta \cdot R^{t \cdot \delta}\}$$

such that

$$f(x) = F(x, \delta \cdot p(x)), \quad x \in D_t,$$

where $\delta \cdot p(x) = (\delta \cdot p_1(x), \ldots, \delta \cdot p_k(x))$.

Take $\theta < 1$ so close to 1 that

$$\{(x, \delta \cdot p(x)) \in C^{N+k}: x \in \bar{D}_s\} \subset \theta \cdot \Delta.$$

Since $\theta \cdot \Delta$ is a relatively compact subset of Δ, the Cauchy inequalities imply

$$|c_{\alpha\beta}| \leqslant M/(\theta \cdot \delta \cdot R^{t \cdot d})^{|\alpha|+|\beta|}, \quad M = M(\theta) = \text{const.}$$

Put

$$f_n(x) := \sum_{|\alpha|+|\beta| \leqslant n} c_{\alpha\beta} x^\alpha (\delta \cdot p(x))^\beta, \quad n \geqslant 1.$$

Then f_n is a polynomial of degree $\leqslant n \cdot d$ and

$$\|f - f_n\|_{D_s} \leqslant M_1 \theta^{-n} (R^{s-t})^{n \cdot d}, \quad M_1 = \text{const},$$

because $|x| \leqslant \delta \cdot R^{s \cdot d}$, $|\delta \cdot p(x)| \leqslant \delta \cdot R^{s \cdot d}$ as $x \in \bar{D}_s$. Hence

(1) $$\limsup_{n \to \infty} \sqrt[n \cdot d]{\varrho_{n \cdot d}(\bar{D}_s, f)} \leqslant R^{s-t}.$$

Now, by a standard reasoning (see e.g. [24]), one may easily show that (1) holds true also for $d = 1$. Q.E.D.

8.5. Theorem [24]. 1° *If E is a polynomially convex compact subset of C^N and f is a function holomorphic in a neighbourhood G of E, then*

$$\varrho(E, f) < 1.$$

2° *Let E be a compact subset of C^N such that the extremal function Φ_E is continuous. Put*

(2) $$D := \{x \in C^N: \Phi_E(x) < R\} \quad (R > 1).$$

Then

(a) *If f is holomorphic in D, then $\varrho(E, f) \leqslant 1/R$;*

(b) *If $\varrho(E, f) < 1$, then f is continuable to a function $\tilde{f}$ holomorphic in D given by* (2) *with $R = 1/\varrho(E, f)$.*

Proof. 1° By the polynomial convexity of E we can find a positive integer d, a positive number t and polynomials $p_1, \ldots, p_k$, $\deg p_i \leqslant d$, such that for a real number $R > 1$

$$E \subset D_t := \{x \in C^N: |p_j(x)|^{1/d} < R^t \ (j = 1, \ldots, k)\} \subset G.$$

Now Lemma 8.4 implies the inequality $\varrho(E, f) < 1$.

2° (a). Let $\xi^{(m_n)}$ denote any system of extremal points of E of order m_n. It follows from 4.7–4.11 that given $\varepsilon > 0$ we may find an integer d so large that

$$\Phi_E(x) R^{-\varepsilon} \leqslant \varphi(x) \leqslant \Phi_E(x), \quad x \in D,$$

where

$$\varphi(x) := \big(\max_{1 \leqslant j \leqslant m_d} |L^{(j)}(x, \xi^{(m_d)})|\big)^{1/d}.$$

Put for $t > 0$

$$D_t := \{x \in C^N: \Phi_E(x) < R^t\}, \quad D_t' := \{x \in C^N: \varphi(x) < R^t\}.$$

Then

$$E \subset D_t \subset D_t' \subset D_{t+\varepsilon}, \quad \text{if } 0 < t \text{ and } t+\varepsilon \leqslant 1.$$

Hence by Lemma 8.4

$$\varrho(E, f) \leqslant \varrho(\bar{D}_\varepsilon', f) \leqslant R^{\varepsilon-(1-\varepsilon)} = R^{2\varepsilon-1},$$

because f is holomorphic in $D_{1-\varepsilon}'$. By the arbitrariness of ε we get $\varrho(E, f) \leqslant R^{-1}$.

(b) Take $\varrho_1 > \varrho = \varrho(E, f)$. Then $\varrho_n(E, f) \leqslant M\varrho_1^n$ $(n \geqslant 1)$ for a positive constant $M = M(\varrho_1)$. Take $p_n \in \mathscr{P}_n$ such that $\varrho_n(E, f) = \|f - p_n\|_E$. Put $Q_1 := p_1$, $Q_n := p_n - p_{n-1}$ $(n \geqslant 2)$. Then

$$|Q_n(x)| \leqslant M(1 + 1/\varrho_1)\varrho_1^n, \quad x \in E, \ n \geqslant 1.$$

Hence

$$|Q_n(x)| \leqslant M(1 + 1/\varrho_1)[\varrho_1 \Phi_E(x)]^n, \quad x \in C^N, \ n \geqslant 1.$$

Therefore the series $\sum Q_n$ is uniformly convergent on compact subsets of $\{x \in C^N: \Phi_E(x) < 1/\varrho_1\}$. By the arbitrariness of ϱ_1 $(\varrho_1 > \varrho)$ the series is uniformly convergent on compact subsets of D given by (2) with $R = 1/\varrho$ and its limit $\tilde{f}$ gives the required continuation of f.

8.6. Corollary. *Let E be a compact subset of C^N such that Φ_E is continuous. If f is a continuous function on E and $\{p_n\}$ is a sequence of polynomials such that* $\deg p_n \leqslant n$ *and*

(3) $$\limsup_{n \to \infty} \sqrt[n]{\|f - p_n\|_E} \leqslant 1/R \quad \textit{with } R > 1,$$

then f is holomorphic in D given by (2) *and for every σ* $(0 < \sigma < 1)$

(4) $$\limsup_{n\to\infty} \sqrt[n]{\|f-p_n\|_{D_\sigma}} \leqslant R^{\sigma-1},$$

where

$$D_\sigma := \{x \in C^N:\ \Phi_E(x) < R^\sigma\}.$$

Proof. Put $Q_n = p_1$, $Q_n := p_n - p_{n-1}$ $(n \geqslant 1)$. It follows from (3) that

$$\limsup_{n\to\infty} \sqrt[n]{\|Q_n\|_E} \leqslant R^{-1}.$$

Hence for every $\varepsilon > 0$ there exists $M > 0$ such that $|Q_n(x)| \leqslant M(R^{\varepsilon-1})^n$ for $x \in E$, $n \geqslant 1$. Now, by the Bernstein–Walsh inequality 2.11, $|Q_n(x)| \leqslant M(R^{\sigma+\varepsilon-1})^n$ for $x \in D_\sigma$. Therefore

$$\|f-p_n\|_{D_\sigma} \leqslant \sum_{j\geqslant n+1} \|Q_j\|_{D_\sigma} \leqslant M_1(R^{\sigma+\varepsilon-1})^n, \qquad M_1 = \text{const},$$

$\varepsilon > 0$ being any positive number such that $\sigma+\varepsilon < 1$. Q.E.D.

8.7. Remark. Point 1° of Theorem 8.5 implies the Oka–Weil approximation theorem. If $N = 1$, Theorem 8.5 is due to Bernstein and Walsh. If $N \geqslant 2$ it has been first proved by a slightly different method (based on the Weil integral formula) in [24]. Yet another proof of Theorem 8.5 was recently given by Zacharjuta [33].

8.8. Let us consider the following three conditions, where E is a compact subset of C^N:

(A) $f \in \mathscr{C}(E)$, $\varrho(E, f) < 1 \Rightarrow f \in \mathscr{A}(E)$;

(B) For every real number $\omega > 1$ there exist an open neighbourhood U of E and a positive constant M such that for all polynomials f of N complex variables

$$\|f\|_U \leqslant M\|f\|_E\, \omega^{\deg f};$$

(Φ) Φ_E is continuous.

$\mathscr{A}(E)$ denotes the space of all holomorphic functions f in a neighbourhood of E that may depend on f. $\mathscr{A}(E)$ may be considered as an inductive limit of the sequence of Banach spaces $\{A_n\}$, where A_n is the Banach space of all bounded holomorphic functions in

$$U_n := \{x \in C^N:\ \operatorname{dist}(z, E) < 1/n\}.$$

We say that E is determining for $\mathscr{A}(E)$, if for every n the following implication is true:

$$f \in A_n, \quad f|E = 0 \Rightarrow f = 0 \quad \text{in } U_n.$$

8.9. Theorem (comp. [3], [27]). *Under the assumption of* 8.8 (B)$\Leftrightarrow$(Φ)$\Rightarrow$(A). *If, moreover, E is determining for $\mathscr{A}(E)$, then the three conditions are equivalent.*

Proof. By Theorem 8.5 it remains to show that (under the additional assumption) (A) $\Rightarrow$ (B). Put

$$\mathscr{C}_w := \{f \in \mathscr{C}(E): \sup_{n \geqslant 0} w^n \varrho_n(E,f) \text{ is finite}\}, \quad w > 1.$$

Then $\|f\| := \|f\|_E + \sup_{n \geqslant 0} w^n \varrho_n(E,f)$ is a norm in $\mathscr{C}_w$ and $\mathscr{C}_w$ endowed with this norm is a Banach space. Consider the following injective linear mappings

$$\varphi\colon \mathscr{C}_w \to \mathscr{A}(E), \quad \psi\colon \mathscr{A}(E) \to \mathscr{C}(E) \quad \text{and} \quad j\colon \mathscr{C}_w \to \mathscr{C}(E),$$

where φ is given by condition (A), $\psi(f)$ denotes the restriction of $f \in \mathscr{A}(E)$ to E, and j denotes the inclusion. Since E is determining, φ is well defined. The mapping j and ψ are continuous and $j = \psi \circ \varphi$. Therefore the graph of φ is closed. Hence by a well-known Grothendick's theorem φ is continuous and there exists s such that $\varphi(\mathscr{C}_w) \subset \mathscr{A}_s$ and $\varphi\colon \mathscr{C}_w \to \mathscr{A}_s$ is continuous. Therefore there exists $M > 0$ such that for all $f \in \mathscr{C}_w$

$$\|f\|_{U_s} \leqslant M \|f\|_w.$$

If f is a polynomial of degree $\leqslant k$, then $f \in \mathscr{C}_w$ and

$$\|f\|_{U_s} \leqslant M\big(\|f\|_E + \sup_{n \geqslant 0} w^n \varrho_n(E,f)\big) \leqslant 2M \|f\|_E w^k,$$

because $\varrho_n(E,f) = 0$ for $n \geqslant k+1$. Q.E.D.

9. Separately analytic functions

9.1. Let E and D be subsets of C^N. We say that the pair (E, D) satisfies *condition* (h^*), if $E \subset D$, D is open and if for every increasing sequence of compact sets $\{E_k\}$ such that $E_k \uparrow E$

$$h^*_{E_k D} \downarrow h^*_{ED} \quad \text{in } D \text{ as } k \to \infty,$$

where $h_{E_k D}$ and h_{ED} are defined by 3.2.

9.2. PROPOSITION. *Let E be a compact subset of C^N such that Φ_E is continuous. Put*

(1) $$D := \{x \in C^N\colon \Phi_E(x) < R\} \quad (R > 1).$$

Then

(i) $h_{ED} = \log \Phi_E / \log R$ *in* D (see [33]);

(ii) *The pair (E, D) satisfies condition (h^*) if and only if for every sequence $\{E_k\}$ of compact subsets of E such that $E_k \uparrow E$ $(k \to \infty)$, we have $V^*_{E_k} \searrow V_E$ in C^N as $k \to \infty$.*

Proof. (i) It is clear that $V_E = \log \Phi_E \leqslant h_{ED} \cdot \log R$ in D. Take $u \in \mathrm{PHS}(D)$ such that $u \leqslant 0$ on E and $u \leqslant 1$ in D, and define

$$v(x) := \max\{u(x) \log R, V_E(x)\} \quad \text{in } D, \quad v(x) := V_E(x) \quad \text{in } C^N \setminus D.$$

Then $v \in L$ and $v \leqslant 0$ on E. Hence $u \log R \leqslant V_E$ in D. Therefore $h_{ED} \times \times \log R \leqslant V_E$ in D. The proof of (i) is concluded.

(ii) First assume that (E, D) satisfies (h^*). Since $E_k \uparrow E$ and $V_E \in L$, by 3.6 and 3.5 there exists k_0 such that $V_{E_k}^* \in L$ for $k \geqslant k_0$. Put $M_k := \sup_D V_{E_k}^*$. Then $M := M_{k_0} \geqslant M_k$ $(k \geqslant k_0)$ and $V_{E_k}^* \leqslant M h_{E_k D}$ in D for $k \geqslant k_0$. Indeed, if $u \in L$ and $u \leqslant 0$ on E_k, then $u \leqslant M h_{E_k D}$ in D. Hence $V_{E_k} \leqslant M h_{E_k D}$ in D. Therefore the function

$$V = \lim_{k \to \infty} V_{E_k}^*$$

belongs to L and $V \leqslant 0$ on E. Hence $V \leqslant V_E$. On the other hand by 2.3

$$V_E \leqslant V_{E_k}^* \quad (k \geqslant 1), \text{ so that } V_E \leqslant V.$$

Now, in order to prove the implication in the opposite direction take any sequence of compact sets $\{E_k\}$ such that $E_k \uparrow E$. It is obvious that

$$(2) \qquad h_{ED} \leqslant h_{E_k D} \leqslant h_{E_k D}^* \quad \text{for } k \geqslant 1.$$

Take any $\varepsilon > 0$ and let k_0 be so large that $V_{E_k}^* \leqslant \varepsilon$ on E for $k \geqslant k_0$. Take $u \in \mathrm{PSH}(D)$ such that $u \leqslant 0$ on E_k and $u \leqslant 1$ in D, and define $v(x) := \max\{u \log(Re^{-\varepsilon}), V_{E_k}^* - \varepsilon\}$ in D, $v(x) := V_{E_k}^* - \varepsilon$ in $C^N \setminus D$. Then $v \in L$ and $v \leqslant 0$ on E. Therefore $v \leqslant V_E$. Hence

$$h_{E_k D}^* \leqslant V_E / \log(Re^{-\varepsilon}) = \frac{\log R}{\log(Re^{-\varepsilon})} h_{ED} \quad \text{in } D \text{ for } k \geqslant k_0,$$

because by (i) $V_E/\log R = h_{ED}$ in D. Now by (2) we get the required result.

9.3. Proposition. *Let $D \subset C^p$, $G \subset C^q$ be open sets, $E \subset D$ and $F \subset G$ compact subsets such that h_{ED} and h_{FG} are continuous. Put*

$$\Omega := \{(x, y) \in D \times G \colon h_{ED}(x) + h_{FG}(y) < 1\}.$$

Then

$$h(x, y) := h_{E \times F, \Omega}(x, y) = h_{ED}(x) + h_{FG}(y), \quad (x, y) \in \Omega.$$

Proof. Fix $(a, b) \in \Omega$. Take $u \in \mathrm{PSH}(\Omega)$ such that $u \leqslant 0$ on $E \times F$ and $u \leqslant 1$ in Ω. If $a \in E$, then of course $u(a, b) \leqslant h(a, b) = h_{FG}(b)$. If $a \notin E$, then $h_{FG}(b) < 1 - h_{ED}(a)$. Therefore

$$\frac{u(a, b) - h_{ED}(a)}{1 - h_{ED}(a)} \leqslant \frac{h_{FG}(y)}{1 - h_{ED}(a)}, \quad \text{i.e.} \quad u(a, b) \leqslant h_{ED}(a) + h_{FG}(b),$$

because $h_{FG_\varepsilon}(y) = h_{FG}(y)/(1-\varepsilon)$, where $G_\varepsilon := \{y \in G \colon h_{FG}(y) < 1 - \varepsilon\}$.

Propositions 9.2, 9.3 and 9.3′ imply the following

9.4. Corollary. *Let D_j be a domain on the complex plane C and let E_j*

be a compact subset of D_j ($j = 1, \ldots, N$). If $h_{E_jD_j}$ is continuous and D is given by

$$(*)\qquad D := \{x \in D_1 \times \ldots \times D_N \colon h_{E_1D_1}(x_1) + \ldots + h_{E_ND_N}(x_N) < 1\},$$

then $h_{ED}(x) = \sum_{j=1}^{N} h_{E_jD_j}(x_j)$ (where $E = E_1 \times \ldots \times E_N$) and the pair (E, D) satisfies condition (h^*).

9.5. THEOREM. *Let E be a compact subset of C^p such that Φ_E is continuous. Put*

$$D := \{x \in C^p \colon \Phi_E(x) < R\} \quad (R > 1).$$

Let G be an open subset of C^q and F a compact subset of G such that h_{FG} is continuous. Assume that the pairs (E, D) and (F, G) satisfy condition (h^*). Put

$$X := E \times G \cup D \times F,$$
$$\Omega := \{(x, y) \in D \times G \colon h_{ED}(x) + h_{FG}(y) < 1\}.$$

Let $f \colon X \to C$ be a separately analytic function on X, i.e.

1° $\forall_{x \in E}$ the function $y \to f(x, y)$ is holomorphic in G;

2° $\forall_{y \in F}$ the function $y \to f(x, y)$ is holomorphic in D.

Then there exists a (unique) holomorphic function $\tilde{f}$ in Ω such that $f = \tilde{f}|X$.

Proof. Put

$$D_\sigma := \{x \in D \colon h_{ED}(x) < \sigma\}, \quad 0 < \sigma \leqslant 1,$$
$$G_\tau := \{y \in G \colon h_{FG}(y) < \tau\}, \quad 0 < \tau \leqslant 1.$$

Since Ω is a union of the open sets $D_\sigma \times G_\tau$, where $\sigma + \tau < 1$, it is sufficient to show that there exists a function q holomorphic in $D_\sigma \times G_\tau$ such that $q|E \times F = f|E \times F$ (then $q|X \cap (D_\sigma \times G_\tau) = f|X \cap (D_\sigma \times G_\tau)$).

Fix $\sigma > 0$ and $\tau > 0$ such that $\sigma + \tau < 1$. Take $\varepsilon > 0$ so small that

$$2\varepsilon < \sigma \quad \text{and} \quad \sigma + \tau + 7\varepsilon < 1.$$

By the Vitali theorem the set

$$(3)\qquad E_k := \{x \in E \colon |f(x, y)| \leqslant k,\ y \in G_\tau\}$$

is closed for every $k \geqslant 1$. By 1°, $E_k \uparrow E$ ($k \to \infty$). Take k so large that

$$(4)\qquad \Phi^*_{E_k}(x) \leqslant \Phi_E(x) R^\varepsilon \quad \text{in } D.$$

The existence of k follows from Proposition 9.2 (ii). It follows from 4.7–4.11 that there exists d so large that

$$(5)\qquad \Phi_{E_k}(x) R^{-\varepsilon} \leqslant \varphi(x) \leqslant \Phi_{E_k}(x) \quad \text{in } D,$$

where

$$\varphi(x) := \max_{1 \leqslant j \leqslant m_d} |L^{(j)}(x, \xi^{(m_d)})|^{1/d},$$

$\xi^{(m_d)} = \{\xi_1, \ldots, \xi_{m_d}\}$ denoting any system of extremal points of E_k of order m_d. Since $\Phi_E \leqslant \Phi_{E_k}$, we get from (4) and (5)

$$\Phi_E(x) R^{-\varepsilon} \leqslant \varphi(x) \leqslant \Phi_E(x) R^{\varepsilon} \quad \text{in } D. \tag{6}$$

Hence

$$E_k \subset E \subset D_\varepsilon \subset D'_{2\varepsilon}, \quad D_\sigma \subset D'_{\sigma+\varepsilon}, \quad D'_{1-2\varepsilon} \subset D_{1-\varepsilon},$$

where

$$D'_t := \{x \in C^p \colon \varphi(x) < R^t\}, \quad t > 0.$$

Let $\xi^{(m_n)}$ be any system of extremal points of E_k of order m_n. Put

$$f_n(x, y) := \sum_{j=1}^{m_n} f(\xi_j, y) L^{(j)}(x, \xi^{(m_n)}), \tag{6}$$

$$Q_n(x, y) := f_n(x, y) - f_{n-1}(x, y), \quad x \in C^p,\ y \in G_\tau,\ n \geqslant 1\ (f_0 \equiv 0). \tag{7}$$

Then by (3), (4) and 2.11

$$|Q_n(x, y)| \leqslant 2k m_n (R^{\sigma+\varepsilon})^n, \quad x \in D_\sigma,\ y \in G_\tau,\ n \geqslant 1. \tag{8}$$

Now for any fixed $y \in F$ put $g(x) := f(x, y)$, $(L_n g)(x) := f_n(x, y)$. Let $p_n(x)$ be a polynomial of degree n such that $\varrho_n(\bar{D}'_{2\varepsilon}, g) = \|g - p_n\|_{D'_{2\varepsilon}}$. Then $|(L_n g - p_n)(x)| = |L_n(g - p_n)(x)| \leqslant m_n \|g - p_n\|_{E_k} \Phi^n_{E_k}(x)$ in C^p. Hence

$$\begin{aligned}\sup_{x \in D_\sigma} |f(x, y) - f_n(x, y)| &= \|g - L_n g\|_{D_\sigma} \leqslant \|g - p_n\|_{D_\sigma} + \|L_n(g - p_n)\|_{D_\sigma} \\ &\leqslant \|g - p_n\|_{D_\sigma} + m_n \|g - p_n\|_{D'_{2\varepsilon}} (R^{\sigma+\varepsilon}).\end{aligned}$$

Now, by Lemma 8.4, $\limsup_{n \to \infty} \sqrt[n]{\|g - p_n\|_{D'_{2\varepsilon}}} \leqslant R^{2\varepsilon - (1 - 2\varepsilon)} = R^{4\varepsilon - 1}$. Hence, since $E \subset D'_{2\varepsilon}$, we get

$$\limsup_{n \to \infty} \sqrt[n]{\|g - p_n\|_{D_\sigma}} \leqslant R_1^{\sigma - 1} = (R^{1-4\varepsilon})^{\sigma - 1} \leqslant R^{\sigma + 4\varepsilon - 1}.$$

Therefore

$$\limsup_{n \to \infty} \big(\sup_{x \in D_\sigma} |Q_n(x, y)|\big)^{1/n} \leqslant R^{\sigma + 5\varepsilon - 1}, \quad y \in F. \tag{9}$$

Hence

$$\limsup_{n \to \infty} \big(\sup_{x \in D_\sigma} |Q_n(x, y)|\big)^{1/n} \leqslant R^{\sigma + 5\varepsilon - 1}, \quad y \in F. \tag{10}$$

Put for every integer $l \geqslant 1$

(11) $$F_l := \{y \in F\colon (\sup_{x \in D_\sigma} |Q_n(x, y)|)^{1/n} \leqslant R^{\sigma+6\varepsilon-1},\ n \geqslant l\}.$$

The set F_l is compact and by (10) $F_l \uparrow F$ $(l \to \infty)$. Take l so large that

$$h_{F_l G}(y) \leqslant \tau + \varepsilon \quad \text{on } G_\tau.$$

Then by (8) and (11)

$$\frac{\frac{1}{n}\log|Q_n(x, y)| - \log R^{\sigma+6\varepsilon-1}}{\log(\sqrt[n]{M_n}\, R^{\sigma+\varepsilon}) - \log R^{\sigma+6\varepsilon-1}} \leqslant h_{F_l G}(y) \leqslant \tau+\varepsilon, \qquad x \in D_\sigma,\ y \in G_\tau,\ n \geqslant l,$$

where $M_n = 2km_n$. Hence for $x \in D_\sigma$, $y \in G_\tau$, $n \geqslant l$

$$|Q_n(x, y)|^{1/n} \leqslant R^{\sigma+\tau+7\varepsilon-1-(5\varepsilon-\varepsilon_n)(\tau+\varepsilon)},$$

where $\varepsilon_n \to 0$ is given by $R^{\varepsilon_n} = \sqrt[n]{M_n}$. Now, if $n \geqslant l_1 \geqslant l$, l_1 being sufficiently large, we get

$$\sigma+\tau+7\varepsilon-1-(5\varepsilon-\varepsilon_n)(\tau+\varepsilon) \leqslant -\alpha < 0, \qquad n \geqslant l_1,$$

so that the series $\sum_{n \geqslant 1} Q_n$ is uniformly convergent on $D_\sigma \times G_\tau$. Its limit q is a holomorphic function in $D_\sigma \times G_\tau$ and, moreover, by (7) and (9), $q(x, y) = f(x, y)$ for $x \in D_\sigma$, $y \in F$. Hence

$$q(x, y) = f(x, y) \quad \text{in } X \cap D_\sigma \times G_\tau,$$

because E (resp. F) is a determining set for functions holomorphic in D_σ (resp. in G_τ). Q.E.D.

As a corollary from the proof we get the following

9.6. PROPOSITION. *Under the assumptions of Theorem* 9.5 *let* $\xi^{(m_n)}$ *denote any extremal points system of* E *of order* m_n. *If* f_n *is given by* (6), *then the sequence* $\{f_n\}$ *is uniformly convergent on compact subsets of* Ω. *Its limit* $\tilde{f}$ *gives the holomorphic continuation of* f *from* X *to* Ω.

By induction with respect to N one can easily deduce from Corollary 9.4 and Theorem 9.5 the following

9.7. THEOREM. *Let domains* D_j *of Corollary* 9.4 *by given by*

$$D_j := \{z \in \mathbf{C}\colon \Phi_{E_j}(z) < R_j\} \quad (R_j > 1).$$

Let f *be defined and separately analytic on the set*

$$X := (D_1 \times E_2 \times \ldots \times E_N) \cup (E_1 \times D_2 \times \ldots \times E_N) \cup \ldots \cup (E_1 \times \ldots \times E_{n-1} \times D_N),$$

i.e. for every $j = 1, \ldots, N$ *and for every point* $(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_N) \in E_1 \times \ldots \times E_{j-1} \times E_{j+1} \times \ldots \times E_N$ *the function* $f(a_1, \ldots, a_{j-1}, z, a_{j+1}, \ldots, a_N)$ *is holomorphic for* $z \in D_j$.

Then the function f is continuable to a (unique) holomorphic function $\tilde{f}$ in

$$\Omega := \{x \in C^N\colon \sum_{j=1}^{N} \log \Phi_{E_j}(x_j)/\log R_j < 1\}.$$

9.8. Remark. Theorem 9.5 holds true if D is given by $(*)$ of Corollary 9.4 (see [24], Theorem 7.1). Recently Zaharjuta [34] has obtained a version of Theorem 9.5 by a different method developed in his earlier papers (see [31] for references). We obtain the required continuation $\tilde{f}$ of f by using the Lagrange interpolation formula with nodes at extremal points, while Zaharjuta obtains $\tilde{f}$ by means of Schauder bases of some spaces of analytic functions. Nguyen Thank Van was the first to point out (in his thesis [20]), that Schauder bases of some spaces of analytic functions may be used to obtain holomorphic continuation of separately analytic functions.

9.9. CONJECTURE. If h_{ED} is continuous, the pair (E, D) satisfies condition (h^*). Observe that the conjecture is true if $N = 1$.

Added in proof. By a recent result due to *E.* Bedford the conjecture holds true for all $N \geqslant 1$.

10. A sufficient condition for single-valuedness of analytic functions of N complex variables

Given a compact subset E of C^N consider the following condition:

(ξ) For every bounded open neighbourhood D of E there exists a point $\xi \in D$ such that $h^*_{ED}(\xi) < 1$.

Observe that by 3.3 and 3.10 condition (ξ) is satisfied by E if and only if E is not C^N-polar.

10.1. THEOREM. (Comp. [9], [10].) *Let U be a domain in C^N and let (f, U) be an analytic element of a complete analytic function f whose natural domain of existence is W_f. Let E be a compact subset of U satisfying (ξ). Let Λ be an infinite set of positive integers.*

If for every $n \in \Lambda$ there exists a polynomial p_n, $\deg p_n \leqslant n$, such that

$$\varepsilon_n := \|f-p_n\|_E^{1/n} \to 0 \quad \text{as } n \to \infty \ (n \in \Lambda),$$

then for every compact subset F of W_f

$$\|f-p_n\|_F^{1/n} \to 0 \quad \text{as } n \to \infty \ (n \in \Lambda),$$

so that f is single-valued and W_f is schlicht.

Proof. 1° By 3.9 it follows from (ξ) that $V_E^* \in L$. Let D be a relatively compact schlicht open neighbourhood of E contained in W_f. Since $\|p_n\|_E \leqslant \|f\|_E + \varepsilon_n^n \leqslant M_1 = \text{const}$,

$$\|f-p_n\|_D \leqslant \|f\|_D + \|p_n\|_D \leqslant \|f\|_D + \sup_{x \in D} [M_1 \exp nV_E(x)] \leqslant M^n,$$

M being a sufficiently large constant. Hence

$$\frac{(1/n)\log|f(x)-p_n(x)|-\log\varepsilon_n}{\log M-\log\varepsilon_n} \leqslant h_{ED}(x) \quad \text{in } D \text{ for } n\in\Lambda.$$

Therefore

$$|f(x)-p_n(x)| \leqslant [M\varepsilon_n^{1-h_{ED}(x)}]^n, \qquad x\in D,\ n\in\Lambda.$$

Take $\varepsilon \geqslant 0$ so small that $h^*_{ED}(\xi)+\varepsilon < 1$. Then $G := \{x\in D\colon h^*_{ED}(x) < 1-\varepsilon\}$ is an open neighbourhood of ξ and

$$\text{(1)} \qquad \|f-p_n\|_G^{1/n} \to 0 \quad \text{as } n\to\infty\ (n\in\Lambda).$$

2° It follows from (1) that without any loss of generality we may assume that E is a compact ball. Let F be any compact subset of W_f and let D be any relatively compact subdomain of W_f such that $E\cup F \subset D$. Since E is a ball, $h^*_{ED}(x) < 1$ in D. Take $\varepsilon > 0$ so small that $G := \{x\in D\colon h^*_{ED}(x) < 1-\varepsilon\}$ contains F. Then by (1)

$$\|f-p_n\|_F^{1/n} \to 0 \quad \text{as } n\to\infty\ (n\in\Lambda). \qquad \text{Q.E.D.}$$

10.2. Corollary. (Generalization of the Ostrowski theorem on lacunary power series.) *Let* $\sum_0^\infty f_n$ *be a series of homogeneous polynomials* ($\deg f_n = n$) *convergent in a neighbourhood* U *of* $0\in \mathbf{C}^N$.

If

$$f_n = 0 \quad \textit{for } n_k < n \leqslant n'_k\ (k\geqslant 1),$$

where $\{n_k\}$ *and* $\{n'_k\}$ *are increasing sequences of positive integers such that* $\lim_{k\to\infty} n'_k/n_k = +\infty$, *then the maximal analytic extension of* $f := \sum f_n$ *is single-valued.*

Proof. Put $\Lambda := \{n_k\}$ and $p_n := f_0+\ldots+f_n$. Let $E := B(0,r)$ be a compact ball contained in U. Then there exist constants θ $(0<\theta<1)$ and $M>0$ such that

$$|f(x)-p_{n_k}(x)| \leqslant \sum_{s\geqslant n'_k} |f_s(x)| \leqslant M\frac{\theta^{n'_k}}{1-\theta}, \qquad |x|\leqslant r,\ k\geqslant 1.$$

Hence $\|f-p_{n_k}\|^{1/n_k} \to 0$ as $k\to\infty$, and by Theorem 10.1 we get the required result.

10.3. Corollary. *If* W *is a domain of convergence of a lacunary series of homogeneous polynomials* $\sum_{k\geqslant 1} f_{n_k}$, *where* $\deg f_{n_k} = n_k$ *and* $\lim_{k\to\infty} n_k/n_{k+1} = 0$, *then* W *is identical with the natural domain of existence of* $f := \sum f_{n_k}$.

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